Relativistic transformations of the electric and magnetic dipole moment vectors

Consider the following problem:

> A time-independent charge density ρ' of zero total charge, but with a nonzero electric dipole moment vector $\vec{\boldsymbol{p}}\,'$ and a steady localized current density $\vec{\boldsymbol{J}}\,'$ with a nonzero magnetic dipole moment vector $\vec{\boldsymbol{m}}\,'$ exists in the rest frame of the dipoles, denoted by K'. The frame K' moves with velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ in the laboratory frame K. Find the charge and current densities ρ and $\vec{\boldsymbol{J}}$ in the frame K and determine the electric and magnetic dipole moment vectors in the frame K.

1. A few useful identities

If $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} = 0$, then the following identities can be derived:

$$\int J^i\, d^3x = \int \partial_k(J^k x^i) d^3x = 0\,, \tag{1}$$

$$\int (J^i x^j + J^j x^i)\, d^3x = \int \partial_k(J^k x^i x^j) d^3x = 0\,. \tag{2}$$

under the assumption that the currents vanish at spatial infinity. We shall use eq. (2) to obtain one additional identity. First we write

$$J^i x^j = \tfrac{1}{2}(J^i x^j + J^j x^i) + \tfrac{1}{2}(J^i x^j - J^j x^i)\,. \tag{3}$$

Next, we note that

$$-\epsilon_{ijk}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}})^k = J^i x^j - J^j x^i\,. \tag{4}$$

The proof of the last identity follows from

$$-\epsilon_{ijk}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}})^k = -\epsilon_{ijk}\epsilon_{k\ell m}x^\ell J^m = (\delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm})x^\ell J^m = J^i x^j - J^j x^i\,. \tag{5}$$

Hence, it follows that

$$J^i x^j = \tfrac{1}{2}(J^i x^j + J^j x^i) - \tfrac{1}{2}\epsilon_{ijk}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}})^k\,. \tag{6}$$

Integrating over all space and using eq. (2), we end up with

$$\int J^i x^j\, d^3x = -\tfrac{1}{2}\epsilon_{ijk}\int (\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}})^k\, d^3x\,. \tag{7}$$

2. Relativistic transformation of the electric dipole moment vector

In frame K', the total charge Q' is zero, i.e.

$$Q' = \int \rho'(\vec{\boldsymbol{x}}\,')\, d^3x' = 0\,, \tag{8}$$

whereas the electric dipole moment in frame K', denoted by $\vec{\boldsymbol{p}}\,'$, is assumed to be nonzero. By definition, the electric dipole moment is given by

$$\vec{\boldsymbol{p}}\,' = \int \vec{\boldsymbol{x}}\,'\, \rho'(\vec{\boldsymbol{x}}\,')\, d^3x'\,, \tag{9}$$

independently of the choice of origin of the coordinate system in light of eq. (8). The continuity equation relates that charge and current densities:

$$\vec{\boldsymbol{\nabla}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,' + \frac{\partial\rho'}{\partial t} = 0\,, \tag{10}$$

In the case of a time-independent charge density, it follows $\vec{\boldsymbol{\nabla}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,' = 0$, corresponding to a steady current density.

Frame K' moves with velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ with respect to frame K. The spacetime coordinates of an observer in frame K' are related to those in frame K by,

$$x_0' = \gamma(x_0 - \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\,, \qquad\qquad \vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{x}} + \frac{\gamma-1}{\beta^2}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}x_0\,, \tag{11}$$

where $x_0 = ct$ and $x_0' = ct'$. Under a Lorentz boost from frame K' back to frame K, the transformed charge and current densities are given by

$$c\rho(\vec{\boldsymbol{x}},t) = \gamma\big[c\rho'(\vec{\boldsymbol{x}}\,') + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,'(\vec{\boldsymbol{x}}\,')\big]\,, \tag{12}$$

