1. [Jackson, problem 11.10]

(a) For the Lorentz boost and rotation matrices $\boldsymbol{K}$ and $\boldsymbol{S}$ show that

$$(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^3 = -\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S}\,, \tag{1}$$
$$(\hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K})^3 = \hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K}\,, \tag{2}$$

where $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\epsilon}}'$ are any real unit 3-vectors.

We are given

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

To prove eq. (1), we evaluate the matrix $\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S}$ explicitly,

$$\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_3 & \epsilon_2 \\ 0 & \epsilon_3 & 0 & -\epsilon_1 \\ 0 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix},$$

and then compute $(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^3$ via matrix multiplication. Indeed,

$$(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\epsilon_2^2-\epsilon_3^2 & \epsilon_1\epsilon_2 & \epsilon_1\epsilon_3 \\ 0 & \epsilon_1\epsilon_2 & -\epsilon_1^2-\epsilon_3^2 & \epsilon_2\epsilon_3 \\ 0 & \epsilon_1\epsilon_3 & \epsilon_2\epsilon_3 & -\epsilon_1^2-\epsilon_2^2 \end{pmatrix},$$

and

$$(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^3 = (\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^2\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S} = -(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_3 & \epsilon_2 \\ 0 & \epsilon_3 & 0 & -\epsilon_1 \\ 0 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix} = -\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S}\,,$$

after using the fact that $\hat{\boldsymbol{\epsilon}}$ is a real unit 3-vector, which implies that $\epsilon_1^2+\epsilon_2^2+\epsilon_3^2=1$.

To prove eq. (2), we evaluate the matrix $\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{K}$ explicitly,

$$\hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K}=\begin{pmatrix}0 & \epsilon'_1 & \epsilon'_2 & \epsilon'_3\\ \epsilon'_1 & 0 & 0 & 0\\ \epsilon'_2 & 0 & 0 & 0\\ \epsilon'_3 & 0 & 0 & 0\end{pmatrix}\,, \tag{3}$$

and then compute $(\hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K})^3$ via matrix multiplication. Indeed,

$$(\hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K})^2=\begin{pmatrix}\epsilon'^2_1+\epsilon'^2_2+\epsilon'^2_3 & 0 & 0 & 0\\ 0 & \epsilon'^2_1 & \epsilon'_1\epsilon'_2 & \epsilon'_1\epsilon'_3\\ 0 & \epsilon'_1\epsilon'_2 & \epsilon'^2_2 & \epsilon'_2\epsilon'_3\\ 0 & \epsilon'_1\epsilon'_3 & \epsilon'_2\epsilon'_3 & \epsilon'^2_3\end{pmatrix}\,,$$

and

$$(\hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K})^3=(\hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K})^2\,\hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K}=(\epsilon'^2_1+\epsilon'^2_2+\epsilon'^2_3)\begin{pmatrix}0 & \epsilon'_1 & \epsilon'_2 & \epsilon'_3\\ \epsilon'_1 & 0 & 0 & 0\\ \epsilon'_2 & 0 & 0 & 0\\ \epsilon'_3 & 0 & 0 & 0\end{pmatrix}=\hat{\boldsymbol{\epsilon}}'\cdot\boldsymbol{K}\,,$$

after using the fact that $\hat{\boldsymbol{\epsilon}}'$ is a real unit 3-vector.

ALTERNATIVE SOLUTION:

The following alternative solution to part (a) is noteworthy. First, observe that the first row and column of S_1, S_2 and S_3 are all zeros. Hence we can simply focus on the remaining 3×3 block. That is, we write the S_i in block matrix form,

$$(S_i)_{jk}=\left(\begin{array}{c|c}0 & \mathbf{0}^{\mathsf{T}}_k\\ \hline \mathbf{0}_j & -\epsilon_{ijk}\end{array}\right)\,, \tag{4}$$

where $\mathbf{0}^{\mathsf{T}}$ is a row vector of three zeros, $\mathbf{0}$ is a column vector of three zeros, and

$$\epsilon_{ijk}=\begin{cases}+1\,, & \text{if } (ijk) \text{ is an even permutation of } (123)\,,\\ -1\,, & \text{if } (ijk) \text{ is an odd permutation of } (123)\,,\\ \ \ 0\,, & \text{otherwise}\,,\end{cases}$$

is the three-dimensional Levi-Civita tensor. After excluding the first row and column, jk labels the three remaining rows and columns of the S_i.

Thus, we can compute $(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^3$ by pretending that the first row and column do not exist. More explicitly,[1]

$$\begin{aligned}(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^3_{jk}&=(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})_{j\ell}(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})_{\ell m}(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})_{mk}=\epsilon_i(S_i)_{j\ell}\,\epsilon_p(S_p)_{\ell m}\,\epsilon_q(S_q)_{mk}\\ &=-\epsilon_i\epsilon_p\epsilon_q\epsilon_{ij\ell}\,\epsilon_{p\ell m}\,\epsilon_{qmk}=\epsilon_i\epsilon_p\epsilon_q\epsilon_{ij\ell}\,\epsilon_{pm\ell}\,\epsilon_{qmk}\\ &=\epsilon_i\epsilon_p\epsilon_q(\delta_{ip}\delta_{jm}-\delta_{im}\delta_{jp})\epsilon_{qmk}=\epsilon_q\epsilon_{qjk}-\epsilon_m\epsilon_j\epsilon_q\epsilon_{qmk}\,,\end{aligned} \tag{5}$$

[1] In eq. (5), we employ the Einstein summation convention. In this derivation, we make use of the antisymmetry properties of the Levi-Civita tensor and employ the identity $\epsilon_{ij\ell}\epsilon_{pm\ell}=\delta_{ip}\delta_{jm}-\delta_{im}\delta_{jp}$.

after noting that $\epsilon_i \epsilon_i = \hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{\epsilon}} = 1$ since $\hat{\boldsymbol{\epsilon}}$ is an arbitrary real unit vector. We now observe that $\epsilon_m \epsilon_j \epsilon_q \epsilon_{qmk} = 0$ since $\epsilon_m \epsilon_q$ is symmetric under the interchange of m and q whereas ϵ_{qmk} is antisymmetric under the same interchange of indices. Thus, eq. (5) yields

$$(\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{S})^3_{jk} = \epsilon_q \epsilon_{qjk} = -\epsilon_q (S_q)_{jk} = -(\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{S})_{jk} \, ,$$

which establishes eq. (1).

To establish eq. (2), we rewrite $\hat{\boldsymbol{\epsilon}}' \cdot \boldsymbol{K}$ given by eq. (3) in block matrix form [analogous to the form of the S_i in eq. (4)],

$$(\hat{\boldsymbol{\epsilon}}' \cdot \boldsymbol{K})_{jk} = \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_j & \mathbf{0}_{jk} \end{array}\right) , \tag{6}$$

where $\mathbf{0}_{jk}$ stands for the matrix elements of the 3×3 zero matrix. In particular, j labels the row and k labels the column. Then,

$$\begin{aligned}(\hat{\boldsymbol{\epsilon}}' \cdot \boldsymbol{K})^3_{jk} &= \left(\begin{array}{c|c} 0 & \epsilon'_\ell \\ \hline \epsilon'_j & \mathbf{0}_{j\ell} \end{array}\right) \left(\begin{array}{c|c} 0 & \epsilon'_i \\ \hline \epsilon'_\ell & \mathbf{0}_{\ell i} \end{array}\right) \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_i & \mathbf{0}_{ik} \end{array}\right) = \left(\begin{array}{c|c} \hat{\boldsymbol{\epsilon}}' \cdot \hat{\boldsymbol{\epsilon}}' & \mathbf{0}_i \\ \hline \mathbf{0}^{\mathsf{T}}_j & \epsilon'_j \epsilon'_i \end{array}\right) \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_i & \mathbf{0}_{ik} \end{array}\right) \\ &= \left(\begin{array}{c|c} 1 & \mathbf{0}_i \\ \hline \mathbf{0}^{\mathsf{T}}_j & \epsilon'_j \epsilon'_i \end{array}\right) \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_i & \mathbf{0}_{ik} \end{array}\right) = \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_j \, \hat{\boldsymbol{\epsilon}}' \cdot \hat{\boldsymbol{\epsilon}}' & \mathbf{0}_{jk} \end{array}\right) = \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_j & \mathbf{0}_{jk} \end{array}\right) = (\hat{\boldsymbol{\epsilon}}' \cdot \boldsymbol{K})_{jk} \, ,\end{aligned}$$

after using the fact that $\hat{\boldsymbol{\epsilon}}'$ is real unit vector. Once again, eq. (2) is established.

(b) Use the result of part (a) to show that:

$$\exp\bigl(-\zeta\, \hat{\boldsymbol{\beta}} \cdot \boldsymbol{K}\bigr) = I - \hat{\boldsymbol{\beta}} \cdot \boldsymbol{K} \, \sinh\zeta + (\hat{\boldsymbol{\beta}} \cdot \boldsymbol{K})^2 \left[\cosh\zeta - 1\right] ,$$

where I is the 4×4 identity matrix.

