1. The energy and the linear momentum of a distribution of electromagnetic fields in vacuum is given (in SI units) by

$$U = \frac{\epsilon_0}{2}\int d^3x\,(\vec{\boldsymbol{E}}^{\,2} + c^2\vec{\boldsymbol{B}}^{\,2})\,, \tag{1}$$

$$\vec{\boldsymbol{P}} = \epsilon_0\int d^3x\,\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}\,, \tag{2}$$

where the integration is over all space. Consider an expansion of the electric field in terms of plane waves:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[E_0(\vec{\boldsymbol{k}},\lambda)\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,e^{i(\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}-\omega t)} + \text{c.c.}\right]\,, \tag{3}$$

where $E_0(\vec{\boldsymbol{k}},\lambda)$ is a complex amplitude and c.c. stands for "complex conjugate" of the preceding term. The polarization vector satisfies:

$$\hat{\boldsymbol{\epsilon}}_\lambda(-\vec{\boldsymbol{k}}) = \hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\,. \tag{4}$$

(a) Show that $\vec{\boldsymbol{P}}$ can be written as

$$\vec{\boldsymbol{P}} = \frac{2\epsilon_0}{c}\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\,|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,\hat{\boldsymbol{k}}\,. \tag{5}$$

Note that all time dependence has canceled out. Explain.

Consider the Coulomb gauge, where $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}} = 0$ [cf. eq. (6.21) of Jackson]. In the absence of external sources ($\rho = \vec{\boldsymbol{J}} = 0$), we also have $\Phi = 0$ [cf. eq. (6.23) of Jackson]. Using eq. (6.9) of Jackson, the electric and magnetic fields are given by,

$$\vec{\boldsymbol{E}} = -\frac{\partial\vec{\boldsymbol{A}}}{\partial t}\,,\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}\,. \tag{6}$$

In class, we showed that one can expand $\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)$ in plane waves,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t) = \int\frac{d^3k}{(2\pi)^3}\left[\vec{\boldsymbol{a}}(\vec{\boldsymbol{k}})\,e^{i(\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}-\omega t)} + \vec{\boldsymbol{a}}^*(\vec{\boldsymbol{k}})\,e^{-i(\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}-\omega t)}\right]\,,$$

where $\omega = kc$ (with $k \equiv |\vec{\boldsymbol{k}}|$) and

$$\vec{\boldsymbol{a}}(\vec{\boldsymbol{k}}) = \sum_\lambda a_\lambda(\vec{\boldsymbol{k}})\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,. \tag{7}$$

The sum over λ is taken over two orthogonal polarization states, labeled by λ, that satisfy:

$$\vec{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}}) = 0\,,\qquad\text{and}\qquad \hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}) = \delta_{\lambda\lambda'}\,. \tag{8}$$

Using eq. (6), it then follows that:

$$\vec{\boldsymbol{E}}_0(\vec{\boldsymbol{k}}) = ikc\,\vec{\boldsymbol{a}}(\vec{\boldsymbol{k}})\,, \qquad \vec{\boldsymbol{B}}_0(\vec{\boldsymbol{k}}) = i\vec{\boldsymbol{k}}\times\vec{\boldsymbol{a}}(\vec{\boldsymbol{k}}) = \frac{1}{c}\,\hat{\boldsymbol{k}}\times\vec{\boldsymbol{E}}_0(\vec{\boldsymbol{k}})\,, \tag{9}$$

where $\hat{\boldsymbol{k}}\equiv\vec{\boldsymbol{k}}/k$ and

$$\vec{\boldsymbol{E}}_0(\vec{\boldsymbol{k}}) = \sum_\lambda E_0(\vec{\boldsymbol{k}},\lambda)\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,.$$

That is, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ is given by eq. (3) and

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{1}{c}\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[E_0(\vec{\boldsymbol{k}},\lambda)\,\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)}+\text{c.c.}\right]\,. \tag{10}$$

Inserting eqs. (3) and (10) into eq. (2) [taking care to employ different dummy variables in the sums and integrals], and expanding out the resulting expression, we obtain:

$$\begin{aligned}\vec{\boldsymbol{P}} = \frac{\epsilon_0}{(2\pi)^6\,c}\sum_\lambda\sum_{\lambda'}\int d^3k\,d^3k'\,d^3x\Big\{&E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]e^{i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{-i(\omega+\omega')t}\\ &+E_0^*(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]e^{-i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{i(\omega+\omega')t}\\ &+E_0(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]e^{-i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{-i(\omega-\omega')t}\\ &+E_0^*(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]e^{-i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{i(\omega-\omega')t}\Big\}\,,\end{aligned} \tag{11}$$

where $\omega\equiv kc$ and $\omega'\equiv k'c$. In our notation, $k\equiv|\vec{\boldsymbol{k}}|$ and $k'\equiv|\vec{\boldsymbol{k}}'|$.

We may now perform the integral over $\vec{\boldsymbol{x}}$, using

$$\frac{1}{(2\pi)^3}\int d^3x\,e^{i(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}} = \delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}')\,, \tag{12}$$

and then use the delta function to facilitate the integration over $\vec{\boldsymbol{k}}'$. Then eq. (11) reduces to

$$\begin{aligned}\vec{\boldsymbol{P}} = \frac{\epsilon_0}{c}\sum_\lambda\sum_{\lambda'}\int\frac{d^3k}{(2\pi)^3}\Big\{&-E_0(\vec{\boldsymbol{k}},\lambda)E_0(-\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]e^{-2i\omega t}\\ &-E_0^*(\vec{\boldsymbol{k}},\lambda)E_0^*(-\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]e^{2i\omega t}\\ &+E_0(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]\\ &+E_0^*(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]\Big\}\,,\end{aligned} \tag{13}$$

where we have used eq. (4) to write:[1]

$$\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}') = \hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}') = \hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\,. \tag{14}$$

[1]Recall that for any well-behaved function $f(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}')$ we have $f(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}')\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}') = f(\vec{\boldsymbol{k}},\pm\vec{\boldsymbol{k}})\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}')$, due to the presence of the delta function. For example, $\omega'\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}') = k'c\,\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}') = kc\,\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}') = \omega\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}')$, since $|\pm\vec{\boldsymbol{k}}| = k$.

We can now make use of the vector identity,

$$\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}}) \times [\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})] = \hat{\boldsymbol{k}}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})] - \hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})[\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})] = \hat{\boldsymbol{k}}\,\delta_{\lambda\lambda'}\,, \tag{15}$$

while employing the properties of the polarization vector given in eq. (8). Using eq. (15) allows us to perform the sum over λ' in eq. (13), which yields

$$\vec{\boldsymbol{P}} = \frac{\epsilon_0}{c}\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\,\hat{\boldsymbol{k}}\left\{-E_0(\vec{\boldsymbol{k}},\lambda)E_0(-\vec{\boldsymbol{k}},\lambda)e^{-2i\omega t} - E^*_0(\vec{\boldsymbol{k}},\lambda)E^*_0(-\vec{\boldsymbol{k}},\lambda)e^{2i\omega t} + 2|E_0(\vec{\boldsymbol{k}},\lambda)|^2\right\}. \tag{16}$$

Noting that $\omega = kc$ where $k \equiv |\vec{\boldsymbol{k}}|$ and $\hat{\boldsymbol{k}} \equiv \vec{\boldsymbol{k}}/k$, it follows that

$$\int d^3k\,\hat{\boldsymbol{k}}\,E_0(\vec{\boldsymbol{k}},\lambda)E_0(-\vec{\boldsymbol{k}},\lambda)e^{-2ikct} = 0\,,$$

since the integrand is an odd function under $\vec{\boldsymbol{k}} \to -\vec{\boldsymbol{k}}$. That is, if we denote the integrand by $f(\vec{\boldsymbol{k}}) \equiv \hat{\boldsymbol{k}}\,E_0(\vec{\boldsymbol{k}},\lambda)E_0(-\vec{\boldsymbol{k}},\lambda)e^{-2ikct}$, then $f(\vec{\boldsymbol{k}}) = -f(-\vec{\boldsymbol{k}})$. It follows that

$$\int f(\vec{\boldsymbol{k}})\,d^3k = -\int f(-\vec{\boldsymbol{k}})\,d^3k = -\int f(\vec{\boldsymbol{k}})\,d^3k = 0\,, \tag{17}$$

after making a change of integration variables $\vec{\boldsymbol{k}} \to -\vec{\boldsymbol{k}}$ and noting that the absolute value of the determinant of the corresponding Jacobian matrix is one. In the final step above, we used the fact that a quantity that is equal to its negative must be zero. Hence, eq. (16) yields

$$\vec{\boldsymbol{P}} = \frac{2\epsilon_0}{c}\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\,\hat{\boldsymbol{k}}\,|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,, \tag{18}$$

which confirms the result of eq. (5).

Note that $\vec{\boldsymbol{P}}$ given in eq. (18) is explicitly time-independent. This is simply an expression of the conservation of momentum, $d\vec{\boldsymbol{P}}/dt = 0$. This is a consequence of eq. (6.122) of Jackson. Since $\rho = \vec{\boldsymbol{J}} = 0$ for a free electromagnetic field, we have $\vec{\boldsymbol{P}}_{\rm mech} = 0$, in which case

$$\frac{d\vec{\boldsymbol{P}}}{dt} = \frac{\vec{\boldsymbol{P}}_{\rm field}}{dt} = \oint_S da\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\overleftrightarrow{\boldsymbol{T}} = 0\,,$$

where $\overleftrightarrow{\boldsymbol{T}}$ is the Maxwell stress tensor. The unit vector $\hat{\boldsymbol{n}}$ is the outward normal to the surface S, where S is the surface of infinity. For any finite energy field configuration, the stress tensor vanishes at the surface of infinity and we recover $d\vec{\boldsymbol{P}}/dt = 0$ as expected.

(b) Obtain the corresponding expression for the total energy U. Employing the photon interpretation for each mode $(\vec{\boldsymbol{k}},\lambda)$ of the electromagnetic field, justify the statement that photons are massless.