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{J}}\,'(\vec{\boldsymbol{x}}\,') + \frac{\gamma-1}{\beta^2}\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,'(\vec{\boldsymbol{x}}\,')\big]\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}c\rho'(\vec{\boldsymbol{x}}\,')\,. \tag{13}$$

Although $\rho'(\vec{\boldsymbol{x}}\,')$ and $\vec{\boldsymbol{J}}\,'(\vec{\boldsymbol{x}}\,')$ are time-independent by assumption, note that the transformed charge and current densities in frame K are time-dependent due to eq. (11). For example, using the chain rule,

$$\frac{\partial\rho}{\partial t} = \gamma\frac{\partial\vec{\boldsymbol{x}}\,'}{\partial t}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,'\left(\rho' + \frac{1}{c}\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,'\right) = -\gamma^2\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,'\big(c\rho' + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,'\big)\,. \tag{14}$$

Recall that the differential proper time is related to the coordinate time in frame K by $d\tau = \gamma^{-1}dt$. In the rest frame K', we can identify $\tau = t'$. Hence, $dt = \gamma dt'$. Since d^4x is a Lorentz invariant quantity, it follows that $d^3x dt = d^3x' dt'$, and we can conclude that $d^3x' = \gamma d^3x$. Another derivation of this result makes use of the determinantal formula, $\det(\delta^{ij} + a^i b^j) = 1 + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}$, which is derived in the class handout entitled *A determinantal identity*. At fixed x_0, we compute the Jacobian of the transformation from x to x',

$$d^3x' = \left|\frac{\partial(x'^1, x'^2, x'^3)}{\partial(x^1, x^2, x^3)}\right| d^3x = \left|\det\left(\delta^{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j\right)\right| d^3x = \big[1 + (\gamma-1)\big] = \gamma\, d^3x\,. \tag{15}$$

If we multiply both sides of eq. (12) by d^3x and integrating over all space, we obtain

$$Q = Q' + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\int \vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\,d^3x'\,. \tag{16}$$

In light of eq. (1), it follows that $Q = Q'$, which is a well-known result since the electric charge is a Lorentz scalar. Eq. (8) then implies that $Q = Q' = 0$. In particular,

$$\int \rho(\vec{\boldsymbol{x}}, t)\,d^3x = 0\,. \tag{17}$$

Next, we multiply eq. (12) by $\vec{\boldsymbol{x}}\,'\,d^3x'$ and integrate over all space. On the left-hand side of the resulting equation, we will re-express $\vec{\boldsymbol{x}}\,'$ in terms of $\vec{\boldsymbol{x}}$, by making use of $d^3x' = \gamma d^3x$ and the Lorentz boost equation for $\vec{\boldsymbol{x}}\,'$ given in eq. (11). We end up with

$$\int \left(\vec{\boldsymbol{x}} + \frac{\gamma - 1}{\beta^2}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}x_0\right)\rho(\vec{\boldsymbol{x}})\,d^3x = \int \vec{\boldsymbol{x}}\,'\,\rho'(\vec{\boldsymbol{x}}\,')\,d^3x' + \frac{1}{c}\int \vec{\boldsymbol{x}}\,'\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\big]\,d^3x'\,. \tag{18}$$

The last term on the left-hand side of eq. (18) vanishes when integrated over all space at fixed time t after imposing eq. (8). Moreover, in light of eq. (7),

$$\frac{1}{c}\int \vec{\boldsymbol{x}}\,'\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\big]\,d^3x' = \frac{1}{2c}\int \vec{\boldsymbol{\beta}} \times \big(\vec{\boldsymbol{x}}\,' \times \vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\big)\,d^3x'\,. \tag{19}$$

The expression for the electric dipole moment in frame K' as given in eq. (9). The magnetic dipole moments in frame K' is given by

$$\vec{\boldsymbol{m}}\,' = \frac{1}{2c}\int \vec{\boldsymbol{x}}\,' \times \vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\,d^3x\,'\,. \tag{20}$$