We employ the series expansion for the exponential (which *defines* the matrix exponential),

$$\exp\bigl(-\zeta\, \hat{\boldsymbol{\beta}} \cdot \boldsymbol{K}\bigr) = \sum_{n=0}^{\infty} \frac{(-\zeta)^n}{n!} (\hat{\boldsymbol{\beta}} \cdot \boldsymbol{K})^n \, . \tag{7}$$

In part (a), we established the following result: $(\hat{\boldsymbol{\beta}} \cdot \boldsymbol{K})^3 = \hat{\boldsymbol{\beta}} \cdot \boldsymbol{K}$. Hence, it follows that

$$(\hat{\boldsymbol{\beta}} \cdot \boldsymbol{K})^{2n} = (\hat{\boldsymbol{\beta}} \cdot \boldsymbol{K})^2 \, , \qquad (\hat{\boldsymbol{\beta}} \cdot \boldsymbol{K})^{2n+1} = \hat{\boldsymbol{\beta}} \cdot \boldsymbol{K} \, , \quad \text{for } n = 1, 2, 3, \ldots .$$

Thus, we can rewrite the series given in eq. (7) as

$$\exp\bigl(-\zeta\, \hat{\boldsymbol{\beta}} \cdot \boldsymbol{K}\bigr) = I - \hat{\boldsymbol{\beta}} \cdot \boldsymbol{K} \sum_{\substack{n \text{ odd} \\ n \geq 1}} \frac{\zeta^n}{n!} + (\hat{\boldsymbol{\beta}} \cdot \boldsymbol{K})^2 \sum_{\substack{n \text{ even} \\ n \geq 2}} \frac{\zeta^n}{n!} \, , \tag{8}$$

after using the fact that $(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K})^0 = I$ is the 4×4 identity matrix. Using,

$$\sum_{n=0}^{\infty}\frac{\zeta^{2n+1}}{(2n+1)!} = \sinh\zeta\,, \qquad\qquad \sum_{n=0}^{\infty}\frac{\zeta^{2n}}{(2n)!} = \cosh\zeta\,,$$

and noting that the last summation in eq. (8) starts at $n = 2$, we end up with

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K}\bigr) = I - \hat{\boldsymbol{\beta}}\cdot\boldsymbol{K}\,\sinh\zeta + (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K})^2\,[\cosh\zeta - 1]\;, \tag{9}$$

which is the desired result.

<u>*REMARKS*</u>:

To understand the significance of eq. (9), let us write it explicitly in matrix form. It is convenient to use the block matrix form of eq. (6), where j labels the row and k labels the column,

$$I = \left(\begin{array}{c|c} 1 & \mathbf{0}_k^{\mathsf{T}} \\ \hline \mathbf{0}_j & \delta_{jk}\end{array}\right)\,, \qquad (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K})_{jk} = \left(\begin{array}{c|c} 0 & \hat{\boldsymbol{\beta}}_k \\ \hline \hat{\boldsymbol{\beta}}_j & \mathbf{0}_{jk}\end{array}\right)\,, \qquad (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K})^2_{jk} = \left(\begin{array}{c|c} 1 & \mathbf{0}_k^{\mathsf{T}} \\ \hline \mathbf{0}_j & \hat{\boldsymbol{\beta}}_j\hat{\boldsymbol{\beta}}_k\end{array}\right)\,. \tag{10}$$

Then, eq. (9) yields

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K}\bigr) = \left(\begin{array}{c|c} \cosh\zeta & -\hat{\boldsymbol{\beta}}_k\sinh\zeta \\ \hline -\hat{\boldsymbol{\beta}}_j\sinh\zeta & \delta_{jk} + \hat{\boldsymbol{\beta}}_j\hat{\boldsymbol{\beta}}_k(\cosh\zeta - 1)\end{array}\right)\,.$$

In class, we identified $\zeta = \tanh^{-1}\beta$ as the *rapidity*, which satisfies

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\zeta\,, \qquad\qquad \beta\gamma = \sinh\zeta\,.$$

Hence, after writing $\vec{\boldsymbol{\beta}} = \beta\,\hat{\boldsymbol{\beta}} = (\beta_1\,,\,\beta_2\,,\,\beta_3)$, it follows that

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K}\bigr) = \left(\begin{array}{c|c} \gamma & -\gamma\beta_k \\ \hline -\gamma\beta_j & \delta_{jk} + (\gamma-1)\dfrac{\beta_j\beta_k}{\beta^2}\end{array}\right)\,, \tag{11}$$

which we recognize as the boost matrix defined in eq. (11.98) of Jackson.

<u>*AN ALTERNATIVE METHOD FOR COMPUTING* $\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K}\bigr)$:</u>

If $\zeta = 0$, then $\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K}\bigr) = I$. Henceforth, we suppose that $\zeta \neq 0$. Using eq. (3),

$$M \equiv -\zeta\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K} = \begin{pmatrix} 0 & -\zeta\beta_1/\beta & -\zeta\beta_2/\beta & -\zeta\beta_3/\beta \\ -\zeta\beta_1/\beta & 0 & 0 & 0 \\ -\zeta\beta_2/\beta & 0 & 0 & 0 \\ -\zeta\beta_3/\beta & 0 & 0 & 0 \end{pmatrix}\,, \tag{12}$$

In order to compute $f(M) = \exp M$, we shall employ the following formula of matrix algebra. Denote the m distinct eigenvalues of the $n \times n$ matrix M by λ_i (noting that $m \leq n$), and define the following polynomial,[2]

$$p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_m)\,. \tag{13}$$

Then, M is diagonalizable if and only if $p(M) = \mathbf{0}_n$, where $\mathbf{0}_n$ is the $n \times n$ zero matrix.[3] In this case, any function of M is given by[4]

$$f(M) = \sum_{i=1}^{m} f(\lambda_i) \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{M - \lambda_j \boldsymbol{I}_n}{\lambda_i - \lambda_j} \right), \tag{14}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identify matrix and m is the number of distinct eigenvalues.[5]

We first compute the eigenvalues of M, which are roots of the characteristic polynomial,

$$\det(M - \lambda \boldsymbol{I}_4) = \lambda^4 + \frac{\zeta\beta_1}{\beta} \det \begin{pmatrix} -\zeta\beta_1/\beta & 0 & 0 \\ -\zeta\beta_2/\beta & -\lambda & 0 \\ -\zeta\beta_3/\beta & 0 & -\lambda \end{pmatrix} - \frac{\zeta\beta_2}{\beta} \det \begin{pmatrix} -\zeta\beta_1/\beta & -\lambda & 0 \\ -\zeta\beta_2/\beta & 0 & 0 \\ -\zeta\beta_3/\beta & 0 & -\lambda \end{pmatrix}$$

$$- \frac{\zeta\beta_3}{\beta} \det \begin{pmatrix} -\zeta\beta_1/\beta & -\lambda & 0 \\ -\zeta\beta_2/\beta & 0 & -\lambda \\ -\zeta\beta_3/\beta & 0 & 0 \end{pmatrix} = \lambda^2(\lambda^2 - \zeta^2)\,, \tag{15}$$

after using $\beta^2 = \beta_1^2 + \beta_2^2 + \beta_3^2$. Thus, the three distinct eigenvalues of M are $\lambda_i = 0,\ \zeta,\ -\zeta$.

We can check that M is diagonalizable by evaluating:

$$p(M) = M(M - \zeta\boldsymbol{I}_4)(M + \zeta\boldsymbol{I}_4)$$

$$= \begin{pmatrix} 0 & -\zeta\beta_1/\beta & -\zeta\beta_2/\beta & -\zeta\beta_3/\beta \\ -\zeta\beta_1/\beta & 0 & 0 & 0 \\ -\zeta\beta_2/\beta & 0 & 0 & 0 \\ -\zeta\beta_3/\beta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\zeta & -\zeta\beta_1/\beta & -\zeta\beta_2/\beta & -\zeta\beta_3/\beta \\ -\zeta\beta_1/\beta & -\zeta & 0 & 0 \\ -\zeta\beta_2/\beta & 0 & -\zeta & 0 \\ -\zeta\beta_3/\beta & 0 & 0 & -\zeta \end{pmatrix}$$

$$\times \begin{pmatrix} \zeta & -\zeta\beta_1/\beta & -\zeta\beta_2/\beta & \zeta\beta_3/\beta \\ -\zeta\beta_1/\beta & \zeta & 0 & 0 \\ -\zeta\beta_2/\beta & 0 & \zeta & 0 \\ -\zeta\beta_3/\beta & 0 & 0 & \zeta \end{pmatrix}. \tag{16}$$

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[2] Since the n eigenvalues of M are roots of the characteristic polynomial of M, and some of these roots can have multiplicity greater than one, it follows that the number of distinct eigenvalues $m \leq n$.

[3] A very nice proof of this result can be found in Section 8.3.2 of James B. Carrell, *Groups, Matrices, and Vector Spaces–A Group Theoretic Approach to Linear Algebra* (Springer Science+Business Media LLC, New York, NY, 2017).

[4] For example, see eqs. (7.36) and (7.3.11) of Carl D. Meyer, *Matrix Analysis and Applied Linear Algebra* (SIAM, Philadelphia, PA, 2000) or Chapter V, Section 2.2 of F.R. Gantmacher, *Theory of Matrices*–Volume I (Chelsea Publishing Company, New York, NY, 1959).