The total energy is given (in SI units) by

$$U = \frac{\epsilon_0}{2}\int d^3x\,(\vec{\boldsymbol{E}}^{\,2} + c^2\vec{\boldsymbol{B}}^{\,2})\,. \tag{19}$$

We first compute

$$\int \vec{\boldsymbol{E}}^{\,2}\, d^3x = \frac{1}{(2\pi)^6}\sum_\lambda\sum_{\lambda'}\int d^3k d^3k' d^3x \biggl\{\Bigl[E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')\,e^{i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\,e^{-i(\omega+\omega')t} + \text{c.c.}\Bigr]$$
$$+\Bigl[E_0(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')\,e^{i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\,e^{-i(\omega-\omega')t} + \text{c.c.}\Bigr]\biggr\}\,,$$

where the computation is similar to that of part (a). Integrating over $\vec{\boldsymbol{x}}$ and using eq. (14) as we did in part (a), it follows that

$$\int \vec{\boldsymbol{E}}^{\,2}\, d^3x = \sum_\lambda\sum_{\lambda'}\int\frac{d^3k}{(2\pi)^3}\biggl\{E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})\,e^{-2i\omega t} + E_0(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}) + \text{c.c.}\biggr\}.$$

Summing over λ' using eq. (8), we obtain

$$\int \vec{\boldsymbol{E}}^{\,2}\, d^3x = \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\Bigl[E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}},\lambda)\,e^{-2i\omega t} + \text{c.c.}\Bigr] + 2\sum_\lambda\int\frac{d^3k}{(2\pi)^3}|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,. \qquad (20)$$

Next, we compute $\int c^2\vec{\boldsymbol{B}}^{\,2}d^3x$. The only difference in the computation compared to the one above is that $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})$ is replaced by $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})$ and $\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')$ is replaced by $\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')$. Thus, instead of obtaining the factor $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')\,\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')$ after the integration over $\vec{\boldsymbol{x}}$, we now have [cf. footnote 1]:

$$\begin{aligned}
[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})]\boldsymbol{\cdot}[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]\,\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}') &= [\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{k}}'][\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')] - [\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')][\hat{\boldsymbol{k}}'\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})]\,\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\\
&= \bigl\{-\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}}) + [\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})][\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})]\bigr\}\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\\
&= -\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})\,\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\,,
\end{aligned}$$

after using eqs. (14) and (8). Similarly, instead of obtaining the factor $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')\,\delta^3(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')$ after the integration over $\vec{\boldsymbol{x}}$, we now have:

$$[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})]\boldsymbol{\cdot}[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]\,\delta^3(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}') = \hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})\,\delta^3(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\,.$$

Hence, it follows that:

$$\int c^2\vec{\boldsymbol{B}}^{\,2}\, d^3x = -\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\Bigl[E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}},\lambda)\,e^{-2i\omega t} + \text{c.c.}\Bigr] + 2\sum_\lambda\int\frac{d^3k}{(2\pi)^3}|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,. \qquad (21)$$

Adding eqs. (20) and (21) yields

$$U = 2\epsilon_0\sum_\lambda\int\frac{d^3k}{(2\pi)^3}|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,. \qquad (22)$$

Note that U given in eq. (22) is explicitly time-independent. This is simply an expression of the conservation of momentum, $dU/dt = 0$. This is a consequence of eq. (6.111) of Jackson. Since $\rho = \vec{\boldsymbol{J}} = 0$ for a free electromagnetic field, we have $\vec{\boldsymbol{P}}_{\rm mech} = 0$, in which case

$$\frac{dU}{dt} = \frac{U_{\rm field}}{dt} = -\oint_S da\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{S}} = 0\,,$$

where $\vec{\boldsymbol{S}}$ is the Poynting vector. For any finite energy field configuration, the Poynting vector vanishes at the surface of infinity and we recover $dU/dt = 0$ as expected.

Finally, consider a fixed wave number vector $\vec{\boldsymbol{k}}_0$, for which $E_0(\vec{\boldsymbol{k}}, \lambda) \equiv E_0(\lambda)\, \delta^3(\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}_0)$. Then, eqs. (18) and (22) yield

$$U = 2\epsilon_0 \sum_\lambda |E_0(\lambda)|^2 \,, \qquad\qquad \vec{\boldsymbol{P}} = \hat{\boldsymbol{k}}_0 \frac{2\epsilon_0}{c} \sum_\lambda |E_0(\lambda)|^2 = \frac{\hat{\boldsymbol{k}}_0}{c} U \,.$$

That is, $U = Pc$. Comparing this result to the relativistic relation between the energy and momentum of a particle, $E = \sqrt{p^2c^2 + m^2c^4}$, we conclude that photons are massless.

2. [Jackson, problem 7.12] The time dependence of electrical disturbances in good conductors is governed by the frequency-dependent conductivity given in Jackson eq. (7.58). Consider longitudinal electric fields in a conductor,[2] using Ohm's law, the continuity equation, and the differential form of Coulomb's law.

(a) Show that the time-Fourier-transformed charge density satisfies the equation

$$\bigl[\sigma(\omega) - i\omega\epsilon_0\bigr]\rho(\vec{\boldsymbol{x}}, \omega) = 0 \,.$$

In this problem, we assume that the electric permittivity is that of the vacuum. First, we perform a Fourier transform of the current vector $\vec{\boldsymbol{J}}$, the electric displacement vector $\vec{\boldsymbol{D}}$, and the current density ρ,

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) e^{i\omega t}\, dt \,, \tag{23}$$

$$\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{\boldsymbol{D}}(\vec{\boldsymbol{x}}, t) e^{i\omega t}\, dt \,, \tag{24}$$

$$\rho(\vec{\boldsymbol{x}}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\vec{\boldsymbol{x}}, t) e^{i\omega t}\, dt \,. \tag{25}$$

Taking the divergence of eq. (23) yields

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, \omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) e^{i\omega t}\, dt = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \rho(\vec{\boldsymbol{x}}, t)}{\partial t} e^{i\omega t}\, dt \\ &= \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\vec{\boldsymbol{x}}, t) e^{i\omega t}\, dt = i\omega \rho(\vec{\boldsymbol{x}}, \omega) \,, \end{aligned} \tag{26}$$

where we have used the continuity equation in the second step above,

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) + \frac{\partial \rho(\vec{\boldsymbol{x}}, t)}{\partial t} = 0 \,. \tag{27}$$

[2]Inside a real imperfect conductor (i.e, with conductivity $\sigma \neq \infty$), a small longitudinal electric field exists, parallel to the current density vector $\vec{\boldsymbol{J}}$, which drives the charges according to Ohm's Law, and allows for plasma oscillations of the free charges within the conducting medium.

After an integration by parts (where the surface term vanishes under the assumption that there is no charge density at infinity), eq. (25) has been employed to obtain the final result.

Next, we apply Ohm's Law for harmonic currents and electric fields,[3]

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},\omega) = \sigma(\omega)\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},\omega) = \frac{\sigma(\omega)}{\epsilon_0}\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}},\omega)\,, \tag{28}$$

where we have used $\vec{\boldsymbol{D}} = \epsilon_0\vec{\boldsymbol{E}}$. Taking the divergence of eq. (28) yields

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},\omega) &= \frac{\sigma(\omega)}{\epsilon_0}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}},\omega) = \frac{\sigma(\omega)}{\epsilon_0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}},t)e^{i\omega t}\,dt \\ &= \frac{\sigma(\omega)}{\epsilon_0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\rho(\vec{\boldsymbol{x}},t)e^{i\omega t}\,dt = \frac{\sigma(\omega)}{\epsilon_0}\rho(\vec{\boldsymbol{x}},\omega)\,,\end{aligned} \tag{29}$$

after making use of the differential form of Coulomb's law, $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}},t)$. Setting eqs. (26) and (29) equal, we obtain

$$\big[\sigma(\omega) - i\omega\epsilon_0\big]\rho(\vec{\boldsymbol{x}},\omega) = 0\,, \tag{30}$$

as requested.

(b) Using the representation $\sigma(\omega) = \sigma_0/(1 - i\omega\tau)$, where $\sigma_0 \equiv \epsilon_0\omega_p^2\tau$ and τ is a damping time, show that in the approximation $\omega_p\tau \gg 1$, any initial disturbance will oscillate with the plasma frequency and decay in amplitude with a decay constant $\lambda = 1/(2\tau)$. Note that if you use $\sigma(\omega) \simeq \sigma(0) = \sigma_0$ in part (a), you will find no oscillations and extremely rapid damping with the (wrong) decay constant $\lambda_w = \sigma_0/\epsilon_0$.

Under the assumption that an electrical disturbance of (positive) frequency ω is present, i.e., $\vec{\boldsymbol{E}}(t) = \vec{\boldsymbol{E}}_0 e^{-i\omega t}$, it follows from the differential form of Coulomb's law that $\rho(\vec{\boldsymbol{x}},\omega) \neq 0$. Hence, eq. (30) yields

$$\sigma(\omega) = i\omega\epsilon_0\,. \tag{31}$$

Inserting $\sigma(\omega) = \epsilon_0\omega_p^2\tau/(1 - i\omega\tau)$ into the above equation, we obtain a quadratic equation for ω,

$$\omega^2\tau + i\omega - \omega_p^2\tau = 0\,. \tag{32}$$

Solving for ω, one obtains:

$$\omega = \frac{-i \pm \sqrt{4\omega_p^2\tau^2 - 1}}{2\tau}\,. \tag{33}$$

In the approximation $\omega_p\tau \gg 1$, eq. (33) simplifies to

$$\omega = \omega_p - \frac{i}{2\tau}\left[1 + \mathcal{O}\left(\frac{1}{\omega_p\tau}\right)\right]\,, \tag{34}$$

[3] For time-independent currents and electric fields in a homogeneous and isotropic conducting medium, Ohm's Law states that $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = \sigma\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$. However, if the currents and electric fields are harmonic with angular frequency ω, then the conducting medium cannot respond instantaneously to the time-varying electric field. In order to satisfy the requirements of causality, the conductivity σ must be a function of ω. In this case, Ohm's Law is given by eq. (28), under suitable assumptions on the functional form of $\sigma(\omega)$.

where we have chosen the plus sign in eq. (33), since the frequency ω is positive. After dropping the term of $\mathcal{O}\bigl(1/(\omega_p\tau)\bigr)$ in eq. (34), the electrical disturbance is given by

$$\vec{\boldsymbol{E}}(t) = \vec{\boldsymbol{E}}_0\, e^{-i\omega_p t}e^{-\lambda t}\,, \quad \text{where } \lambda \equiv (2\tau)^{-1}. \tag{35}$$

That is, the disturbance oscillates with the plasma frequency ω_p, and the amplitude, $\vec{\boldsymbol{E}}_0\, e^{-\lambda t}$, is damped out with a decay constant $\lambda = (2\tau)^{-1}$.

Note that if you use $\sigma(\omega) \simeq \sigma(0) = \sigma_0$ in part (a), then eq. (30) yields $\omega = -i\sigma_0/\epsilon_0$. Inserting this result into $\vec{\boldsymbol{E}}(t) = \vec{\boldsymbol{E}}_0 e^{-i\omega t}$ would then yield $\vec{\boldsymbol{E}}(t) = \vec{\boldsymbol{E}}_0 e^{-\lambda_w t}$, where $\lambda_w \equiv \sigma_0/\epsilon_0$. One would then wrongly conclude that the electric field disturbance does not oscillate and the amplitude is rapidly damped out with the wrong decay constant.