Note that the definition of $\vec{\boldsymbol{m}}\,'$ is independent of the choice of origin of the coordinate system in light of eq. (1). To evaluate the electric dipole moment in frame K, we note that the vector that points from the location of the dipole to the spacetime point $\vec{\boldsymbol{x}}$ is given by $\vec{\boldsymbol{r}} = \vec{\boldsymbol{x}} - c\vec{\boldsymbol{\beta}}t$. Thus

$$\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{r}}\,\rho(\vec{\boldsymbol{x}}, t)\,d^3x = \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}}, t)\,d^3x\,. \tag{21}$$

Thus, we again find that the definition of $\vec{\boldsymbol{p}}$ is independent of the choice of origin of the coordinate system, in light of eq. (8).

Using the definitions of the electric and magnetic moments given above, eq. (19) reduces to:

$$\vec{\boldsymbol{p}} + \frac{\gamma - 1}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})\vec{\boldsymbol{\beta}} = \vec{\boldsymbol{p}}\,' + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}\,'\,. \tag{22}$$

If we now take the dot product of eq. (22) with $\vec{\boldsymbol{\beta}}$, we obtain $\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,' = \gamma\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}$. Inserting this result back into eq. (22) and making use of the identity $\beta^2\gamma^2 = \gamma^2 - 1$, we end up with

$$\boxed{\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}\,' - \frac{\gamma}{\gamma + 1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}\,'\,.} \tag{23}$$

The same result can be achieved with a slightly different method. First, we invert eq. (11) to obtain:

$$x_0 = \gamma(x'_0 + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'), \qquad \vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}\,' + \frac{\gamma-1}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}x'_0\,. \tag{24}$$

However, since $\rho(\vec{\boldsymbol{x}}, t)$ depends on the time coordinate in reference frame K, we shall hold x_0 fixed in the equation for $\vec{\boldsymbol{x}}$ above. That is, we must insert $x'_0 = x_0/\gamma - \vec{\boldsymbol{x}}\,'$ into the equation above for $\vec{\boldsymbol{x}}$, which yields

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}\,' + \left(\frac{\gamma-1}{\beta^2} - \gamma\right)(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}}x_0 = \vec{\boldsymbol{x}}\,' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}}x_0\,, \tag{25}$$

after making use of $\beta^2\gamma^2 = \gamma^2 - 1$. If we now multiply both sides of eq. (12) by $\vec{\boldsymbol{x}}\,d^3x$ and use $d^3x' = \gamma d^3x$ on the right hand side, we obtain

$$\begin{aligned}\vec{\boldsymbol{p}} &= \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}}, t)\,d^3x = \int d^3x' \left(\vec{\boldsymbol{x}}\,' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}}x_0\right)\left[\rho'(\vec{\boldsymbol{x}}\,') + \frac{1}{c}\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,'(\vec{\boldsymbol{x}}\,')\right] \\ &= \vec{\boldsymbol{p}}\,' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,')\vec{\boldsymbol{\beta}} + \frac{1}{c}\int \vec{\boldsymbol{x}}\,'\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,'(\vec{\boldsymbol{x}}\,')\big]\,d^3x'\,, \end{aligned} \tag{26}$$

after making use of eq. (1). Finally, the evaluation of the last integral above is the same as before, and we again arrive at

$$\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}\,' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}\,'\,. \tag{27}$$

Note that although $\rho'(\vec{\boldsymbol{x}}\,')$ and therefore $\vec{\boldsymbol{p}}\,'$, are time-independent, the corresponding charge density $\rho(\vec{\boldsymbol{x}}, t)$, in frame K, is time-dependent [as emphasized below eq. (13)]. However $\vec{\boldsymbol{p}}$ is time-independent, as there is no time-dependence on the right-hand side of eq. (23). This last assertion van be verified by evaluating

$$\begin{aligned}\frac{d\vec{\boldsymbol{p}}}{dt} &= \frac{\partial}{\partial t}\int d^3x\,\vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}}, t) = \frac{1}{\gamma}\int d^3x'\,\vec{\boldsymbol{x}}\,\frac{\partial\rho}{\partial t} \\ &= -\gamma\int d^3x'\left(\vec{\boldsymbol{x}}\,' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}}x_0\right)\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,'\big(c\rho'(\vec{\boldsymbol{x}}\,') + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,'(\vec{\boldsymbol{x}}\,')\big) = 0\,. \end{aligned} \tag{28}$$

After employing eq. (25) above, we integrated by parts and then made use of eqs. (8), (1) and (14), under the assumption of localized charges and currents that vanish at infinity.