[5] If the $n \times n$ matrix M is *not* diagonalizable then $p(M) \neq \mathbf{0}_n$, in which case the formula for $f(M)$ is more complicated than the one given in eq. (14). A generalization of the formula for $f(M)$ when M is not diagonalizable can be found in the references cited in footnote 4 above (although the more general formula is not needed here).

After multiplying the first two matrices on the right-hand side of eq. (16), we are left with

$$p(M) = \zeta^3 \begin{pmatrix} 1 & \beta_1/\beta & \beta_2/\beta & \beta_3/\beta \\ \beta_1/\beta & \beta_1^2/\beta^2 & \beta_1\beta_2/\beta^2 & \beta_1\beta_3/\beta^2 \\ \beta_2/\beta & \beta_1\beta_2/\beta^2 & \beta_2^2/\beta^2 & \beta_2\beta_3/\beta^2 \\ \beta_3/\beta & \beta_1\beta_3/\beta^2 & \beta_2\beta_3/\beta^2 & \beta_3^2/\beta^2 \end{pmatrix} \begin{pmatrix} 1 & -\beta_1/\beta & -\beta_2/\beta & \beta_3/\beta \\ -\beta_1/\beta & 1 & 0 & 0 \\ -\beta_2/\beta & 0 & 1 & 0 \\ -\beta_3/\beta & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

Carrying out the final matrix multiplication above yields:

$$p(M) = \mathbf{0}_4 \,, \tag{18}$$

where $\mathbf{0}_4$ is the 4×4 zero matrix. This computation again confirms that M is diagonalizable, in which case the formula for $f(M)$ given in eq. (14) is applicable.

We now apply eq. (14) to $f(M) = \exp M$. It then follows that

$$\begin{aligned} \exp M &= -\frac{1}{\zeta^2}(M - \zeta \boldsymbol{I}_4)(M + \zeta \boldsymbol{I}_4) + e^{\zeta}\frac{1}{2\zeta^2}M(M + \zeta \boldsymbol{I}_4) + e^{-\zeta}\frac{1}{2\zeta^2}M(M - \zeta \boldsymbol{I}_4) \\ &= \boldsymbol{I}_4 + \left(\frac{\sinh\zeta}{\zeta}\right) M + \left(\frac{\cosh\zeta - 1}{\zeta^2}\right) M^2 \,. \end{aligned} \tag{19}$$

Inserting $M = -\zeta\, \hat{\boldsymbol{\beta}} \boldsymbol{\cdot} \boldsymbol{K}$, we recover the result of eq. (9). Note that in the limit of $\zeta \to 0$, we obtain $\exp M = \boldsymbol{I}_4$, as expected.

<u>*REMARK*</u>: The method employed above can be generalized to the computation of the most general proper orthochronous Lorentz transformation (which combines boosts and three-dimensional proper rotations). This computation is explicitly carried out in Howard E. Haber, *Explicit form for the most general Lorentz transformation revisited*, Symmetry 2024, 16, 1155 [arXiv:2312.12969 [physics.class-ph]].

2. [Jackson, problem 11.13] An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. The frame K' (and the wire) move with velocity $\vec{\boldsymbol{v}}$ parallel to the direction of the wire with respect to the laboratory frame K.

(a) Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

In the rest frame of the wire (i.e. frame K'), choose the z-axis to point along the wire. Then, to compute the electric field, we draw a cylinder of length L and radius r', whose symmetry axis coincides with the z-axis. Applying Gauss' law in gaussian units,

$$\oint_S \vec{\boldsymbol{E}}' \boldsymbol{\cdot} \hat{\boldsymbol{n}}\, da = 4\pi Q \,, \tag{20}$$

where Q is the total charge enclosed inside the cylinder. In cylindrical coordinates (r', ϕ', z'),[6] the symmetry of the problem implies that $\vec{\boldsymbol{E}}'(r', \phi', z') = E'(r')\hat{\boldsymbol{r}}'$, where $E'(r')$ depends only on the radial distance from the symmetry axis. Choosing the surface S to be the surface of the cylinder, we have $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}'$, and so eq. (20) reduces to

$$2\pi r' L E'(r') = 4\pi Q\,.$$

Defining the linear charge density (i.e. charge per unit length) by $q_0 = Q/L$, we conclude that[7]

$$\vec{\boldsymbol{E}}'(r') = \frac{2q_0}{r'}\,\hat{\boldsymbol{r}}'\,. \tag{21}$$

Since there are no moving charges in the rest frame of the wire, it follows that $\vec{\boldsymbol{B}}' = 0$.

The transformation laws for the electric and magnetic field between reference frames K and K' are given by[8]

$$\vec{\boldsymbol{E}} = \gamma\left[\vec{\boldsymbol{E}}' - \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}'\right] - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}')\,,$$

$$\vec{\boldsymbol{B}} = \gamma\left[\vec{\boldsymbol{B}}' + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}'\right] - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}')\,.$$

For this problem, $\vec{\boldsymbol{\beta}} = \beta\,\hat{\boldsymbol{z}}$. Using the results of part (a), and noting that $r = r'$ (since the radial direction is perpendicular to the direction of the velocity of frame K' with respect to K), it follows that

$$\vec{\boldsymbol{E}} = \frac{2\gamma q_0}{r}\hat{\boldsymbol{r}}\,, \qquad\qquad \vec{\boldsymbol{B}} = \frac{2\gamma\beta q_0}{r}\,\hat{\boldsymbol{\phi}}\,, \tag{22}$$

where we have used $\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{r}} = 0$ and $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}} = \hat{\boldsymbol{\phi}}$.

(b) What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

In reference frame K' there are no moving charges, so that $\vec{\boldsymbol{J}}' = 0$. The corresponding charge density is

$$\rho'(r') = \frac{q_0}{2\pi r'}\delta(r')\,. \tag{23}$$

To check this, let us integrate over a cylinder of length L and arbitrary nonzero radius, whose symmetry axis coincides with the z-axis. Then,

$$\int \rho'(r')\,dV = \int \rho'(r')\,r'\,dr'\,d\phi\,dz' = q_0\int dr'\delta(r')dz' = q_0 L = Q\,.$$

[6]We denote the radial coordinate of cylindrical coordinates in frame K' to be r' rather than the more traditional ρ', since we reserve the letter ρ for charge density.

[7]The direction of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{z}}$ are the same in frames K and K', so no extra primed-superscript is required on these quantities.

[8]Eq. (11.149) of Jackson provides the equations to transform the fields from reference frame K to reference frame K'. To transform the fields from K' to K, simply change the sign of $\vec{\boldsymbol{\beta}}$.

Since $J^\mu = (c\rho\,;\,\vec{\boldsymbol{J}})$ is a four-vector, the relevant transformation law between frames K and K' are:

$$c\rho = \gamma(c\rho' + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}')\,, \tag{24}$$

$$\vec{\boldsymbol{J}} = \vec{\boldsymbol{J}}' + \frac{\gamma-1}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}')\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}c\rho'\,. \tag{25}$$

Plugging in $\vec{\boldsymbol{J}}' = 0$ and the result of eq. (23), and noting that $\vec{\boldsymbol{\beta}} = \beta\,\hat{\boldsymbol{z}}$ and $r' = r$, it follows that[9]

$$\rho(r) = \frac{\gamma q_0}{2\pi r}\,\delta(r)\,, \qquad\qquad \vec{\boldsymbol{J}} = \frac{\gamma\beta c\, q_0}{2\pi r}\,\hat{\boldsymbol{z}}\,\delta(r) = \rho(r)v\hat{\boldsymbol{z}} = \rho(r)\vec{\boldsymbol{v}}\,, \tag{26}$$

after using $v \equiv \beta\, c$.

(c) From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part (a).

This is an electrostatics and magnetostatics problem, so we can use Gauss' law to compute $\vec{\boldsymbol{E}}$ and Ampère's law to compute $\vec{\boldsymbol{B}}$. The computation of $\vec{\boldsymbol{E}}$ is identical to the one given in part (a) with q_0 replaced by γq_0. Hence, it immediately follows from eq. (21) that

$$\vec{\boldsymbol{E}}(r) = \frac{2\gamma q_0}{r}\,\hat{\boldsymbol{r}}\,,$$

in agreement with eq. (22). Ampère's law in gaussian units is

$$\oint_C \vec{\boldsymbol{B}}\boldsymbol{\cdot} d\vec{\boldsymbol{\ell}} = \frac{4\pi I}{c}\,,$$

where I is the current enclosed in the loop C. With $\vec{\boldsymbol{J}}$ given by eq. (26),

$$I = \int_A \vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,da = \int \rho(r)v\,r\,dr\,d\phi = \gamma q_0 v\,,$$

after noting that $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ points along the direction of the current flow and $da = rdrd\phi$ is the infinitesimal area element perpendicular to the current flow. Using the symmetry of the problem, $\vec{\boldsymbol{B}} = B(r)\hat{\boldsymbol{\phi}}$. Thus, evaluating Ampère's law with a contour C given by a circle centered at $r = 0$ that lies in a plane perpendicular to the current flow, $d\vec{\boldsymbol{\ell}} = rd\phi\hat{\boldsymbol{\phi}}$ and

$$2\pi r B(r) = \frac{4\pi I}{c} = \frac{4\pi\gamma q_0 v}{c}\,,$$

which yields

$$\vec{\boldsymbol{B}}(r) = \frac{2\gamma\beta q_0 v}{r}\,\hat{\boldsymbol{\phi}}\,,$$

after using $v = \beta c$, in agreement with eq. (22).