3. [Jackson, problem 7.22]. Use the Kramers-Kronig relations to calculate the real part of $\epsilon(\omega)$, given the imaginary part of $\epsilon(\omega)$ for positive ω as[4]

(a) $\operatorname{Im}\epsilon(\omega)/\epsilon_0 = \lambda\bigl[\Theta(\omega-\omega_1) - \Theta(\omega-\omega_2)\bigr]\,, \qquad \omega_2 > \omega_1 > 0,$

(b) $\operatorname{Im}\epsilon(\omega)/\epsilon_0 = \dfrac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\,, \qquad \gamma > 0 \text{ and } \omega_0 > \tfrac{1}{2}\gamma.$

In each case, sketch the behavior of $\operatorname{Im}\epsilon(\omega)$ and the results for $\operatorname{Re}\epsilon(\omega)$ as functions of ω. Comment on the reasons for similarities or differences of your results as compared with the curves in Fig. 7.8 of Jackson. In part (a), the step function is defined as $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$.

The Kramers-Kronig relation for computing $\operatorname{Re}\epsilon(\omega)/\epsilon_0$ given $\operatorname{Im}\epsilon(\omega)/\epsilon_0$ has been given in eq. (7.119) of Jackson:

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 + \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{Im}\epsilon(\omega')/\epsilon_0}{\omega'-\omega}\,d\omega'\,. \tag{36}$$

An alternative form of eq. (36) can be obtained by breaking up the integration region into two regimes: (i) $-\infty < \omega' < 0$ and (ii) $0 < \omega' < \infty$. In the first regime, one can change the integration variable via $\omega' \to -\omega'$. In light of the fact that $\operatorname{Im}\epsilon(-\omega') = -\operatorname{Im}\epsilon(\omega')$, one can combine the resulting two integrals to obtain eq. (7.120) of Jackson,

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 + \frac{2}{\pi}P\int_{0}^{\infty}\frac{\omega'\operatorname{Im}\epsilon(\omega')/\epsilon_0}{\omega'^2-\omega^2}\,d\omega'\,. \tag{37}$$

(a) $\operatorname{Im}\epsilon(\omega)/\epsilon_0$ is given by:

$$\operatorname{Im}\epsilon(\omega')/\epsilon_0 = \begin{cases} 0\,, & \text{for } \omega > \omega_2, \\ \lambda\,, & \text{for } \omega_1 < \omega < \omega_2, \\ 0\,, & \text{for } 0 \leq \omega < \omega_1, \end{cases} \tag{38}$$

[4] As noted by Jackson, one can extend these results to negative frequencies by imposing $\operatorname{Re}\epsilon(-\omega) = \operatorname{Re}\epsilon(\omega)$ and $\operatorname{Im}\epsilon(-\omega) = -\operatorname{Im}\epsilon(\omega)$.

under the assumption that $\omega_2 > \omega_1 > 0$ and λ is a constant independent of ω. Plugging this result into eq. (37) yields:

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 + \frac{2\lambda}{\pi} P \int_{\omega_1}^{\omega_2} \frac{\omega' d\omega'}{\omega'^2 - \omega^2} = 1 + \frac{\lambda}{\pi} P \int_{\omega_1^2}^{\omega_2^2} \frac{d\omega'^2}{\omega'^2 - \omega^2} .$$

If $\omega^2 > \omega_2^2$ or $0 < \omega^2 < \omega_1^2$, then the denominator of the integrand is never zero over the range of integration. In this case, we can drop the principal value symbol P and carry out the integration. It then follows that

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 + \frac{\lambda}{\pi} \ln \left| \frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2} \right| , \qquad \text{for } \omega^2 > \omega_2^2 \text{ or } \omega^2 < \omega_1^2. \tag{39}$$

On the other hand, if $\omega_1^2 < \omega^2 < \omega_2^2$, then the denominator of the integrand will vanish when $\omega' = \omega$. In this case, we must use the definition of the principal value prescription to obtain:

$$\begin{aligned}\operatorname{Re}\epsilon(\omega)/\epsilon_0 &= 1 + \frac{\lambda}{\pi} \lim_{\varepsilon\to 0} \left\{ \int_{\omega_1^2}^{\omega^2-\varepsilon} \frac{d\omega'^2}{\omega'^2 - \omega^2} + \int_{\omega^2+\varepsilon}^{\omega_2^2} \frac{d\omega'^2}{\omega'^2 - \omega^2} \right\} \\ &= 1 + \frac{\lambda}{\pi} \lim_{\varepsilon\to 0} \left\{ \ln\left(\frac{\varepsilon}{\omega^2 - \omega_1^2} \right) + \ln\left(\frac{\omega_2^2 - \omega^2}{\varepsilon} \right) \right\} \\ &= 1 + \frac{\lambda}{\pi} \ln\left(\frac{\omega_2^2 - \omega^2}{\omega^2 - \omega_1^2} \right) , \qquad \text{for } \omega_1^2 < \omega^2 < \omega_2^2. \end{aligned} \tag{40}$$

Comparing the results of eqs. (39) and (40), it follows that the result of eq. (39) is correct for *all* values of ω. That is,

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 + \frac{\lambda}{\pi} \ln \left| \frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2} \right| , \qquad \text{for } 0 \le \omega < \infty. \tag{41}$$

A sketch of the behavior of $\operatorname{Im}\epsilon(\omega)/\epsilon_0$ [left panel] and $\operatorname{Re}\epsilon(\omega)/\epsilon_0$ [right panel] as a function of ω is exhibited in Figure 1 below. As expected, $\operatorname{Re}\epsilon(\omega)/\epsilon_0 \to 1$ as $\omega \to \infty$. However, this

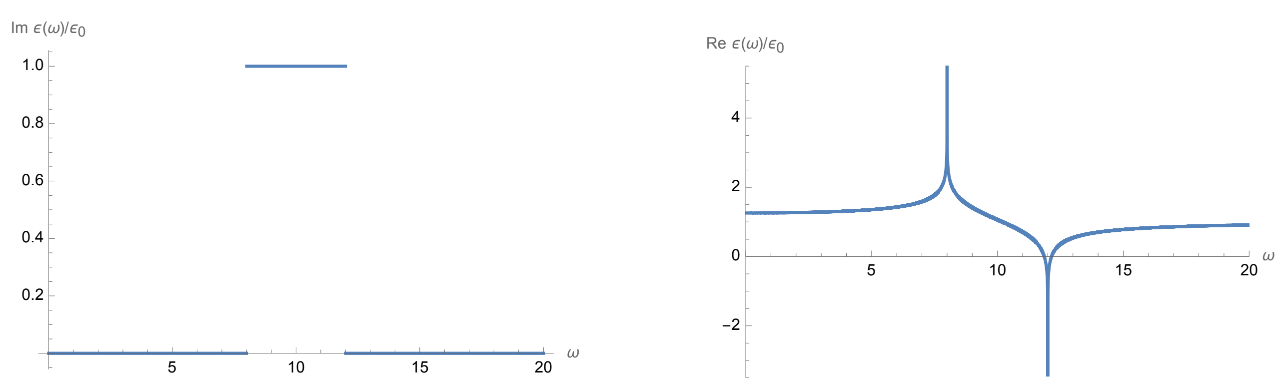

Figure 1: A sketch of the behavior of $\operatorname{Im}\epsilon(\omega)/\epsilon_0$ [left panel] and $\operatorname{Re}\epsilon(\omega)/\epsilon_0$ [right panel] as a function of ω. The representative values of $\lambda = 1$, $\omega_1 = 8$, and $\omega_2 = 12$ have been chosen for illustrative purposes. Note that $\operatorname{Re}\epsilon(\omega)/\epsilon_0$ diverges logarithmically as $\omega \to \omega_1$ and as $\omega \to \omega_2$.

example is somewhat unrealistic since $\mathrm{Re}\,\epsilon(\omega)/\epsilon_0$ diverges (albeit logarithmically) as $\omega \to \omega_1$ or as $\omega \to \omega_2$. This behavior can be attributed to the discontinuity in $\mathrm{Im}\,\epsilon(\omega)/\epsilon_0$ at $\omega = \omega_1$ and at $\omega = \omega_2$. In a more realistic model, this discontinuity would be smoothed out, which would then remove the corresponding divergent behavior of $\mathrm{Re}\,\epsilon(\omega)/\epsilon_0$ at $\omega = \omega_1$ and at $\omega = \omega_2$. The end result would look more like the resonant behavior exhibited in Figure 7.8 of Jackson.

(b) $\mathrm{Im}\,\epsilon(\omega)/\epsilon_0$ is given by:

$$\mathrm{Im}\,\epsilon(\omega)/\epsilon_0 = \frac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\,. \tag{42}$$

Since eq. (42) satisfies $\mathrm{Im}\,\epsilon(-\omega') = -\,\mathrm{Im}\,\epsilon(\omega')$, it follows that eq. (42) can be used for both positive and negative frequencies. In this case, it is more convenient to employ eq. (36), which yields:

$$\mathrm{Re}\,\epsilon(\omega)/\epsilon_0 = 1 + \frac{\lambda\gamma}{\pi} P\int_{-\infty}^{\infty} \frac{\omega'\,d\omega'}{(\omega'-\omega)\big[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2\big]} \tag{43}$$

To evaluate eq. (43), we first employ the method of partial fractions to write:

$$\begin{aligned}
\frac{\gamma\omega'}{(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2} &= \frac{\gamma\omega'}{(\omega_0^2-\omega'^2+i\gamma\omega')(\omega_0^2-\omega'^2-i\gamma\omega')} \\
&= \frac{A}{\omega_0^2-\omega'^2+i\gamma\omega'} + \frac{B}{\omega_0^2-\omega'^2-i\gamma\omega'} \\
&= \frac{A(\omega_0^2-\omega'^2-i\gamma\omega') + B(\omega_0^2-\omega'^2+i\gamma\omega')}{(\omega_0^2-\omega'^2+i\gamma\omega')(\omega_0^2-\omega'^2-i\gamma\omega')} \\
&= \frac{(\omega_0^2-\omega'^2)(A+B) - i\gamma\omega'(A-B)}{(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2}\,. \qquad (44)
\end{aligned}$$