3. Relativistic transformation of the magnetic dipole moment vector

To evaluate the magnetic moment in frame K, we again note that the vector that points from the location of the dipole to the spacetime point $\vec{\boldsymbol{x}}$ is given by $\vec{\boldsymbol{r}} = \vec{\boldsymbol{x}} - c\vec{\boldsymbol{\beta}}t$. Thus

$$\vec{\boldsymbol{m}} = \frac{1}{2c}\int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)\,d^3x = \frac{1}{2c}\int \vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)\,d^3x - \frac{t}{2}\vec{\boldsymbol{\beta}} \times \int \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)\,d^3x\,. \tag{29}$$

However, note that eq. (13) implies that $\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')$. Hence, it follows that[1]

$$\vec{\boldsymbol{\beta}}\times\int\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)\,d^3x=\frac{1}{\gamma}\,\vec{\boldsymbol{\beta}}\times\int\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\,d^3x'=0\,, \tag{30}$$

after making use of eq. (1). Hence, eq. (29) simplifies to

$$\vec{\boldsymbol{m}}=\frac{1}{2c}\int\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)\,d^3x\,. \tag{31}$$

Using $d^3x'=\gamma d^3x$ at fixed t and integrating over all space, we obtain

$$\begin{aligned}\frac{1}{2c}\int d^3x\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)=\frac{1}{2c\gamma}&\int d^3x'\left(\vec{\boldsymbol{x}}\,'-\frac{\gamma}{\gamma+1}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}}+\vec{\boldsymbol{\beta}}x_0\right)\times\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\\ &+\frac{\gamma-1}{2c\gamma\beta^2}\int d^3x'\,\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\big]\left(\vec{\boldsymbol{x}}\,'-\frac{\gamma}{\gamma+1}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}}+\vec{\boldsymbol{\beta}}x_0\right)\times\vec{\boldsymbol{\beta}}\\ &+\frac{1}{2}\int d^3x'\,\left(\vec{\boldsymbol{x}}\,'-\frac{\gamma}{\gamma+1}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}}+\vec{\boldsymbol{\beta}}x_0\right)\times\vec{\boldsymbol{\beta}}\,\rho'(\vec{\boldsymbol{x}}\,')\,.\end{aligned} \tag{32}$$

After using eqs. (1), (21) and (31), we obtain

$$\vec{\boldsymbol{m}}=\frac{1}{\gamma}\vec{\boldsymbol{m}}\,'+\tfrac{1}{2}\vec{\boldsymbol{p}}\,'\times\vec{\boldsymbol{\beta}}-\frac{1}{2c(\gamma+1)}\int d^3x'\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')+\frac{\gamma-1}{2c\gamma\beta^2}\int d^3x'\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\,\big(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{\beta}}\big). \tag{33}$$

Using eq. (7),

$$\begin{aligned}\frac{1}{c}\epsilon_{ijk}\beta^m\beta^j\int d^3x'\;J'^kx'^m=&\frac{1}{2c}\epsilon_{ijk}\epsilon_{mkn}\beta^m\beta^j\int d^3x'\,(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}\,')^n\\ =&\frac{1}{2c}\tfrac{1}{2}(\delta_{in}\delta_{jm}-\delta_{im}\delta_{jn})\beta^m\beta^j\int d^3x'\,(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}\,')^n\\ =&\frac{1}{2c}\beta^2\int d^3x'\,(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}\,')^i-\frac{1}{2c}\beta^i\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\int d^3x'\,(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}\,')\,.\end{aligned} \tag{34}$$