[9] We can interpret $q \equiv \gamma q_0$ as the linear charge density as observed in reference frame K. This is not unexpected due to the phenomenon of length contraction.

3. [Jackson, problem 11.15] In a certain reference frame, a static uniform electric field E_0 is parallel to the x-axis, and a static uniform magnetic field $B_0 = 2E_0$ lies in the x–y plane, making an angle θ with respect to the x-axis. Determine the relative velocity of a reference frame in which the electric and magnetic fields are parallel. What are the fields in this frame for $\theta \ll 1$ and $\theta \to \frac{1}{2}\pi$?

In frame K, we have

$$\vec{\boldsymbol{E}} = E_0\,\hat{\boldsymbol{x}}\,, \qquad \vec{\boldsymbol{B}} = B_x\,\hat{\boldsymbol{x}} + B_y\,\hat{\boldsymbol{y}}\,, \tag{27}$$

with

$$\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = |\vec{\boldsymbol{E}}|\,|\vec{\boldsymbol{B}}|\cos\theta = E_0 B_0\cos\theta = 2E_0^2\cos\theta\,, \tag{28}$$

after writing $|\vec{\boldsymbol{E}}| = E_0$ and $|\vec{\boldsymbol{B}}| = B_0 = 2E_0$. It follows that

$$B_x = 2E_0\cos\theta\,, \qquad B_y = 2E_0\sin\theta\,. \tag{29}$$

The electric and magnetic fields are parallel in a reference frame K' which is moving at a velocity $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$ with respect to reference frame K. That is, the fields in K' satisfy,

$$\vec{\boldsymbol{E}}\,' \times \vec{\boldsymbol{B}}\,' = 0\,. \tag{30}$$

The electric and magnetic fields in frame K' are related to the corresponding fields in frame K by eq. (11.149) of Jackson,

$$\vec{\boldsymbol{E}}\,' = \gamma(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}) - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\,, \tag{31}$$

$$\vec{\boldsymbol{B}}\,' = \gamma(\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}) - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\,. \tag{32}$$

These relations can be rewritten in the following form,

$$\vec{\boldsymbol{E}}'_{\parallel} = \vec{\boldsymbol{E}}_{\parallel}\,, \qquad \vec{\boldsymbol{B}}'_{\parallel} = \vec{\boldsymbol{B}}_{\parallel}\,, \tag{33}$$

$$\vec{\boldsymbol{E}}'_{\perp} = \gamma\left(\vec{\boldsymbol{E}}_{\perp} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}_{\perp}\right)\,, \qquad \vec{\boldsymbol{B}}'_{\perp} = \gamma\left(\vec{\boldsymbol{B}}_{\perp} - \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}_{\perp}\right)\,. \tag{34}$$

In eqs. (33) and (34), fields with a $\parallel$ subscript are parallel to $\vec{\boldsymbol{\beta}}$ and fields with a $\perp$ subscript are perpendicular to $\vec{\boldsymbol{\beta}}$. For example, $\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}_{\parallel} = 0$ and $\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{\perp} = 0$, which implies that

$$\vec{\boldsymbol{E}}_{\parallel} = \frac{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}}{\beta^2} \quad\text{and}\quad \vec{\boldsymbol{E}}_{\perp} = \vec{\boldsymbol{E}} - \frac{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}}{\beta^2} = \frac{\vec{\boldsymbol{\beta}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{\beta}})}{\beta^2}\,.$$

The form of eqs. (33) and (34) suggests that the relative velocity $\vec{\boldsymbol{v}}$ should point in the z-direction. That is, $\vec{\boldsymbol{\beta}} = \beta\,\hat{\boldsymbol{z}}$, in which case $\vec{\boldsymbol{E}}_{\parallel} = E_z\hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{B}}_{\parallel} = B_z\hat{\boldsymbol{z}}$. Since $E_z = B_z = 0$, it follows from eq. (33) that $E'_z = B'_z = 0$. Using eq. (34), the transverse fields are given by

$$E'_x = \gamma(E_x - \beta B_y) = \gamma E_0(1 - 2\beta\sin\theta)\,, \qquad E'_y = \gamma(E_y + \beta B_x) = 2\beta\gamma E_0\cos\theta\,, \tag{35}$$

$$B'_x = \gamma(B_x + \beta E_y) = 2\gamma E_0\cos\theta\,, \qquad B'_y = \gamma(E_y + \beta B_x) = \gamma E_0(2\sin\theta - \beta)\,, \tag{36}$$

after using eqs. (27)–(29). Moreover, eq. (30) implies that $E'_x B'_y - E'_y B'_x = (\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}')_z = 0$. Inserting the results for the primed fields in this last equation, it then follows that

$$\gamma^2 E_0^2 (1 - 2\beta \sin\theta)(2\sin\theta - \beta) - 4\beta\gamma^2 E_0^2 \cos^2\theta = 0\,.$$

Multiplying out the factors above and writing $\cos^2\theta = 1 - \sin^2\theta$, the above equation simplifies to

$$2\beta^2 \sin\theta - 5\beta + 2\sin\theta = 0\,. \tag{37}$$

This is a quadratic equation in β which is easily solved. The larger of the two roots is greater than 1, which we reject since $0 \leq \beta \leq 1$ (i.e., $0 \leq v \leq c$). The smaller of the two roots is non-negative and less than 1. Thus, we conclude that

$$\beta = \frac{v}{c} = \frac{5 - \sqrt{25 - 16\sin^2\theta}}{4\sin\theta}\,. \tag{38}$$

The two limiting cases are easily analyzed. In the case of $\theta \ll 1$, we can work to first order in θ. From eq. (38) we find that $\beta \simeq \frac{2}{5}\theta$. Since $\theta \ll 1$ it follows that $\beta \ll 1$, in which case

$$\gamma = (1 - \beta^2)^{-1/2} \simeq 1 + \mathcal{O}(\beta^2)\,.$$

Since we are working to first order in θ, we also must work to first order in β. In particular we can neglect terms such as $\beta\theta$. Hence, in this limiting case, eqs. (35) and (36) yield

$$\vec{\boldsymbol{E}}' = \tfrac{1}{2}\vec{\boldsymbol{B}}' = E_0(\hat{\boldsymbol{x}} + 2\beta\hat{\boldsymbol{y}})\,, \qquad \text{for } \beta \simeq \tfrac{2}{5}\theta \ll 1\,, \tag{39}$$

where we have neglected terms that are second order (or higher) in β. Finally, in the limit of $\theta \to \frac{1}{2}\pi$, eq. (38) yields $\beta = \frac{1}{2}$. Then $\gamma = 2/\sqrt{3}$, and eqs. (35) and (36) yield

$$\vec{\boldsymbol{E}}' = 0\,, \qquad \vec{\boldsymbol{B}}' = \sqrt{3}E_0\hat{\boldsymbol{y}}\,, \qquad \text{for } \theta = \tfrac{1}{2}\pi\,. \tag{40}$$

<u>*REMARK 1*</u>:

Recall that in class, we showed that the quantity $F^{\mu\nu}\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} = -4\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}$ is a Lorentz invariant. This means that if $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are perpendicular in one frame, then they must be perpendicular in all frames. Thus, if $\theta = \frac{1}{2}\pi$ in frame K and $\theta = 0$ in frame K', then it must be true that either the electric field or the magnetic field (or both) vanish in frame K', since the only vector that is both perpendicular and parallel to a given fixed nonzero vector is the zero vector. This is indeed the case here, as can be seen in eq. (40).

<u>*REMARK 2*</u>:

It is easy to show that eq. (38) implies that $0 \leq \beta \leq \frac{1}{2}$. If we multiply the numerator and denominator of eq. (38) by $5 + \sqrt{25 - 16\sin^2\theta}$, we obtain,

$$\beta = \frac{4\sin\theta}{5 + \sqrt{25 - 16\sin^2\theta}}\,.$$

Since the polar angle lies in the range $0 \leq \theta \leq \pi$ or equivalently $0 \leq \sin\theta \leq 1$, it follows immediately that $\beta \geq 0$ (where $\beta = 0$ corresponds to $\sin\theta = 0$ as expected). Finally, it is easy to verify that

$$\frac{4\sin\theta}{5+\sqrt{25-16\sin^2\theta}} \leq \frac{1}{2}\,. \tag{41}$$

Since the denominator on the left hand side above is positive, we can rewrite eq. (41) as

$$4\sin\theta \leq \tfrac{1}{2}\left(5+\sqrt{25-16\sin^2\theta}\right)\,. \tag{42}$$

This inequality is manifestly true for $\sin\theta = 0$. For $\sin\theta > 0$, eq. (42) can be rearranged into the following form

$$8\sin\theta - 5 \leq \sqrt{25-16\sin^2\theta}\,. \tag{43}$$

Squaring both sides and simplifying the resulting expression then yields $\sin\theta\,(\sin\theta - 1) \leq 0$. Dividing both sides of the equation by $\sin\theta$ (which is assumed positive) yields $0 \leq \sin\theta \leq 1$, which is valid for all polar angles θ. Hence, eq. (41) is established. The inequality becomes an equality if $\sin\theta = 1$, in which case $\beta = \frac{1}{2}$.