Hence, we can conclude that the following two polynomials must be identical,

$$\gamma\omega' = (\omega_0^2-\omega'^2)(A+B) - i\gamma\omega'(A-B)\,, \tag{45}$$

which yields two equations for A and B,

$$A+B=0\,, \qquad\qquad A-B=i\,. \tag{46}$$

These two equations are easily solved, and we get

$$A=-B=\tfrac{1}{2}i\,. \tag{47}$$

It then follows that:

$$\begin{aligned}
\frac{\gamma\omega'}{(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2} &= \frac{i}{2}\left\{\frac{1}{\omega_0^2-\omega'^2+i\gamma\omega'} - \frac{1}{\omega_0^2-\omega'^2-i\gamma\omega'}\right\} \\
&= \frac{1}{2i}\left\{\frac{1}{\omega'^2-\omega_0^2-i\gamma\omega'} - \frac{1}{\omega'^2-\omega_0^2+i\gamma\omega'}\right\} \\
&= \mathrm{Im}\left(\frac{1}{\omega'^2-\omega_0^2-i\gamma\omega'}\right)\,, \qquad (48)
\end{aligned}$$

after making use of the identity $\operatorname{Im} z = (z - z^*)/(2i)$, which is valid for any complex number z. Hence, we can write:

$$\frac{\gamma\omega'}{(\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2} = \operatorname{Im}\left(\frac{1}{(\omega' - \omega_+)(\omega' - \omega_-)}\right)\,, \tag{49}$$

where the roots of the quadratic equation $\omega'^2 - \omega_0^2 - i\gamma\omega' = 0$, denoted by $\omega_\pm$, are given by

$$\omega_\pm \equiv \tfrac{1}{2}i\gamma \pm \sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}\,. \tag{50}$$

Using eq. (49), we find that eq. (43) can be written in the following form:

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 + \operatorname{Im}\left\{\frac{\lambda}{\pi}\, P\int_{-\infty}^{\infty}\frac{d\omega'}{(\omega' - \omega)(\omega' - \omega_+)(\omega' - \omega_-)}\right\}\,. \tag{51}$$

The principal value is required because the integrand above is singular when $\omega' = \omega$. To evaluate this integral, we shall employ eq. (65) of the class handout entitled *Generalized Functions for Physics*:

$$P\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x - x_0} = \lim_{\varepsilon\to 0}\frac{1}{2}\left\{\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x - x_0 + i\varepsilon} + \int_{-\infty}^{\infty}\frac{f(x)\,dx}{x - x_0 - i\varepsilon}\right\}, \tag{52}$$

where ε is a positive infinitesimal quantity. It then follows that

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 + \operatorname{Im}\left\{\frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{(\omega' - \omega_+)(\omega' - \omega_-)}\left(\frac{1}{\omega' - \omega + i\varepsilon} + \frac{1}{\omega' - \omega - i\varepsilon}\right)\right\}\,, \tag{53}$$

where it is understood that the limit $\varepsilon \to 0$ is taken at the end of the computation.

We can now evaluate the integrals in eq. (53) using the Cauchy residue theorem by closing the contour in the lower half complex plane with a semicircular arc of radius $R \to \infty$. This step is justified since for $\omega' = Re^{i\theta}$, the integrand vanishes on the semicircular arc as $R \to \infty$. By assumption, $\gamma > 0$ and $\omega_0 > \frac{1}{2}\gamma$. Then, the only pole that lies inside the closed contour is at $\omega' = \omega - i\varepsilon$. Since the integration path along the closed contour is in the clockwise direction, we must multiply the residue at the pole by $-2\pi i$. Hence,

$$\begin{aligned}\operatorname{Re}\epsilon(\omega)/\epsilon_0 &= 1 + \operatorname{Im}\left\{\frac{\lambda}{2\pi}(-2\pi i)\frac{1}{(\omega - \omega_+)(\omega - \omega_-)}\right\} \\ &= 1 + \lambda\operatorname{Im}\left\{\frac{-i}{\omega^2 - \omega_0^2 - i\gamma\omega}\right\} = 1 + \lambda\operatorname{Im}\left\{\frac{-i(\omega^2 - \omega_0^2 + i\gamma\omega)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}\right\},\end{aligned} \tag{54}$$

after setting $\varepsilon = 0$. Taking the imaginary part of the above expression yields

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 - \frac{\lambda(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}\,. \tag{55}$$

Note that eqs. (42) and (55) imply that

$$\epsilon(\omega)/\epsilon_0 = 1 + \frac{\lambda}{\omega_0^2 - \omega^2 - i\omega\gamma}\,, \tag{56}$$

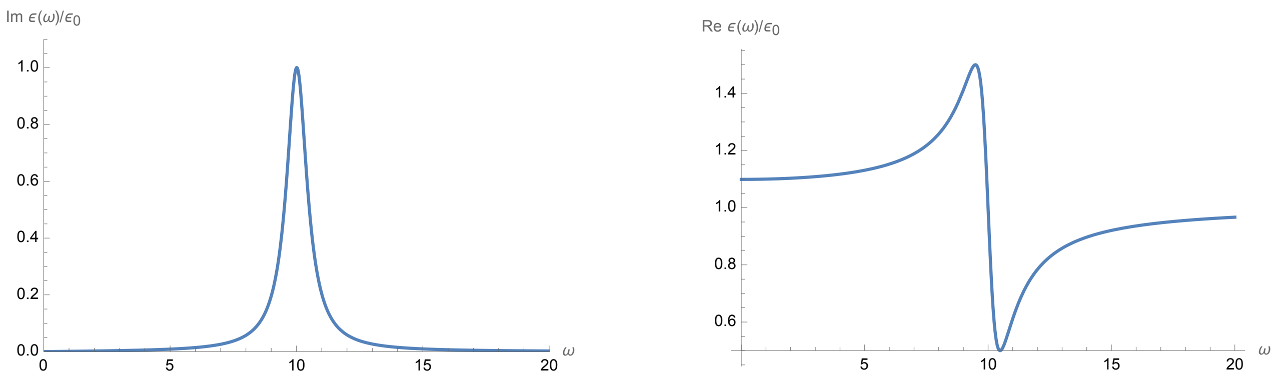

Figure 2: A sketch of the behavior of $\operatorname{Im}\epsilon(\omega)/\epsilon_0$ [left panel] and $\operatorname{Re}\epsilon(\omega)/\epsilon_0$ [right panel] as a function of ω. The representative values of $\lambda = 10$, $\omega_0 = 10$, and $\gamma = 1$ have been chosen for illustrative purposes.

which coincides with eq. (7.51) of Jackson if we identify $\lambda = Ne^2 f/(\epsilon_0 m)$ with a single binding frequency for all molecules.

A sketch of the behavior of $\operatorname{Im}\epsilon(\omega)/\epsilon_0$ [left panel] and $\operatorname{Re}\epsilon(\omega)/\epsilon_0$ [right panel] as a function of ω is exhibited in Figure 2 above. Again, we note that $\operatorname{Re}\epsilon(\omega)/\epsilon_0 \to 1$ as $\omega \to \infty$. This case exhibits the typical resonant behavior seen in Figure 7.8 of Jackson.

EXTRA CREDIT: An alternative derivation of $\operatorname{Re}\epsilon(\omega)/\epsilon_0$

In solving part (b) of this problem, suppose we were to employ eq. (37). Then,

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 + \frac{2\lambda\gamma}{\pi} P\int_0^\infty \frac{\omega'^2\, d\omega'}{(\omega'^2-\omega^2)\big[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2\big]}\,. \tag{57}$$

To evaluate this integral, we use the method of partial fractions to rewrite the integrand in eq. (57) as follows:

$$\begin{aligned}\frac{\omega'^2}{(\omega'^2-\omega^2)\big[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2\big]} &= \frac{A}{\omega'^2-\omega^2} + \frac{B\omega'^2+C}{(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2} \\ &= \frac{A\big[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2\big] + (B\omega'^2+C)(\omega'^2-\omega^2)}{(\omega'^2-\omega^2)\big[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2\big]}\,. \end{aligned} \tag{58}$$

It then follows that

$$\begin{aligned} A\omega_0^4 + C\omega^2 &= 0\,, \\ A(\gamma^2 - 2\omega_0^2) + C - B\omega^2 &= 1\,, \\ A + B &= 0\,. \end{aligned} \tag{59}$$

The solutions to these equations are easily derived:

$$A = -B = \frac{\omega^2}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}\,, \qquad C = \frac{\omega_0^4}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}\,. \tag{60}$$

Hence, we have derived the identity:

$$\frac{\omega'^2}{(\omega'^2-\omega^2)\big[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2\big]}=\frac{1}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}\left[\frac{\omega^2}{\omega'^2-\omega^2}+\frac{\omega_0^4-\omega^2\omega'^2}{(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2}\right]. \tag{61}$$

Plugging this result into eq. (57) yields:

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0=1+\frac{2\lambda\gamma\omega^2}{\pi\big[(\omega^2-\omega_0^2)^2+\gamma^2\omega^2\big]}\left[P\int_0^\infty\frac{d\omega'}{\omega'^2-\omega^2}+\int_0^\infty\frac{\big[(\omega_0^4/\omega^2)-\omega'^2\big]d\omega'}{(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2}\right]. \tag{62}$$

Note that we can drop the principal value symbol in the second integral on the right hand side of eq. (62) since its denominator never vanishes over the range of integration.

Using the definition of the principal value prescription,

$$P\int_0^\infty\frac{d\omega'}{\omega'^2-\omega^2}=\lim_{\varepsilon\to 0}\left\{\int_0^{\omega-\varepsilon}\frac{d\omega'}{\omega'^2-\omega^2}+\int_{\omega+\varepsilon}^\infty\frac{d\omega'}{\omega'^2-\omega^2}\right\}, \tag{63}$$

where ε is a *positive* infinitesimal quantity. Consulting any decent integral table yields the following indefinite integral:

$$\int\frac{d\omega'}{\omega'^2-\omega^2}=-\frac{1}{2\omega}\ln\left|\frac{\omega+\omega'}{\omega-\omega'}\right|. \tag{64}$$

Thus, it follows that

$$\begin{aligned}P\int_0^\infty\frac{d\omega'}{\omega'^2-\omega^2}&=\lim_{\varepsilon\to 0}\left\{-\frac{1}{2\omega}\left[\ln\left(\frac{2\omega-\varepsilon}{\epsilon}\right)-\ln\left(\frac{2\omega+\varepsilon}{\epsilon}\right)\right]\right\}\\&=\lim_{\varepsilon\to 0}\left\{-\frac{1}{2\omega}\ln\left(\frac{2\omega-\varepsilon}{2\omega+\varepsilon}\right)\right\}=0\,.\end{aligned} \tag{65}$$

Hence, eq. (62) reduces to:

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0=1+\frac{2\lambda\gamma\omega^2}{\pi\big[(\omega^2-\omega_0^2)^2+\gamma^2\omega^2\big]}\int_0^\infty\frac{\big[(\omega_0^4/\omega^2)-\omega'^2\big]d\omega'}{(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2}. \tag{66}$$

To evaluate the remaining integral, we shall make use of the following two results:[5]

$$\int_0^\infty\frac{dx}{b^2x^4+2ax^2+c^2}=\frac{\pi}{2\sqrt{2}\,|c|\sqrt{a+|bc|}}\,,\qquad\text{for } a+|bc|>0, \tag{67}$$

$$\int_0^\infty\frac{x^2\,dx}{b^2x^4+2ax^2+c^2}=\frac{\pi}{2\sqrt{2}\,|b|\sqrt{a+|bc|}}\,,\qquad\text{for } a+|bc|>0. \tag{68}$$

[5]Eqs. (67) and (68) are derived in Chapter 7 of George Boros and Victor H. Moll, *Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals* (Cambridge University Press, Cambridge, UK, 2004). The condition $a+|bc|>0$ ensures that $b^2x^4+2ax^2+c^2>0$ for $0\leq x<\infty$. Eq. (67) is also provided by formula 857.11 on p. 214 of Herbert B. Dwight, *Table of Integrals and Other Mathematical Data* (Macmillan Publishing Co., Inc., New York, 1961). Then, eq. (68) can be obtained from eq. (67) by performing a change of the integration variable, $x\to 1/x$.