and

$$\begin{aligned}\frac{1}{c}\epsilon_{ijk}\beta^k\beta^m\int d^3x'\,x'^j\,J'^m=&\frac{1}{2c}\epsilon_{ijk}\epsilon_{jmn}\beta^k\beta^m\int d^3x\,(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}\,')^n\\ =&\frac{1}{2c}(\delta_{in}\delta_{km}-\delta_{im}\delta_{kn})\beta^k\beta^m\int d^3x\,(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}\,')^n\\ =&\frac{1}{2c}\beta^2\int d^3x'\,(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}\,')^i-\frac{1}{2c}\beta^i\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\int d^3x'\,(\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}\,')\,.\end{aligned} \tag{35}$$

[1]Although one cannot make use of eq. (1) to eliminate the last term in eq. (29), since the current density in frame K is no longer steady (i.e., $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\neq 0$ since $\partial\rho/\partial t\neq 0$), one can instead make use of a modified version of eq. (1) as follows. Using the continuity equation [cf. eq. (10)], it follows that

$$\int J^i\,d^3x=\int\big[\partial_k(J^kx^i)-x^i\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\big]d^3x=\int x^i\frac{\partial\rho(\vec{\boldsymbol{x}},t)}{\partial t}\,d^3x=\frac{dp^i}{dt}\,.$$

However, in light of eqs. (14) and (28), it follows that $dp^i/dt=0$. Hence, the last term in eq. (29) vanishes even before one takes the cross product with $\vec{\boldsymbol{\beta}}$.

That is,

$$\begin{aligned}
\frac{1}{c}\int d^3x'\,(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}')\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}') &= \frac{1}{c}\int d^3x'\,\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}')\left(\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{\beta}}\right) \\
&= \frac{1}{2c}\int d^3x'\left[\beta^2(\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}') - \vec{\boldsymbol{\beta}}\cdot(\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}')\,\vec{\boldsymbol{\beta}}\right] \\
&= \beta^2\vec{\boldsymbol{m}}' - (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{m}}')\vec{\boldsymbol{\beta}} = -\vec{\boldsymbol{\beta}}\times\left(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{m}}'\right). \qquad (36)
\end{aligned}$$

Inserting this result back into eq. (33) yields,[2]

$$\boxed{\vec{\boldsymbol{m}} = \frac{1}{\gamma}\vec{\boldsymbol{m}}' - \frac{\gamma-1}{2(\gamma+1)}\vec{\boldsymbol{\beta}}\times\left(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{m}}'\right) + \tfrac{1}{2}\vec{\boldsymbol{p}}'\times\vec{\boldsymbol{\beta}}.} \qquad (37)$$

Note that although $\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}')$ and therefore $\vec{\boldsymbol{m}}'$, are time-independent, the corresponding current density in frame K is time-dependent [as emphasized below eq. (13)]. However $\vec{\boldsymbol{m}}$ is time-independent, as indicated explicitly in eq. (37), as there is no time-dependence on the right-hand side of eq. (23). The time-independence of $\vec{\boldsymbol{m}}$ can also be checked by a calculation that is similar to the one employed in obtaining eq. (28).

4. The dispute over the relativistic transformation of $\vec{\boldsymbol{m}}$

The result obtained in eq. (37) has been disputed in the literature. In the case of $\vec{\boldsymbol{m}}' = 0$, eq. (37) reduces to

$$\vec{\boldsymbol{m}} = \tfrac{1}{2}\vec{\boldsymbol{p}}'\times\vec{\boldsymbol{\beta}}. \qquad (38)$$

Although this result is obtained by Jackson in problem 11.27 of his textbook, the factor of 1/2 is controversial. In the literature, one often finds the result of eq. (38) quoted without the factor of 1/2. For a discussion of this discrepancy, see V. Hnizdo, *Magnetic dipole moment of a moving electric dipole*, Am. J. Phys, **80**, 645 (2012), with more details given in V. Hnizdo, and K.T. McDonald, *Fields and Moments of a Moving Electric Dipole*.[3]