<u>*REMARK 3*</u>: Non-uniqueness of the solution

In our analysis above, we found one solution to the problem. However, it is easy to see that there are an infinite number of solutions. That is, there are an infinite number of Lorentz boost matrices such that

$$F'^{\mu\nu} = \Lambda(\vec{\boldsymbol{\beta}})^\mu{}_\alpha \Lambda(\vec{\boldsymbol{\beta}})^\nu{}_\beta F^{\alpha\beta}\,, \tag{44}$$

where $F^{\alpha\beta}$ is the electromagnetic field strength tensor made up of the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields given in eqs. (27) and (29), $F'^{\mu\nu}$ is the electromagnetic field strength tensor made up of the $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ fields such that $\vec{\boldsymbol{E}}\,' \times \vec{\boldsymbol{B}}\,' = 0$, and $\Lambda(\vec{\boldsymbol{\beta}})$ is the Lorentz boost matrix in the direction of $\vec{\boldsymbol{\beta}}$ given in eq. (11). We have already found one such boost matrix, namely $\Lambda(\beta\hat{\boldsymbol{z}})$, where β is given by eq. (38). This boost matrix produces the $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ fields given in eqs. (35) and (36). Since $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ are parallel in the primed reference frame, we can write

$$\vec{\boldsymbol{E}}\,' = E'\hat{\boldsymbol{n}}\,, \qquad\qquad \vec{\boldsymbol{B}}\,' = B'\hat{\boldsymbol{n}}\,, \tag{45}$$

where $\hat{\boldsymbol{n}}$ is the common direction of $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$. Using eq. (35), one obtains an explicit form for $\hat{\boldsymbol{n}}$ that is given by,

$$\hat{\boldsymbol{n}} = \frac{(1-2\beta\sin\theta)\hat{\boldsymbol{x}} + 2\beta\cos\theta\,\hat{\boldsymbol{y}}}{\sqrt{1-4\beta\sin\theta+4\beta^2}} = \frac{(\sin\theta - 2\beta)\vec{\boldsymbol{E}} + \beta\cos\theta\,\vec{\boldsymbol{B}}}{E_0\sin\theta\sqrt{1-4\beta\sin\theta+4\beta^2}}\,, \tag{46}$$

where β is given by eq. (38). We used eqs. (27)–(29) to obtain the final expression above.

If one applies the following Lorentz transformation to reference frame K,

$$\Lambda = \Lambda(\beta'\hat{\boldsymbol{n}})\Lambda(\beta\hat{\boldsymbol{z}})\,, \tag{47}$$

then in the resulting reference frame K'' the $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ fields are also parallel, for *any* choice of β'. This result follows from eq. (33), which states that the components of the electric and magnetic field that are parallel to the boost direction are unaffected by the Lorentz transformation. Having found the reference frame K' after applying $\Lambda(\beta\hat{\boldsymbol{z}})$ where $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ are parallel and point in the $\hat{\boldsymbol{n}}$ direction, one can perform an arbitrary boost in the direction parallel to $\hat{\boldsymbol{n}}$ without modifying $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ further.

One can evaluate the right hand side of eq. (47) explicitly. Here, I will make use of Paweł Klimas, *Lecture Notes on Classical Electrodynamics*, which has been posted to the Physics 214 webpage. Using eqs. (1.73) and (1.78) of Klimas' notes,

$$\Lambda(\beta'\hat{\boldsymbol{n}})\Lambda(\beta\hat{\boldsymbol{z}}) = \mathcal{O}\Lambda(\vec{\boldsymbol{\beta}}''), \tag{48}$$

where $\mathcal{O}$ is a Lorentz transformation corresponding to a pure rotation[10] and

$$\vec{\boldsymbol{\beta}}'' = \frac{1}{1+\beta\beta'\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}}\left[\frac{\beta'\hat{\boldsymbol{n}}}{\gamma} + \left(1+\frac{\gamma\beta\beta'}{\gamma+1}\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}\right)\beta\hat{\boldsymbol{z}}\right], \tag{49}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$. In light of eq. (46), it follows that $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{z}} = 0$, and eq. (49) simplifies to[11]

$$\vec{\boldsymbol{\beta}}'' = \beta'(1-\beta^2)^{1/2}\,\hat{\boldsymbol{n}} + \beta\hat{\boldsymbol{z}}\,. \tag{50}$$

Note that the parallel electric and magnetic field remain parallel if one transforms the reference frame by a pure rotation. Thus, we can neglect the pure rotation $\mathcal{O}$ in eq. (47) to conclude that starting from reference frame K, the application of the boost $\Lambda(\vec{\boldsymbol{\beta}}'')$ to produce reference frame K'' yields $\vec{\boldsymbol{E}}''$ and $\vec{\boldsymbol{B}}''$ fields that are parallel.

To summarize, the complete answer to the problem posed by Jackson (although probably not what Jackson meant to ask) is that any boost of the form $\Lambda\big(\beta'(1-\beta^2)^{1/2}\,\hat{\boldsymbol{n}} + \beta\hat{\boldsymbol{z}}\big)$, where β and $\hat{\boldsymbol{n}}$ are fixed by eqs. (38) and (46), respectively, will yield a reference frame K'' such that the $\vec{\boldsymbol{E}}''$ and $\vec{\boldsymbol{B}}''$ fields are parallel, for *any choice of the parameter* β', where $0 \leq \beta' \leq 1$.

REMARK 4:

An alternative solution to Jackson, Problem 11.15 is provided in an Appendix at the end of this Solution Set.

4. [Jackson, problem 11.18] The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$, given by eq. (11.52) of Jackson, become more and more concentrated as $\beta \to 1$, as indicated in Fig. 11.9 on p. 561 of Jackson. Choose axes so that the charge moves along the z axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define the transverse vector $\vec{\boldsymbol{r}}_\perp$, with components x and y. Consider the fields and the source in the limit of $\beta = 1$.

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[10] The rotation $\mathcal{O}$ is called the Wigner rotation. As explained below eq. (50), the parallel electric and magnetic fields remain parallel under a pure rotation, and thus we will not require an explicit expression for the Wigner rotation in this problem.

[11] If we define $\beta'' \equiv |\vec{\boldsymbol{\beta}}''|$, then $\beta''^2 = \beta'^2(1-\beta^2) + \beta^2$. One can then check that $0 \leq \beta^2, \beta'^2 \leq 1$ implies that $0 \leq \beta''^2 \leq 1$, as required by special relativity.

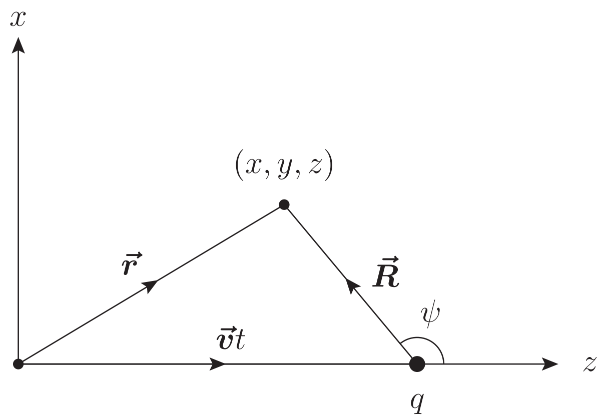

Figure 1: A charge q moving at constant velocity $\vec{\boldsymbol{v}}$ in the z-direction as seen from reference frame K. The angle ψ is defined so that $\hat{\boldsymbol{v}} \boldsymbol{\cdot} \hat{\boldsymbol{R}} = \cos\psi$.

(a) Show that the fields can be written as

$$\vec{\boldsymbol{E}} = 2q\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct - z)\,, \qquad\qquad \vec{\boldsymbol{B}} = 2q\frac{\hat{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct - z)\,, \tag{51}$$

where $\hat{\boldsymbol{v}}$ is a unit vector in the direction of the particle's velocity.

We begin with eq. (11.154) on p. 560 of Jackson,

$$\vec{\boldsymbol{E}} = \frac{q\vec{\boldsymbol{R}}}{R^3\gamma^2(1 - \beta^2\sin^2\psi)^{3/2}}\,, \tag{52}$$

where ψ is the angle between the vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{R}}$. I have modified Jackson's notation by employing the symbol $\vec{\boldsymbol{R}}$ for the vector that points from the charge q to the observation point $\vec{\boldsymbol{r}} = (x, y, z)$ in reference frame K.[12] Eq. (52) was also derived in class along with the corresponding result for the magnetic field,

$$\vec{\boldsymbol{B}} = \frac{q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{R}})}{cR^3\gamma^2(1 - \beta^2\sin^2\psi)^{3/2}}\,. \tag{53}$$

The reference frame K is exhibited in Fig. 1. It is evident from this figure that

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{v}}t\,. \tag{54}$$

The velocity vector is taken to lie along the z-direction. That is, $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{z}}$.