It then follows that:

$$\int_0^\infty \frac{\big[(\omega_0^4/\omega^2) - \omega'^2\big]d\omega'}{(\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2} = \frac{\pi}{2\gamma}\left(\frac{\omega_0^2}{\omega^2} - 1\right) . \tag{69}$$

Plugging this result back into eq. (66), we obtain our final result:

$$\operatorname{Re}\epsilon(\omega)/\epsilon_0 = 1 - \frac{\lambda(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} , \tag{70}$$

in agreement with the result obtained in eq. (55).

Now that you have seen both derivations of $\operatorname{Re}\epsilon(\omega)/\epsilon_0$, you can decide for yourself which is simpler.

Derivation of eqs. (67) and (68):

Eqs. (67) and (68) were obtained from the references provided in footnote 5. But these results are not difficult to derive as we now show. First, we define

$$\mathcal{I} \equiv \int_0^\infty \frac{dx}{x^4 + 2ax^2 + c^2} . \tag{71}$$

Next, we factor the denominator:

$$\begin{aligned} x^4 + 2ax^2 + c^2 &= \left(x^2 + a - \sqrt{a^2 - c^2}\,\right)\left(x^2 + a + \sqrt{a^2 - c^2}\,\right) \\ &= \left(x^2 + \tfrac{1}{2}\big[\sqrt{a + |c|} - \sqrt{a - |c|}\,\big]^2\right)\left(x^2 + \tfrac{1}{2}\big[\sqrt{a + |c|} + \sqrt{a - |c|}\,\big]^2\right) . \end{aligned} \tag{72}$$

Assuming that $a > -|c|$, it follows that $x^4 + 2ax^2 + c^2 > 0$ for all real values of x, which guarantees that the integral $\mathcal{I}$ is well-defined. One can now apply the method of partial fractions to obtain:

$$\frac{1}{x^4 + 2ax^2 + c^2} = \frac{1}{2\sqrt{a^2 - c^2}}\left[\frac{1}{x^2 + \frac{1}{2}\big[\sqrt{a + |c|} - \sqrt{a - |c|}\,\big]^2} - \frac{1}{x^2 + \frac{1}{2}\big[\sqrt{a + |c|} + \sqrt{a - |c|}\,\big]^2}\right] . \tag{73}$$

Inserting this result into eq. (71), and employing the well know result (for real numbers A),

$$\int_0^\infty \frac{dy}{y^2 + A^2} = \frac{1}{A}\tan^{-1}\left(\frac{y}{A}\right)\bigg|_0^\infty = \frac{\pi}{2|A|} , \tag{74}$$

it follows that

$$\mathcal{I} = \frac{\pi}{2\sqrt{2}\sqrt{a^2 - c^2}}\left[\frac{1}{\sqrt{a + |c|} - \sqrt{a - |c|}} - \frac{1}{\sqrt{a + |c|} + \sqrt{a - |c|}}\right] = \frac{\pi}{2\sqrt{2}|c|\sqrt{a + |c|}} , \tag{75}$$

under the assumption of $a + |c| > 0$, as previously noted.

Using eq. (75), we can now evaluate eq. (67):

$$\begin{aligned} \int_0^\infty \frac{dx}{b^2x^4 + 2ax^2 + c^2} &= \frac{1}{b^2}\int_0^\infty \frac{dx}{x^4 + (2a/b^2)x^2 + (c^2/b^2)} = \frac{\pi}{2\sqrt{2}|bc|\sqrt{\dfrac{a}{b^2} + \left|\dfrac{c}{b}\right|}} \\ &= \frac{\pi}{2\sqrt{2}\,|c|\sqrt{a + |bc|}} , \qquad \text{for } a + |bc| > 0, \end{aligned} \tag{76}$$

which establishes eq. (67). Next, we make a change of variables $x = y^{-1}$ in eq. (76) to obtain

$$\int_0^\infty \frac{y^2 dy}{c^2 y^4 + 2ay^2 + b^2} = \frac{\pi}{2\sqrt{2}\,|c|\sqrt{a+|bc|}}\,, \qquad \text{for } a + |bc| > 0. \tag{77}$$

Relabeling $y \to x$ and interchanging $b \leftrightarrow c$ then yields:

$$\int_0^\infty \frac{x^2 dx}{b^2 x^4 + 2ax^2 + c^2} = \frac{\pi}{2\sqrt{2}\,|b|\sqrt{a+|bc|}}\,, \qquad \text{for } a + |bc| > 0, \tag{78}$$

which establishes eq. (68).

4. [Jackson, problem 7.27] The angular momentum of a distribution of electromagnetic fields in vacuum (in SI units) is given by

$$\vec{\boldsymbol{L}} = \frac{1}{\mu_0 c^2}\int d^3x\, \vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}})\,, \tag{79}$$

where the integration is over all space.

(a) For fields produced a finite time in the past (and so localized to a finite region of space) show that, provided the magnetic field is eliminated in favor of the vector potential $\vec{\boldsymbol{A}}$, the angular momentum can be written in the form

$$\vec{\boldsymbol{L}} = \frac{1}{\mu_0 c^2}\int d^3x \left[\vec{\boldsymbol{E}} \times \vec{\boldsymbol{A}} + \sum_{\ell=1}^{3} E_\ell (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}}) A_\ell\right]. \tag{80}$$

The first term above is sometimes identified with the "spin" of the photon and the second with the "orbital" angular momentum because of the presence of the angular momentum operator $\vec{\boldsymbol{L}}_{\rm op} = -i(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}})$.

The magnetic field can be written in terms of the vector potential, $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$. Hence, we need to evaluate $\vec{\boldsymbol{x}} \times [\vec{\boldsymbol{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})]$. Using the Einstein summation convention, where there is an implicit summation over a pair of identical indices, we can write $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}})_i = \epsilon_{ijk} a_j b_k$, where the indices take on the values $i, j, k = 1, 2, 3$ and there is an implicit sum over the repeated indices j and k. The Levi-Civita tensor is defined as

$$\epsilon_{ijk} = \begin{cases} +1\,, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1\,, & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ \;\;0\,, & \text{otherwise.} \end{cases}$$

Thus, we obtain

$$\bigl\{\vec{\boldsymbol{x}} \times [\vec{\boldsymbol{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})]\bigr\}_i = \epsilon_{ijk} x_j [\vec{\boldsymbol{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})]_k = \epsilon_{ijk} x_j \epsilon_{k\ell m} E_\ell (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})_m = \epsilon_{ijk} x_j \epsilon_{k\ell m} E_\ell \epsilon_{mpq} \nabla_p A_q\,,$$

where $\vec{\boldsymbol{x}} \equiv (x_1, x_2, x_3)$ and $\nabla_p \equiv \partial/\partial x_p$. We now employ the following ϵ-identity,

$$\epsilon_{k\ell m}\epsilon_{mpq} = \delta_{kp}\delta_{\ell q} - \delta_{kq}\delta_{\ell p}\,.$$

Hence, it follows that

$$\left\{\vec{\boldsymbol{x}} \times [\vec{\boldsymbol{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})]\right\}_i = \epsilon_{ijk}x_j E_\ell(\delta_{kp}\delta_{\ell q} - \delta_{kq}\delta_{\ell p})\nabla_p A_q = \epsilon_{ijk}x_j E_\ell \nabla_k A_\ell - \epsilon_{ijk}x_j E_\ell \nabla_\ell A_k\,. \quad (81)$$

We recognize $\epsilon_{ijk}x_j E_\ell \nabla_k A_\ell = E_\ell(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}})_i A_\ell$ which corresponds to the second term in eq. (80). To obtain the first term in eq. (80) will require an integration by parts. That is, we first write:

$$\epsilon_{ijk}x_j E_\ell \nabla_\ell A_k = \epsilon_{ijk}\left[\nabla_\ell(x_j E_\ell A_k) - A_k \nabla_\ell(x_j E_k)\right]\,,$$

which is an identity that follows from the rule for differentiating products. Next, we note that

$$\epsilon_{ijk}A_k \nabla_\ell(x_j E_\ell) = \epsilon_{ijk}A_k\left[x_j(\nabla_\ell E_\ell) + E_\ell(\nabla_\ell x_j)\right] = \epsilon_{ijk}A_k E_\ell \delta_{\ell j} = \epsilon_{ijk}A_k E_j = (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{A}})_i\,,$$

where we used $\nabla_\ell x_j \equiv \partial x_j/\partial x_\ell = \delta_{\ell j}$ and $\nabla_\ell E_\ell = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 0$ (in vacuum). Thus, eq. (81) yields the vector identity,

$$\left\{\vec{\boldsymbol{x}} \times [\vec{\boldsymbol{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})]\right\}_i = E_\ell(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}})_i A_\ell + (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{A}})_i - \epsilon_{ijk}\nabla_\ell(x_j E_\ell A_k)\,, \quad (82)$$

where there is an implicit sum over the repeated index ℓ. An alternative proof of eq. (82) is given at the end of the solution to part (a) of this problem [see eqs. (85)–(88)].

When we integrate over all of space, we can use the divergence theorem [given in the inside cover of Jackson's textbook]:

$$\int_V d^3x\, \epsilon_{ijk}\nabla_\ell(x_j E_\ell A_k) = \oint_S da\, \epsilon_{ijk}n_\ell x_j E_\ell A_k = \oint_S da\, \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{A}})_i = 0\,, \quad (83)$$

where n_ℓ is the outward normal at the surface of infinity S. Since the fields are assumed to be localized to a finite region of space, the integral above vanishes. Hence, inserting the results of eqs. (82) and (83) into eq. (79) [after putting $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$] immediately yields

$$\int d^3x\, \vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}) = \int d^3x \left[\vec{\boldsymbol{E}} \times \vec{\boldsymbol{A}} + \sum_{\ell=1}^{3} E_\ell(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}})A_\ell\right]\,.$$

Therefore, eq. (80) is proven.