However, it has been argued that eqs. (37) and (38) are both wrong due to a subtle effect that invalidates the computation presented in Section 3. Although the equation for the magnetic moment given in eq. (31) is valid when the current density is steady, some authors argue that eq. (31) is *not* valid when $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}}\neq 0$. Indeed, a detailed computation in A.L. Kholmetskii, O.V. Missevitch, and T. Yarman, Eur. Phys. J. Plus, **131**, 316 (2016) shows that a more appropriate definition of $\vec{\boldsymbol{m}}$ in the laboratory frame yields.[4]

$$\vec{\boldsymbol{m}} = \vec{\boldsymbol{m}}' - \frac{\gamma-1}{\gamma\beta^2}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{m}}')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{p}}'\times\vec{\boldsymbol{\beta}}. \qquad (39)$$

[2]Eq. (37) was first obtained in G.P. Fisher, Am. J. Phys. **39**, 1528 (1971).

[3]This reference can be obtained from `http://kirkmcd.princeton.edu/examples/movingdipole.pdf`.

[4]See also A.L. Kholmetskii, O.V. Missevitch, and T. Yarman, Progress In Electromagnetics Research B **47**, 263 (2013) and Int. J. Mod. Phys. A **35**, 2050135 (2020).

After using $\gamma^2\beta^2 = \gamma^2 - 1$, eq. (39) takes on a form that is very similar to that of eq. (23),

$$\boxed{\vec{\boldsymbol{m}} = \vec{\boldsymbol{m}}' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}}')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{p}}' \times \vec{\boldsymbol{\beta}}\,.} \tag{40}$$

5. A summary of results

Collecting the results obtained in Sections 2 and 4, the relativistic transformations of the electric and magnetic dipole moment vectors from their rest frame K' (which is moving with velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ with respect to a laboratory observer) to the laboratory frame K are given by

$$\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}'\,, \tag{41}$$

$$\vec{\boldsymbol{m}} = \vec{\boldsymbol{m}}' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}}')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{p}}' \times \vec{\boldsymbol{\beta}}\,. \tag{42}$$

To invert these formulae, we can compute the following quantities:

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} = \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}'\left[1 - \frac{\gamma\beta^2}{\gamma+1}\right] = \frac{1}{\gamma}\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}'\,, \tag{43}$$

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}} = \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}}'\left[1 - \frac{\gamma\beta^2}{\gamma+1}\right] = \frac{1}{\gamma}\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}}'\,. \tag{44}$$

Hence,

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}' = \gamma\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,, \qquad\qquad \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}}' = \gamma\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}}\,. \tag{45}$$

Likewise, after using the above results,

$$\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{p}}' = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{p}} - \gamma\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}}) + \beta^2\vec{\boldsymbol{m}}'\,, \tag{46}$$

$$\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}' = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}} + \gamma\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}) - \beta^2\vec{\boldsymbol{p}}'\,. \tag{47}$$

Plugging these results back into eqs. (41) and (42), and using $1 - \beta^2 = \gamma^{-2}$, we end up with

$$\vec{\boldsymbol{p}}' = \gamma^2\left[\vec{\boldsymbol{p}} - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})\vec{\boldsymbol{\beta}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}\right], \tag{48}$$

$$\vec{\boldsymbol{m}}' = \gamma^2\left[\vec{\boldsymbol{m}} - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{m}})\vec{\boldsymbol{\beta}} - \vec{\boldsymbol{p}} \times \vec{\boldsymbol{\beta}}\right]. \tag{49}$$

In particular, notice that it is not enough to start from eqs. (41) and (42), and interchange $\vec{\boldsymbol{p}} \leftrightarrow \vec{\boldsymbol{p}}'$, $\vec{\boldsymbol{m}} \leftrightarrow \vec{\boldsymbol{m}}'$, and $\vec{\boldsymbol{\beta}} \leftrightarrow -\vec{\boldsymbol{\beta}}$.