It is convenient to introduce the notation where

$$\vec{\boldsymbol{r}}_\perp = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}\,, \qquad\qquad \vec{\boldsymbol{r}}_\parallel = z\vec{\boldsymbol{z}}\,, \tag{55}$$

so that $\vec{\boldsymbol{r}}_\perp \boldsymbol{\cdot} \vec{\boldsymbol{v}} = 0$ and $\vec{\boldsymbol{r}}_\parallel \times \vec{\boldsymbol{v}} = 0$. Likewise, we can resolve the vector $\vec{\boldsymbol{R}}$ into components parallel and perpendicular to the velocity vector,

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{R}}_\parallel + \vec{\boldsymbol{R}}_\perp\,,$$

[12]Jackson denotes the vector that points from the charge q to the observation point (x, y, z) by $\vec{\boldsymbol{r}}$. However, I prefer to employ $\vec{\boldsymbol{r}}$ to represent the vector that points from the origin of reference frame K to the observation point, as shown in Fig. 1.

where

$$\vec{\boldsymbol{R}}_{\parallel} \equiv R_{\parallel}\hat{\boldsymbol{z}} = (z - vt)\hat{\boldsymbol{z}}\,, \qquad\qquad \vec{\boldsymbol{R}}_{\perp} = \vec{\boldsymbol{r}}_{\perp}\,. \tag{56}$$

after making use of eq. (54). In particular, note that $|\vec{\boldsymbol{R}}_{\perp}| \equiv R_{\perp} = R\sin\psi$. It follows that

$$\begin{aligned} R^3(1-\beta^2\sin^2\psi)^{3/2} &= (R^2 - \beta^2 R^2\sin^2\psi)^{3/2} = (R_\perp^2 + R_\parallel^2 - \beta^2 R_\perp^2)^{3/2} \\ &= [R_\parallel^2 + R_\perp^2(1-\beta^2)]^{3/2} = (R_\parallel^2 + R_\perp^2/\gamma^2)^{3/2}\,. \end{aligned} \tag{57}$$

Note that in obtaining eq. (57) we used $R^2 = R_\perp^2 + R_\parallel^2$ and $\gamma \equiv (1-\beta^2)^{-1/2}$. Moreover, since $\vec{\boldsymbol{R}}_{\perp} = \vec{\boldsymbol{r}}_{\perp}$ [cf. eq. (56)], we may replace $R_\perp$ with $r_\perp \equiv |\vec{\boldsymbol{r}}_\perp| = (x^2+y^2)^{1/2}$ in the above formulae. Eqs. (52), (56) and (57) then yield

$$\vec{\boldsymbol{E}} = \frac{\gamma q\big[\vec{\boldsymbol{r}}_\perp + (z-vt)\hat{\boldsymbol{z}}\big]}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}}\,. \tag{58}$$

Likewise, eqs.(53), (56) and (57) yield

$$\vec{\boldsymbol{B}} = \frac{\gamma q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp)}{c(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}}\,. \tag{59}$$

Consider the limit of $\beta \to 1$. In this limit, $\gamma \to \infty$, and we see that

$$\lim_{\gamma\to\infty} \frac{\gamma}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}} = \begin{cases} 0\,, & \text{if } R_\parallel \neq 0\,, \\ \infty\,, & \text{if } R_\parallel = 0\,. \end{cases}$$

This implies that

$$\lim_{\gamma\to\infty} \frac{\gamma}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}} = K\delta(R_\parallel)\,, \tag{60}$$

for some constant K. Note that in light of eq. (56),

$$\lim_{\gamma\to\infty} R_\parallel = z - ct\,,$$

since $\gamma \to \infty$ in the limit of $v \to c$. To determine K, we integrate eq. (60) from $-\infty$ to ∞, since $R_\parallel$ can be any real number (either positive, negative or zero) depending on the value of the time t. Thus, employing the substitution $u = \gamma R_\parallel$,

$$K = \int_{-\infty}^{\infty} \frac{\gamma\, dR_\parallel}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}} = \int_{-\infty}^{\infty} \frac{du}{(u^2 + r_\perp^2)^{3/2}} = \frac{u}{r_\perp^2(u^2 + r_\perp^2)^{1/2}}\bigg|_{-\infty}^{\infty} = \frac{2}{r_\perp^2}\,. \tag{61}$$

Hence, we conclude that

$$\lim_{\gamma\to\infty} \frac{\gamma}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}} = \frac{2}{r_\perp^2}\delta(z - ct)\,. \tag{62}$$

Note that in the limit of $v \to c$, we can insert the result of eq. (62) back into eq. (58), and make use of the well-known property of the delta function,

$$(z - ct)\delta(z - ct) = 0\,, \tag{63}$$

to obtain

$$\lim_{v\to c} E_z = \frac{2q}{r_\perp^2}(z-ct)\delta(z-ct) = 0\,. \tag{64}$$

It therefore follows that

$$\lim_{v\to c} \vec{\boldsymbol{E}} = 2q\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\,. \tag{65}$$

Likewise, in light of the observation that $\lim_{v\to c}\vec{\boldsymbol{v}}/c = \hat{\boldsymbol{v}}$, when we insert the result of eq. (62) back into eq. (59), we end up with

$$\lim_{v\to c} \vec{\boldsymbol{B}} = 2q\frac{\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\,. \tag{66}$$

Since the delta function is an even function of its argument, we can write $\delta(z-ct) = \delta(ct-z)$ in eqs. (65) and (66), and eq. (51) is verified.

An alternate solution to part (a)

I cannot resist sharing an alternate solution suggested by one of the Physics 214 students (with additional modifications to the original analysis added below).

Starting from eq. (58) and defining a new variable $w \equiv z - vt$, consider the following Fourier transform of $\vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, w)$,

$$\vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, k) = \int_{-\infty}^{\infty} \vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, w)e^{-ikw}\,dw = \gamma q\vec{\boldsymbol{r}}_\perp \int_{-\infty}^{\infty} \frac{dw}{(\gamma^2 w^2 + r_\perp^2)^{3/2}}\,e^{-ikw}\,. \tag{67}$$

We can change the integration variable $w \to \gamma w$ to obtain,

$$\vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, k) = q\vec{\boldsymbol{r}}_\perp \int_{-\infty}^{\infty} \frac{dw}{(w^2 + r_\perp^2)^{3/2}}\,e^{-ikw/\gamma}\,. \tag{68}$$

If we formally take the limit $v \to c$, or equivalently $\gamma \to \infty$, then $e^{-ikw/\gamma} \to 1$. This limit needs justification, since for finite but very large γ, it is not clear that one can ignore higher order terms. I will attempt to justify this step at the end. Meanwhile, if we go ahead and take the formal limit of $e^{-ikw/\gamma} \to 1$, then

$$\lim_{\gamma\to\infty} \vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, k) = q\vec{\boldsymbol{r}}_\perp \int_{-\infty}^{\infty} \frac{dw}{(w^2 + r_\perp^2)^{3/2}} = \frac{2q\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,. \tag{69}$$

If we now compute the inverse Fourier integral to determine the $\gamma \to \infty$ limit of $E_\perp(\vec{\boldsymbol{r}}_\perp, w)$, then

$$\lim_{\gamma\to\infty} \vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, w) = \lim_{\gamma\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, k)\,e^{ikw} = \frac{2q\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikw}dk = \frac{2q\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta(w)\,. \tag{70}$$

Since $\lim_{\gamma\to\infty} w = \lim_{v\to c} w = z - ct$, we end up with

$$\lim_{v\to c} \vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, w) = \frac{2q\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta(z-ct)\,. \tag{71}$$

In essence, the somewhat mathematical dubious steps above have effectively provided a method for determining the coefficient of the delta function in eq. (71). Indeed, this is not surprising, as the integration carried out in eq. (69) is precisely the same integration that was used in determining the constant K in eq. (61).