REMARK: The identification of

$$\vec{\boldsymbol{L}}_{\rm spin} = \frac{1}{\mu_0 c^2}\int d^3x\, \vec{\boldsymbol{E}} \times \vec{\boldsymbol{A}}\,, \quad (84)$$

as the spin angular momentum is problematical, as eq. (84) is not invariant under gauge transformations. In fact, a gauge-invariant expression for the spin angular momentum can be constructed that reduces to eq. (84) in the radiation (Coulomb) gauge.[6]

[6] See e.g., Iwo Bialynicki-Birula and Zofia Bialynicki-Birula, Journal of Optics **13**, 064014 (2011) and references therein.

Vector identities revisited

Using the well-known vector identity, $\vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}})=\vec{\boldsymbol{B}}(\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{C}})-\vec{\boldsymbol{C}}(\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})$, it follows that

$$\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})=E_i\vec{\boldsymbol{\nabla}}A_i-(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}\,, \tag{85}$$

where there is an implicit sum over i, and we have been careful with the location of the differential operator $\vec{\boldsymbol{\nabla}}$ which is only acting on the vector $\vec{\boldsymbol{A}}$. It follows that

$$\vec{\boldsymbol{x}}\times[\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})]=E_i(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}})A_i-\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}\,. \tag{86}$$

Next, we observe that summing over the repeated index i yields,

$$\begin{aligned}\nabla_i(E_i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})&=(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})+\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})\\&=\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})=E_i\nabla_i(\epsilon_{jk\ell}x_jA_k)=E_i\epsilon_{jk\ell}(\delta_{ij}A_k+x_j\nabla_iA_k)\\&=\vec{\boldsymbol{E}}\times\vec{\boldsymbol{A}}+\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}\,,\end{aligned} \tag{87}$$

after using $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}=0$ (in vacuum) and $\nabla_ix_j=\delta_{ij}$. Combining eqs. (86) and (87) yields

$$\vec{\boldsymbol{x}}\times[\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})]=E_i(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}})A_i+\vec{\boldsymbol{E}}\times\vec{\boldsymbol{A}}-\nabla_i(E_i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})\,, \tag{88}$$

which coincides with eq. (82).

(b) Consider an expansion of the vector potential in the radiation (Coulomb) gauge in terms of plane waves,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)=\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})a_\lambda(\vec{\boldsymbol{k}})e^{i(\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}-i\omega t)}+\text{c.c.}\right]\,. \tag{89}$$

The vectors $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})$ are conveniently chosen as the positive and negative helicity polarization vectors[7]

$$\hat{\boldsymbol{\epsilon}}_\pm=\mp\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\epsilon}}_1\pm i\hat{\boldsymbol{\epsilon}}_2\right)\,, \tag{90}$$

where $\hat{\boldsymbol{\epsilon}}_1$ and $\hat{\boldsymbol{\epsilon}}_2$ are the real orthogonal vectors in the plane whose positive normal is in the direction of $\vec{\boldsymbol{k}}$. Show that the time average of the first (spin) term of $\vec{\boldsymbol{L}}$ can be written as

$$\vec{\boldsymbol{L}}_{\text{spin}}=\frac{2}{\mu_0c}\int\frac{d^3k}{(2\pi)^3}\vec{\boldsymbol{k}}\left[|a_+(\vec{\boldsymbol{k}})|^2-|a_-(\vec{\boldsymbol{k}})|^2\right]\,. \tag{91}$$

Can the term "spin" angular momentum be justified from this expression? Calculate the energy of the field in terms of the plane wave expansion of $\vec{\boldsymbol{A}}$ and compare.

[7] Jackson omits the overall factor of $\mp$ in the definition of $\hat{\boldsymbol{\epsilon}}_\pm$. I prefer to maintain this phase convention, but you are free to choose any convention that suits you.

In the Coulomb gauge, the electric field is (in SI units):

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=-\frac{\partial\vec{\boldsymbol{A}}}{\partial t}=i\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\,\omega\,\left[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})a_\lambda(\vec{\boldsymbol{k}})e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-i\omega t)}-\text{c.c.}\right], \tag{92}$$

where $\omega=ck$ and $k\equiv|\vec{\boldsymbol{k}}|$. Note that due to the overall factor of i, we must subtract the complex conjugate inside the square brackets in order to ensure that $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ is a real field. Inserting eqs. (89) and (92) into eq. (84) and expanding out the integrand, we obtain:

$$\begin{aligned}\vec{\boldsymbol{L}}_{\rm spin}=\frac{1}{\mu_0c^2}\,\frac{i}{(2\pi)^6}\sum_\lambda\sum_{\lambda'}\int\omega\,d^3k\,d^3k'\,d^3x\Big\{&[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]a_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}}')e^{i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{-i(\omega+\omega')t}\\&+[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]a_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}}')e^{i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{-i(\omega-\omega')t}\\&-[\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]a^*_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}}')e^{i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{-i(\omega-\omega')t}\\&-[\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]a^*_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}}')e^{i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{-i(\omega+\omega')t}\Big\},\end{aligned}$$

where $\omega=kc$ and $\omega'=k'c$.

We may now perform the integral over $\vec{\boldsymbol{x}}$, using eq. (12), and then use the delta function to integrate over $\vec{\boldsymbol{k}}'$. The end result is[8]

$$\begin{aligned}\vec{\boldsymbol{L}}_{\rm spin}=\frac{i}{\mu_0c^2}\sum_\lambda\sum_{\lambda'}\int\frac{\omega d^3k}{(2\pi)^3}\Big\{&[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}})-[\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]a^*_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}})\\&+[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(-\vec{\boldsymbol{k}})\,e^{-2i\omega t}-[\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(-\vec{\boldsymbol{k}})]a^*_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(-\vec{\boldsymbol{k}})\,e^{2i\omega t}\Big\}.\end{aligned} \tag{93}$$

However, the last two terms above vanish when integrated over $\vec{\boldsymbol{k}}$, since the corresponding integrands are odd functions of $\vec{\boldsymbol{k}}$. For example, under $\vec{\boldsymbol{k}}\to-\vec{\boldsymbol{k}}$,

$$\begin{aligned}\sum_\lambda\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(-\vec{\boldsymbol{k}})\,e^{-2i\omega t}\longrightarrow&\sum_\lambda\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_\lambda(-\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]a_\lambda(-\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}})\,e^{-2i\omega t},\\=&\sum_\lambda\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda}(\vec{\boldsymbol{k}})]a_{\lambda'}(-\vec{\boldsymbol{k}})a_{\lambda}(\vec{\boldsymbol{k}})\,e^{-2i\omega t},\\=&-\sum_\lambda\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(-\vec{\boldsymbol{k}})\,e^{-2i\omega t},\end{aligned}$$

where we interchanged λ and λ' in the penultimate step (which is justified since these are dummy labels that are being summed over), and used the antisymmetry of the cross product

[8]Indeed, Jackson only asks that we show that the time-average of $\vec{\boldsymbol{L}}_{\rm spin}$ is given by eq. (96). In such a calculation, the last two terms in eq. (93) are immediately set to zero when taking the time-average since the time-averaged values

$$\langle e^{\pm 2i\omega t}\rangle=\frac{1}{T}\int_0^T e^{\pm 2i\omega t}\,dt=0\,,\qquad\text{when }\omega\neq 0\,,$$

where $T=2\pi/\omega$ is the time for one oscillation cycle. The case of $\omega=0$ corresponds to $\vec{\boldsymbol{k}}=0$, in which case the last two terms in eq. (93), when summed over λ and λ', are each manifestly equal to zero, since eq. (90) implies that $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})=0$ for $\lambda=\pm$ (and the cross-terms vanish). However, our result above is more general since no time-averaging is required to obtain the final result quoted in eq. (91).

in the final step. Note that $\omega = |\vec{\boldsymbol{k}}|c$ does not change sign when $\vec{\boldsymbol{k}} \to -\vec{\boldsymbol{k}}$.

Likewise,

$$\sum_{\lambda}\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}}) - [\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]a^*_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}})$$
$$= 2\sum_{\lambda}\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}})\,,$$

after interchanging λ and λ' and using the antisymmetry of the cross product.

Hence, eq. (93) simplifies to

$$\vec{\boldsymbol{L}}_{\rm spin} = \frac{2i}{\mu_0 c^2}\sum_{\lambda}\sum_{\lambda'}\int\frac{\omega d^3k}{(2\pi)^3}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}})\,. \tag{94}$$

Using the definition of the polarization vectors given in eq. (90), it is straightforward to verify that[9]

$$\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}) = -i\lambda\,\hat{\boldsymbol{k}}\,\delta_{\lambda\lambda'}\,,\quad \text{for } \lambda,\lambda' = \pm\,. \tag{95}$$

This result allows us to sum over λ' in eq. (94). Both terms in eq. (94) contribute equally and the end result is:

$$\vec{\boldsymbol{L}}_{\rm spin} = \frac{2}{\mu_0 c^2}\int\frac{d^3k}{(2\pi)^3}\,\vec{\boldsymbol{k}}\left\{|a_+(\vec{\boldsymbol{k}})|^2 - |a_-(\vec{\boldsymbol{k}})|^2\right\}, \tag{96}$$

after using $\omega = kc$ and $\vec{\boldsymbol{k}} = k\hat{\boldsymbol{k}}$. Note that $\vec{\boldsymbol{L}}_{\rm spin}$ is time-independent and thus conserved. This is a stronger condition than the conservation of angular momentum, which only requires that the sum $\vec{\boldsymbol{L}} = \vec{\boldsymbol{L}}_{\rm orbital} + \vec{\boldsymbol{L}}_{\rm spin}$ is conserved. Eq. (96) implies that the spin angular momentum of the electromagnetic field is *separately* a constant of the motion. If we interpret each mode $(\vec{\boldsymbol{k}},\lambda)$ as a photon, then the two possible photon spin states (in a spherical basis) correspond to positive and negative helicity, i.e. states of definite spin angular momentum in which $\vec{\boldsymbol{L}}_{\rm spin}$ points in a direction parallel or antiparallel to the direction of propagation $\hat{\boldsymbol{k}}$, respectively.