However, we still must determine $\lim_{\gamma\to\infty} E_z(\vec{\boldsymbol{r}}_\perp, w)$. Following the steps that yielded eq. (67),

$$E_z(\vec{\boldsymbol{r}}_\perp, k) = \int_{-\infty}^{\infty} E_z(\vec{\boldsymbol{r}}_\perp, w)e^{-ikw}\,dw = \gamma q \int_{-\infty}^{\infty} \frac{w\,dw}{(\gamma^2 w^2 + r_\perp^2)^{3/2}}\, e^{-ikw}\,. \tag{72}$$

This time, if we change the integration variable $w \to \gamma w$, then

$$E_z(\vec{\boldsymbol{r}}_\perp, k) = \frac{q}{\gamma} \int_{-\infty}^{\infty} \frac{w\,dw}{(w^2 + r_\perp^2)^{3/2}}\, e^{-ikw/\gamma}\,, \tag{73}$$

Hence, in the limit of $\gamma \to \infty$, it follows that

$$\lim_{\gamma\to\infty} E_z(\vec{\boldsymbol{r}}_\perp, k) = 0\,. \tag{74}$$

The inverse Fourier transform then yields,

$$\lim_{\gamma\to\infty} E_z(\vec{\boldsymbol{r}}_\perp, w) = \lim_{\gamma\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_z(\vec{\boldsymbol{r}}_\perp, k)\, e^{ikw} = 0\,. \tag{75}$$

Finally, let's try to justify the $\gamma \to \infty$ limit taken below eq. (68). First, we break up the integral in eq. (68) from $-\infty$ to 0 and then from 0 to ∞. In the first integral, we change variables $w \to -w$. We can now combine the sum of the resulting two integrals into

$$\vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, k) = 2q\vec{\boldsymbol{r}}_\perp \int_0^{\infty} \frac{\cos\bigl(kw/\gamma\bigr)dw}{(w^2 + r_\perp^2)^{3/2}}\,. \tag{76}$$

We first note formula 3.754 no. 2 on p. 439 of I.S. Gradschteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (8th edition), edited by Daniel Zwillinger (Academic Press, Waltham, MA, 2015), henceforth to be denoted by G&R,

$$K_0(ab) = \int_0^{\infty} \frac{\cos(ax)\,dx}{\sqrt{x^2 + b^2}}\,, \qquad \text{for } a > 0 \text{ and } \operatorname{Re} b > 0\,, \tag{77}$$

where K_0 is the modified Bessel function of the 2nd kind. Taking the derivative of this formula using G&R formula 8.486 no. 18 on p. 938,

$$\frac{d}{dz} K_0(z) = -K_1(z)\,, \tag{78}$$

we obtain

$$K_1(ab) = \frac{b}{a} \int_0^{\infty} \frac{\cos(ax)\,dx}{(x^2 + b^2)^{3/2}}\,, \qquad \text{for } a > 0 \text{ and } \operatorname{Re} b > 0\,, \tag{79}$$

Identifying $x \to w$, $a \to k/\gamma$ and $b \to r_\perp$, it follows that

$$\vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, k) = \frac{2kq\vec{\boldsymbol{r}}_\perp}{\gamma r_\perp}\, K_1\left(\frac{kr_\perp}{\gamma}\right)\,. \tag{80}$$

In the limit of $\gamma \to \infty$, we need to make use of the small argument approximation of $K_1(z)$. G&R formulae 8.445–8.446 on p. 928 provides the necessary expansions. However, for me, it is easier to first write down the small argument expansion of $K_0(z)$, and then differentiate using eq. (78) to get the small argument expansion of $K_1(z)$. G&R formula 8.447 gives

$$K_0(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2} \left[-\ln\left(\frac{z}{2}\right) - \gamma_E + \sum_{j=1}^{k} \frac{1}{j} \right], \tag{81}$$

after making use of G&R, formula 8.365 nos. 1 and 4 on p. 913, where $\gamma_E = 0.5772156649\ldots$ is the Euler-Mascheroni constant, and by convention $\sum_{j=1}^{k}$ is assigned the value zero when $k = 0$ (corresponding to the case of the "empty sum"). Using eqs. (78) and (81), one obtains

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left[\ln\left(\frac{z}{2}\right) + \gamma_E - \frac{1}{2}\right] + \mathcal{O}(z^3 \ln z)\,. \tag{82}$$

Inserting this expansion into eq. (80),

$$\vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, k) = \frac{2q\vec{\boldsymbol{r}}_\perp}{r_\perp^2} + \mathcal{O}\left(\frac{\ln\gamma}{\gamma^2}\right), \tag{83}$$

which justifies the $\gamma \to \infty$ limit quoted in eq. (69). It is still not clear that one can mathematically justify obtaining $\vec{\boldsymbol{E}}_\perp(\vec{\boldsymbol{r}}_\perp, w)$ as we did in eq. (70) for very large but finite γ (due, in part, to the presence of terms that go as $\ln\gamma$), although the leading term at large γ does yield the correct result in the sense of distributions.

(b) Show by substitution into the Maxwell equations that these fields are consistent with the 4-vector source density

$$J^\alpha = qcv^\alpha \delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(ct - z)\,,$$

where the 4-vector $v^\alpha = (1\,;\,\hat{\boldsymbol{v}})$.

The four-vector current is given by $J^\mu = (c\rho\,;\,\vec{\boldsymbol{J}})$. Hence, using the Maxwell equations in gaussian units,

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = 4\pi\rho = \frac{4\pi J^0}{c}\,.$$

Using eqs. (65) and (66) and noting that $E_z = 0$, it follows that

$$J^0 = \frac{c}{4\pi}\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = \frac{c}{4\pi}\left(\vec{\boldsymbol{\nabla}}_\perp\cdot\vec{\boldsymbol{E}} + \frac{\partial E_z}{\partial z}\right) = \frac{qc}{2\pi}\delta(z - ct)\vec{\boldsymbol{\nabla}}_\perp\cdot\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right), \tag{84}$$

where

$$\vec{\boldsymbol{\nabla}}_\perp \equiv \hat{\boldsymbol{x}}\,\partial/\partial x + \hat{\boldsymbol{y}}\,\partial/\partial y\,. \tag{85}$$

For $\vec{\boldsymbol{r}}_\perp \equiv x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} \neq 0$, an elementary computation yields

$$\vec{\boldsymbol{\nabla}}_\perp\cdot\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0\,. \tag{86}$$

To determine the behavior at $\vec{\boldsymbol{r}}_\perp = 0$, we consider the two-dimensional analogue of the divergence theorem,

$$\int_A dxdy\, \vec{\boldsymbol{\nabla}}_\perp \boldsymbol{\cdot} \left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right) = \oint_C r_\perp d\phi\, \frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2} \boldsymbol{\cdot} \hat{\boldsymbol{r}}_\perp = \int_0^{2\pi} d\phi = 2\pi\,, \tag{87}$$

where A is a circular disk and C is the circular boundary of the disk. Note that $\hat{\boldsymbol{r}}_\perp = \vec{\boldsymbol{r}}_\perp/r_\perp$ is the outward normal to the circular boundary.

Eqs. (86) and (87) imply that

$$\vec{\boldsymbol{\nabla}}_\perp \boldsymbol{\cdot} \left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right) = 2\pi\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\,, \tag{88}$$

where $\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)$ is a two-dimensional delta function. Inserting this result into eq. (84), we end up with

$$J^0 = qc\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z - ct)\,. \tag{89}$$

Next, we employ the Maxwell equation,

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} - \frac{1}{c}\frac{\partial \vec{\boldsymbol{E}}}{\partial t} = \frac{4\pi}{c}\vec{\boldsymbol{J}}, \tag{90}$$

to evaluate $\vec{\boldsymbol{J}}$. First, we compute

$$\hat{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp = \hat{\boldsymbol{z}} \times (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}) = x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}\,, \tag{91}$$

where we have used the fact that $\vec{\boldsymbol{v}}$ points in the z direction. It then follows that

$$\vec{\boldsymbol{\nabla}} \times \left[\frac{\hat{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z - ct)\right] = \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2}\,\delta(z-ct) & \dfrac{x}{x^2+y^2}\,\delta(z-ct) & 0 \end{pmatrix}. \tag{92}$$

Evaluating the determinant and making use of eqs. (55), (85) and (88) yields,

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \times \left[\frac{\hat{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z - ct)\right] &= -\frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}}{x^2+y^2}\delta'(z-ct) + \left\{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\}\delta(z-ct) \\ &= -\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta'(z-ct) + \hat{\boldsymbol{z}}\,\vec{\boldsymbol{\nabla}}_\perp \boldsymbol{\cdot} \left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right)\delta(z-ct) \\ &= -\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta'(z-ct) + 2\pi\hat{\boldsymbol{z}}\,\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct)\,. \end{aligned} \tag{93}$$

The prime refers to differentiation with respect to z. Finally, we compute

$$\frac{\partial}{\partial t}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right) = -c\frac{\partial}{\partial z}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right) = -\frac{c\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta'(z-ct)\,. \tag{94}$$

Inserting eqs. (65) and (66) into eq. (90) and using eqs. (93) and (94), we obtain

$$\begin{aligned}
\vec{\boldsymbol{J}} &= \frac{qc}{2\pi}\vec{\boldsymbol{\nabla}}\times\left[\frac{\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right] - \frac{q}{2\pi}\frac{\partial}{\partial t}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right) \\
&= \frac{qc}{2\pi}\left\{-\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta'(z-ct) + 2\pi\hat{\boldsymbol{z}}\,\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct) + \frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta'(z-ct)\right\} \\
&= qc\hat{\boldsymbol{v}}\,\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct)\,, \qquad (95)
\end{aligned}$$

after using the fact that $\hat{\boldsymbol{v}}=\hat{\boldsymbol{z}}$. Combining eqs. (89) and (95), we can write

$$J^\alpha = qcv^\alpha\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct)\,,$$

where the four-vector $v^\alpha=(1\,;\,\hat{\boldsymbol{v}})$.

(c) Show that the fields of part (a) are derivable from either of the following 4-vector potentials:

$$A^0 = A^z = -2q\delta(ct-z)\ln(\lambda r_\perp)\,, \qquad \vec{\boldsymbol{A}}_\perp = 0\,, \qquad (96)$$

or

$$A^0 = A^z = 0\,, \qquad \vec{\boldsymbol{A}}_\perp = -2q\Theta(ct-z)\vec{\boldsymbol{\nabla}}_\perp\ln(\lambda r_\perp)\,, \qquad (97)$$

where λ is an irrelevant parameter setting the scale of the logarithm. Show that the two potentials differ by a gauge transformation and find the corresponding gauge function χ.