It is instructive to consider the energy of the electromagnetic fields, which was obtained in problem 1. In particular, eq. (22) yields

$$U = 2\epsilon_0\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\,\omega^2|a_\lambda(\vec{\boldsymbol{k}})|^2\,, \tag{97}$$

where we have used eq. (9) to write $E_0(\vec{\boldsymbol{k}},\lambda) = i\omega a_\lambda(\vec{\boldsymbol{k}})$. Consider a fixed mode of positive helicity $(\vec{\boldsymbol{k}}_0, \lambda = +1)$. Then, $a_\lambda(\vec{\boldsymbol{k}}) = a_+(\vec{\boldsymbol{k}}_0)\delta^3(\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}_0)\delta_{\lambda,+1}$, in which case eq. (97) yields

$$U = \frac{2\epsilon_0\omega_0^2}{(2\pi)^3}|a_\lambda(\vec{\boldsymbol{k}}_0)|^2\,,$$

and

$$\vec{\boldsymbol{L}}_{\rm spin} = \frac{2}{\mu_0 c}\cdot\frac{1}{(2\pi)^3}\,\vec{\boldsymbol{k}}_0|a_\lambda(\vec{\boldsymbol{k}}_0)|^2 = \frac{2\epsilon_0\omega_0}{(2\pi)^3}\,\hat{\boldsymbol{k}}_0|a_\lambda(\vec{\boldsymbol{k}}_0)|^2\,,$$

[9]To prove eq. (95), use the fact that $\hat{\boldsymbol{\epsilon}}_1\times\hat{\boldsymbol{\epsilon}}_2 = -\hat{\boldsymbol{\epsilon}}_2\times\hat{\boldsymbol{\epsilon}}_1 = \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{\epsilon}}_1\times\hat{\boldsymbol{\epsilon}}_1 = \hat{\boldsymbol{\epsilon}}_2\times\hat{\boldsymbol{\epsilon}}_2 = 0$.

after using $\epsilon_0\mu_0 = 1/c^2$ and $\vec{\boldsymbol{k}}_0 = (\omega_0/c)\hat{\boldsymbol{k}}_0$. That is,

$$\vec{\boldsymbol{L}}_{\rm spin} = \lambda \frac{U}{\omega_0}\,\hat{\boldsymbol{k}}_0\,, \qquad \text{for } \lambda = +1\,. \tag{98}$$

For a fixed mode of negative helicity ($\vec{\boldsymbol{k}}_0, \lambda = -1$), we again obtain eq. (98) with $\lambda = -1$. For a single photon of frequency ω_0, quantum mechanics states that $U = \hbar\omega_0$, and eq. (98) yields

$$\vec{\boldsymbol{L}}_{\rm spin} = \pm\hbar\hat{\boldsymbol{k}}_0\,,$$

corresponding to a spin-one particle of helicity ± 1, with its spin parallel or antiparallel to the direction of propagation $\hat{\boldsymbol{k}}_0$.

5. (a) Assume that the vector potential in the Lorenz gauge is given by:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}\,,\,t) = A_0(x,y)(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})e^{i(kz-\omega t)}\,, \tag{99}$$

where $A_0(x,y)$ is a very slowly varying function of position. "Slowly varying" means that the second spatial derivatives of $A_0(x,y)$ can be neglected; however, one must *not* neglect first derivatives of $A_0(x,y)$. Derive the approximate forms for the electric and magnetic fields given in Jackson, problem 7.28,

$$\vec{\boldsymbol{E}}(x,y,z,t) \simeq \left[E_0(x,y)(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{\boldsymbol{z}}\right]e^{ikz-i\omega t}\,, \tag{100}$$

$$\vec{\boldsymbol{B}}(x,y,z,t) \simeq \mp i\sqrt{\mu\epsilon}\,\vec{\boldsymbol{E}}(x,y,z,t)\,, \tag{101}$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ are unit vectors in the x, y and z directions, respectively.

The Lorenz gauge condition (in SI units) in an isotropic nonconducting medium characterized by electric permittivity ϵ and magnetic permeability μ is[10]

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}} + \mu\epsilon\frac{\partial\Phi}{\partial t} = 0\,,$$

where the phase velocity of the wave [eq. (7.5) of Jackson] is $v = \omega/k = 1/\sqrt{\mu\epsilon}$. Using eq. (99),

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}} = \left(\frac{\partial A_0}{\partial x} \pm i\frac{\partial A_0}{\partial y}\right)e^{ikz-i\omega t} = -\mu\epsilon\frac{\partial\Phi}{\partial t}\,.$$

Integrating, we get

$$\Phi(\vec{\boldsymbol{x}}\,,\,t) = -\frac{i}{\mu\epsilon\omega}\left(\frac{\partial A_0}{\partial x} \pm i\frac{\partial A_0}{\partial y}\right)e^{ikz-i\omega t}\,, \tag{102}$$

where we have dropped the integration "constant" that is independent of time, as such a term would not correspond to the propagation of the circularly polarized wave.

[10] In vacuum, $v = c$ and $\mu\epsilon = \mu_0\epsilon_0 = 1/c^2$, and we recover the usual form of the Lorenz gauge condition.

The electric and magnetic fields are given by

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\Phi - \frac{\partial\vec{\boldsymbol{A}}}{\partial t}\,, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}\,.$$

Plugging in eqs. (99) and (102),

$$\vec{\boldsymbol{\nabla}}\Phi \simeq \frac{k}{\mu\epsilon\omega}\left(\frac{\partial A_0}{\partial x} \pm i\frac{\partial A_0}{\partial y}\right)\hat{\boldsymbol{z}}\, e^{ikz-i\omega t}\,,$$

where we have dropped second spacial derivatives of A_0 and

$$\frac{\partial\vec{\boldsymbol{A}}}{\partial t} = -i\omega A_0(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})e^{ikz-i\omega t}\,,$$

If we define $E_0(x,y) \equiv i\omega A_0(x,y)$ and make use of $\omega = k/\sqrt{\mu\epsilon}$, we end up with,[11]

$$\vec{\boldsymbol{E}}(x,y,z,t) \simeq \left[E_0(x,y)(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{\boldsymbol{z}}\right] e^{ikz-i\omega t}\,. \tag{103}$$

Next, we evaluate

$$\vec{\boldsymbol{B}}(x,y,z,t) = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_0 e^{ikz-i\omega t} & \pm i A_0 e^{ikz-i\omega t} & 0 \end{pmatrix}$$

$$= \mp i\left[(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})ikA_0(x,y) - \left(\frac{\partial A_0}{\partial x} \pm i\frac{\partial A_0}{\partial y}\right)\hat{\boldsymbol{z}}\right] e^{ikz-i\omega t}\,,$$

$$= \mp i\sqrt{\mu\epsilon}\,\vec{\boldsymbol{E}}(x,y,z,t)\,, \tag{104}$$

after using $E_0(x,y) \equiv i\omega A_0(x,y) = ikA_0(x,y)/\sqrt{\mu\epsilon}$.

<u>REMARK</u>: According to eq. (104), the *complex* $\vec{\boldsymbol{B}}$ vector is proportional to the *complex* $\vec{\boldsymbol{E}}$ vector. Nevertheless, it is easy to check that $\mathrm{Re}\,\vec{\boldsymbol{E}}$ and $\mathrm{Re}\,\vec{\boldsymbol{B}}$ are orthogonal vectors [i.e., $(\mathrm{Re}\,\vec{\boldsymbol{E}})\cdot(\mathrm{Re}\,\vec{\boldsymbol{B}}) = 0$], as expected for the physical $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields of an electromagnetic wave.

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[11] An alternative solution to Jackson, problem (7.28) is to propose an electric field of the form,

$$\vec{\boldsymbol{E}}(x,y,z,t) \simeq \left[E_0(x,y)(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) + F_0(x,y)\hat{\boldsymbol{z}}\right]e^{ikz-i\omega t}\,.$$

This corresponds to a wave that has a finite extent in the transverse directions but is not a simple plane wave. Since the amplitude modulation is slowly varying, one would expect that its form is dominated by the transverse part, but a small longitudinal part can also be present as indicated in the form above. If one now imposes $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = \rho/\epsilon = 0$ (as there is no free charge density present), then one easily derives

$$F_0(x,y) = \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\,,$$

in an approximation where the second order spatial derivatives are neglected. One can the compute the $\vec{\boldsymbol{B}}$ field of the circularly polarized wave by using the Maxwell equation $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} + \partial\vec{\boldsymbol{B}}/\partial t = 0$.

(b) [Jackson, problem 7.29] For the circularly polarized wave given by eqs. (100) and (101), with $E_0(x,y)$ a real function of x and y, calculate the time-averaged component of the angular momentum parallel to the direction of propagation. Show that the ratio of this component of angular momentum to the energy of the wave in vacuum is,

$$\frac{L_3}{U} = \pm\omega^{-1}\,.$$

Interpret this result in terms of quanta of radiation (photons). Show that for a cylindrically symmetric, finite plane wave, the transverse components of angular momentum vanish.

The angular momentum density of the electromagnetic field is given by [cf. problem 6.10 on p. 288 of Jackson].

$$\vec{\boldsymbol{\mathcal{L}}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{g}} = \mu\epsilon\,\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}})\,. \tag{105}$$

Eq. (105) was obtained under the assumption that $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ are *real* physical fields. Using the vector identity,

$$\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{H}}) - \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\,,$$

the z component of the angular momentum density (denoted below by $\mathcal{L}_3$) is given by

$$\mathcal{L}_3 = \mu\epsilon\big[x(E_zH_x - E_xH_z) + y(E_zH_y - E_yH_z)\big]\,. \tag{106}$$

Note that the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields obtained in eqs. (103) and (104) are *complex* fields. The corresponding real physical fields are obtained by taking the real part of the complex fields. Since Jackson specifies that $E_0(x,y)$ is a *real* function of x and y, the corresponding real physical fields are given by

$$E_x = E_0\cos(kz-\omega t)\,, \qquad\qquad E_y = \mp E_0\sin(kz-\omega t)\,, \tag{107}$$

$$E_z = -\frac{i}{k}\left[\frac{\partial E_0}{\partial x}\sin(kz-\omega t) \pm \frac{\partial E_0}{\partial y}\cos(kz-\omega t)\right] \tag{108}$$

$$H_x = \pm\sqrt{\frac{\epsilon}{\mu}}E_0\sin(kz-\omega t)\,, \qquad\qquad H_y = E_0\cos(kz-\omega t)\,, \tag{109}$$

$$H_z = \frac{i}{k}\sqrt{\frac{\epsilon}{\mu}}\left[\pm\frac{\partial E_0}{\partial x}\cos(kz-\omega t) - \frac{\partial E_0}{\partial y}\sin(kz-\omega t)\right]\,, \tag{110}$$

after using $\vec{\boldsymbol{B}} = \mu\vec{\boldsymbol{H}}$. To avoid notational clutter, we have omitted the symbol Re on the left-hand sides of eqs. (107)–(110). That is, $\mathrm{Re}\,\vec{\boldsymbol{E}} = (E_x, E_y, E_z)$ and $\vec{\boldsymbol{H}} = (H_x, H_y, H_z)$ now denote the real physical electric and magnetic fields. Inserting the above results into eq. (106) yields

$$\mathcal{L}_3 = \mp\frac{\epsilon}{k}\sqrt{\mu\epsilon}\left[xE_0\frac{\partial E_0}{\partial x} + yE_0\frac{\partial E_0}{\partial y}\right] = \mp\frac{\epsilon}{2k}\sqrt{\mu\epsilon}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)E_0^2\,. \tag{111}$$

As previously noted, $k = \sqrt{\mu\epsilon}\,\omega$ in the medium. Thus, we can rewrite eq. (111) as

$$\mathcal{L}_3 = \mp\frac{\epsilon}{2\omega}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)E_0^2\,. \tag{112}$$

It is noteworthy that in obtaining eq. (111), the time-dependence has dropped out. Thus, there is no need to time-average the result (even though Jackson only asks for the time-averaged angular momentum).

Next, we compute the energy density [cf. eq. (6.106) of Jackson],

$$u = \tfrac{1}{2}\big(\epsilon|\vec{\boldsymbol{E}}|^2 + \mu|\vec{\boldsymbol{H}}|^2\big)\,.$$

As in part (a), we shall assume that $E_0(x,y)$ is slowly varying so that we can neglect the second spatial derivatives of E_0. That is, we may discard terms proportional to $(\partial E_0/\partial x)^2$, $(\partial E_0/\partial y)^2$ and $(\partial E_0/\partial x)(\partial E_0/\partial y)$ as compared to terms proportional to E_0^2. In particular, in evaluating $|\vec{\boldsymbol{E}}|^2$ and $|\vec{\boldsymbol{H}}|^2$, we can drop the contributions from E_z and H_z. Hence,

$$u \simeq \tfrac{1}{2}\left[\epsilon E_0^2 + \mu\left(\frac{\epsilon}{\mu}E_0^2\right)\right] = \epsilon E_0^2\,.$$

Finally, we compute the total energy and the z-component of the total angular momentum,

$$U = \int d^3x\, u = \epsilon\int d^3x\, E_0^2(x,y)\,,$$

$$L_3 = \int d^3x\,\mathcal{L}_3 = \mp\frac{\epsilon}{2\omega}\int d^3x\left[x\frac{\partial}{\partial x}E_0^2 + y\frac{\partial}{\partial y}E_0^2\right] = \pm\frac{\epsilon}{\omega}\int d^3x\, E_0^2(x.y)\,,$$

after integrating by parts and using the fact that $E_0(x,y)$ vanishes when $|x|$, $|y|\to\infty$. We conclude that[12]

$$\frac{L_3}{U} = \pm\frac{1}{\omega}\,. \tag{113}$$

The interpretation in terms of the photon is clear. Since a photon has an energy $U=\hbar\omega$, it follows that $L_3=\pm\hbar$ for the photon. The two possible signs correspond to positive and negative helicity.

To complete the problem, we compute the x and y components of the angular momentum density (denoted below by $\mathcal{L}_1$ and $\mathcal{L}_2$).

$$\mathcal{L}_1 = \mu\epsilon\big[y(E_xH_y - E_yH_x) + z(E_xH_z - E_zH_x)\big]\,,$$

$$\mathcal{L}_2 = \mu\epsilon\big[x(E_yH_x - E_xH_y) + z(E_yH_z - E_zH_y)\big]\,.$$

Inserting the fields given in eqs. (107)–(110), we end up with

$$\mathcal{L}_1 = \mu\epsilon\left[\sqrt{\frac{\epsilon}{\mu}}\,E_0^2 y \pm \frac{1}{k}\sqrt{\frac{\epsilon}{\mu}}\,E_0 z\frac{\partial E_0}{\partial x}\right]\,,$$

$$\mathcal{L}_2 = \mu\epsilon\left[-\sqrt{\frac{\epsilon}{\mu}}\,E_0^2 x \pm \frac{1}{k}\sqrt{\frac{\epsilon}{\mu}}\,E_0 z\frac{\partial E_0}{\partial y}\right]\,.$$

By assumption, the plane wave is cylindrically symmetric, which implies that

$$E_0(x,y) = E_0(-x,y)\,, \qquad E_0(x,y) = E_0(x,-y)\,.$$

[12] Although Jackson asks us to derive eq. (113) for the wave in vacuum, our calculation is equally valid for the wave in an isotropic nonconducting medium.

Thus,

$$L_1 = \mu\epsilon\sqrt{\frac{\epsilon}{\mu}}\int d^3x\left[yE_0^2(x,y) \pm \frac{z}{2k}\frac{\partial}{\partial x}E_0^2(x,y)\right] = 0\,,$$

since the integrand is an odd function under $x \to -x$, $y \to -y$. Likewise,

$$L_2 = \mu\epsilon\sqrt{\frac{\epsilon}{\mu}}\int d^3x\left[-xE_0^2(x,y) \pm \frac{z}{2k}\frac{\partial}{\partial y}E_0^2(x,y)\right] = 0\,.$$

Hence, we conclude that for a cylindrically symmetric finite circularly polarized electromagnetic wave,

$$L_1 = L_2 = 0\,, \quad \text{and} \quad L_3 = \pm\frac{U}{\omega}\,.$$

An alternative method for obtaining eq. (111)

Since eqs. (103) and (104) provide the *complex* $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields, it is convenient to make use of the corresponding complex angular momentum density [which is defined in analogy with eq. (6.132) of Jackson]:

$$\vec{\boldsymbol{\mathcal{L}}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{g}} = \tfrac{1}{2}\mu\epsilon\,\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}^*)\,.$$

The *physical* angular momentum density, time-averaged over a cycle, can then be identified as $\mathrm{Re}\,\vec{\boldsymbol{\mathcal{L}}}$. Using the vector identity,

$$\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}^*) = \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{H}}^*) - \vec{\boldsymbol{H}}^*(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\,,$$

the z component of the angular momentum density (denoted below by $\mathcal{L}_3$) is given by

$$\mathcal{L}_3 = \tfrac{1}{2}\mu\epsilon\big[x(E_zH_x^* - E_xH_z^*) + y(E_zH_y^* - E_yH_z^*)\big]\,. \tag{114}$$

Using eqs. (103) and (104),

$$\vec{\boldsymbol{E}}(x,y,z,t) \simeq \left[E_0(x,y)(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{\boldsymbol{z}}\right]e^{ikz-i\omega t}\,, \tag{115}$$

$$\vec{\boldsymbol{H}}(x,y,z,t) = \frac{1}{\mu}\vec{\boldsymbol{B}}(x,y,z,t) = \mp i\sqrt{\frac{\epsilon}{\mu}}\,\vec{\boldsymbol{E}}(x,y,z,t)\,. \tag{116}$$

Inserting eqs. (115) and (116) into eq. (114), we obtain

$$\begin{aligned}\mathcal{L}_3 &= \pm\tfrac{1}{2}i\epsilon\sqrt{\mu\epsilon}\big[x(E_zE_x^* - E_xE_z^*) + y(E_zE_y^* - E_yE_z^*)\big]\\ &= \mp\epsilon\sqrt{\mu\epsilon}\big[x\,\mathrm{Im}(E_zE_x^*) + y\,\mathrm{Im}(E_zE_y^*)\big]\,.\end{aligned} \tag{117}$$

An explicit computation yields

$$\mathrm{Im}(E_zE_x^*) = \frac{1}{k}\,\mathrm{Re}\left(E_0^*\frac{\partial E_0}{\partial x}\right) \mp \frac{i}{k}\,\mathrm{Im}\left(E_0^*\frac{\partial E_0}{\partial y}\right)\,, \tag{118}$$

$$\mathrm{Im}(E_zE_y^*) = \frac{1}{k}\,\mathrm{Re}\left(E_0^*\frac{\partial E_0}{\partial y}\right) \pm \frac{i}{k}\,\mathrm{Im}\left(E_0^*\frac{\partial E_0}{\partial x}\right)\,. \tag{119}$$

The first term on the right-hand sides of eqs. (118) and (119) can be simplified by noting that

$$\operatorname{Re}\left(E_0^*\frac{\partial E_0}{\partial x}\right)=\frac{1}{2}\frac{\partial}{\partial x}|E_0|^2\,,\qquad\qquad \operatorname{Re}\left(E_0^*\frac{\partial E_0}{\partial y}\right)=\frac{1}{2}\frac{\partial}{\partial y}|E_0|^2\,. \tag{120}$$

Inserting these results back into eq. (117), we end up with

$$\mathcal{L}_3=\mp\frac{\epsilon\sqrt{\mu\epsilon}}{2k}\left\{\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)|E_0|^2\mp 2\operatorname{Im}\left[E_0^*\left(x\frac{\partial E_0}{\partial y}-y\frac{\partial E_0}{\partial x}\right)\right]\right\}\,. \tag{121}$$

Note that $\mathcal{L}_3$ is a manifestly real quantity, so it corresponds to the third component of the physical angular momentum density.

Likewise, the harmonic energy density is given by [cf. eq. (6.133) of Jackson]

$$u=\tfrac{1}{4}\bigl(\epsilon\,\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}^*+\mu\vec{\boldsymbol{H}}\boldsymbol{\cdot}\vec{\boldsymbol{H}}^*\bigr)=\tfrac{1}{2}\epsilon\,\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}^*\,, \tag{122}$$

after making use of eq. (116). Inserting eqs. (115) and (116) into eq. (122), and neglecting terms involving either the second spatial derivatives of E_0 or a product of two first spatial derivatives of E_0 (and its complex conjugate), we end up with

$$u=\epsilon|E_0|^2\,. \tag{123}$$

Jackson specifies in this problem that the amplitude $E_0(x,y)$ is a *real* function of x and y. That is, $E_0^*=E_0$, which implies that $\operatorname{Im}\bigl[E_0^*(x\partial E_0/\partial y-y\partial E_0/\partial x)\bigr]=0$. Hence, it follows that

$$\mathcal{L}_3=\mp\frac{\epsilon\sqrt{\mu\epsilon}}{2k}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)E_0^2\,, \tag{124}$$

which reproduces the result previously obtained in eq. (111). Integrating eqs. (122) and (124) over all space to obtain the total energy and angular momentum, respectively, yields

$$U=\pm L_3\,\omega\,. \tag{125}$$

Note that the assumption that $E_0(x,y)$ is a real function of x and y is critical in this problem. If E_0 were a more general complex amplitude, the result would be an extra term in $\mathcal{L}_3$ proportional to $\operatorname{Im}\bigl[E_0^*(x\partial E_0/\partial y-y\partial E_0/\partial x)\bigr]$, as indicated by eq. (121). This extra term would then ruin the simple relation between U and L_3 obtained in eq. (125).