The four-vector potential is $A^\mu=(\Phi\,;\,\vec{\boldsymbol{A}})$. Given the four-vector potential, the electromagnetic fields are determined by

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}A^0 - \frac{1}{c}\frac{\partial\vec{\boldsymbol{A}}}{\partial t}\,, \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}\,.$$

Inserting the scalar and vector potentials given in eq. (96),

$$\begin{aligned}
\vec{\boldsymbol{E}} &= 2q\vec{\boldsymbol{\nabla}}\Big[\delta(ct-z)\ln(\lambda r_\perp)\Big] + \frac{2q}{c}\hat{\boldsymbol{z}}\ln(\lambda r_\perp)\frac{\partial}{\partial t}\delta(ct-z) \\
&= 2q\delta(z-ct)\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial}{\partial y}\right)\left[\tfrac{1}{2}\ln(x^2+y^2)+\ln\lambda\right] + 2q\hat{\boldsymbol{z}}\ln(\lambda r_\perp)\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\delta(ct-z) \\
&= 2q\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta(z-ct)\,,
\end{aligned}$$

after using $\vec{\boldsymbol{r}}_\perp = x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}$ and $r_\perp^2=x^2+y^2$. In particular, note that

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)f(ct-z)=0\,,$$

for any function of $ct - z$. Using eq. (96) to compute the magnetic field,

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & -2q\ln(\lambda r_\perp)\delta(ct-z) \end{pmatrix}$$

$$= -2q\delta(ct-z)\left\{\hat{\boldsymbol{x}}\frac{\partial}{\partial y}\ln(\lambda r_\perp) - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\ln(\lambda r_\perp)\right\}$$

$$= -2q\delta(ct-z)\left\{\hat{\boldsymbol{x}}\frac{\partial}{\partial y}\left[\tfrac{1}{2}\ln(x^2+y^2)+\ln\lambda\right] - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\left[\tfrac{1}{2}\ln(x^2+y^2)+\ln\lambda\right]\right\}$$

$$= -\frac{2q}{r_\perp^2}(y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}})\delta(ct-z) = 2q\frac{\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct-z)\,,$$

after employing eq. (91).

Repeating these calculations using eq. (97),

$$\vec{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial\vec{\boldsymbol{A}}_\perp}{\partial t} = 2q\delta(ct-z)\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\right)\left[\tfrac{1}{2}\ln(x^2+y^2)+\ln\lambda\right] = 2q\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct-z)\,,$$

after using the relation between the delta function and the step function, $\delta(x) = \dfrac{d}{dx}\Theta(x)$. In the computation of the magnetic field, we require the following result:

$$\vec{\boldsymbol{\nabla}}_\perp \ln(\lambda r_\perp) = \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\right)\left[\tfrac{1}{2}\ln(x^2+y^2)+\ln\lambda\right] = \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}}{x^2+y^2}\,.$$

Hence, it follows that

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = -2q\det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \Theta(ct-z)\dfrac{x}{x^2+y^2} & \Theta(ct-z)\dfrac{y}{x^2+y^2} & 0 \end{pmatrix}$$

$$= \frac{y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}}{x^2+y^2}\delta(ct-z) + \hat{\boldsymbol{z}}\,\Theta(ct-z)\left\{\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right\}$$

$$= 2q\frac{\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct-z)\,,$$

after employing eq. (91) and noting that

$$\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0\,.$$

Finally, we demonstrate that eqs. (96) and (97) differ by a gauge transformation. Under a gauge transformation (using gaussian units),

$$\vec{\boldsymbol{A}} \to \vec{\boldsymbol{A}}' = \vec{\boldsymbol{A}} + \vec{\boldsymbol{\nabla}}\chi\,, \qquad\qquad A^0 \to A'^0 = A^0 - \frac{1}{c}\frac{\partial\chi}{\partial t}\,.$$

Denoting A^μ by eq. (96) and A'^μ by eq. (97), it follows that

$$\begin{aligned}
\frac{\partial\chi}{\partial t} &= -2qc\delta(ct-z)\ln(\lambda r_\perp)\,,\\
\vec{\boldsymbol{\nabla}}_\perp\chi &= -2q\Theta(ct-z)\vec{\boldsymbol{\nabla}}_\perp\ln(\lambda r_\perp)\,,\\
\frac{\partial\chi}{\partial z} &= 2q\delta(ct-z)\ln(\lambda r_\perp)\,.
\end{aligned}$$

The solution to these equations can be determined by inspection,

$$\chi(\vec{\boldsymbol{x}},t) = -2q\Theta(ct-z)\ln(\lambda r_\perp)\,,$$

up to an overall additive constant.

5. [Jackson, problem 11.22] The presence in the universe of an apparently uniform "sea" of blackbody radiation at a temperature of roughly 3K gives one mechanism for an upper limit on the energies of photons that have traveled an appreciable distance since their creation. Photon-photon collisions can result in the creation of a charged particle and its antiparticle ("pair creation") if there is sufficient energy in the center of "mass" of the two photons. The lowest threshold and also the largest cross section occurs for an electron-positron pair.

(a) Taking the energy of a typical 3K photon to be $E = 2.5\times10^{-4}$ eV, calculate the energy for an incident photon such that there is energy just sufficient to make an electron-positron pair. For photons with energies larger than this threshold value, the cross section increases to a maximum of the order of $(e^2/mc^2)^2$ and then decreases slowly at higher energies. This interaction is one mechanism for the disappearance of such photons as they travel cosmological distances.

Since the photon is massless, it can be described by a four-vector of the form $k = E(1\,;\,\hat{\boldsymbol{n}})$, where E is the photon energy and $\hat{\boldsymbol{n}}$ is the unit vector that points along the direction of the photon three-momentum. Note that $k^2 \equiv g_{\mu\nu}k^\mu k^\nu = 0$, which indicates that the photon is massless.

Denote the four-momentum vectors of the two photons by

$$k_1 = E_1(1\,;\,\hat{\boldsymbol{n}}_1)\,, \qquad\qquad k_2 = E_2(1\,;\,\hat{\boldsymbol{n}}_2)\,, \tag{98}$$

where the subscripts 1 and 2 above label the kinematic quantities of the two photons. We are given $E_1 = 2.5\times10^4$ eV, and we are asked to find the minimum allowed energy E_2 such that the process $\gamma\gamma\to e^+e^-$ is kinematically allowed. Let p_1 and p_2 denote the four-momentum vectors of the electron and positron, respectively. Using the conservation of four-momentum,

$$k_1 + k_2 = p_1 + p_2\,. \tag{99}$$

If the process $\gamma\gamma \to e^+e^-$ is kinematically allowed, then the minimal value that $(p_1+p_2)^2$ can take occurs when the electron and positron (each with equal mass $m_e = 511\ \mathrm{keV}/c^2 = 5.11\times 10^5\ \mathrm{eV}/c^2$) are produced at rest in the center-of mass reference frame of the e^+e^- pair. That is, the minimal value that $(p_1+p_2)^2$ can take occurs when $p_1 = p_2 = (m_e c^2\,;\,0,0,0)$. It follows that the minimal value of $(p_1+p_2)^2$ is equal to $4m_e^2c^4$. Since $(p_1+p_2)^2$ is Lorentz invariant (and therefore can be evaluated in any reference frame with the same result), one can conclude that

$$(k_1+k_2)^2 \geq 4m_e^2c^4\,, \tag{100}$$

after squaring both sides of eq. (99). Using eq. (98), we obtain

$$(E_1+E_2)^2 - (E_1\boldsymbol{\hat{n}_1} + E_2\boldsymbol{\hat{n}_2})^2 \geq 4m_e^2c^4\,. \tag{101}$$

Simplifying the above equation yields

$$E_1E_2(1-\cos\theta) \geq 2m_e^2c^4\,, \tag{102}$$

where $\cos\theta = \boldsymbol{\hat{n}}_1\boldsymbol{\cdot}\boldsymbol{\hat{n}}_2$. Since $|\cos\theta| \leq 1$, the minimum value of E_2 arises when $\cos\theta = -1$. In this case, we end up with $E_1E_2 \geq m_e^2c^4$. Putting in the numbers,

$$E_2 \geq \frac{m_e^2c^4}{E_1} = \frac{(5.11\times 10^5\ \mathrm{eV})^2}{2.5\times 10^{-4}\ \mathrm{eV}} = 1.04\times 10^{15}\ \mathrm{eV} = 1.04\ \mathrm{PeV}\,. \tag{103}$$

(b) There is some evidence for a diffuse x-ray background with photons having energies of several hundred electron volts or more. Beyond 1 keV the spectrum falls as E^{-n} with $n \simeq 1.5$. Repeat the calculation of the threshold incident energy, assuming that the energy of the photon in the "sea" is 500 eV.

In this case, we use eq. (103) but we replace the denominator with 500 eV. The end result is

$$E_2 \geq \frac{m_e^2c^4}{E_1} = \frac{(5.11\times 10^5\ \mathrm{eV})^2}{5\times 10^2\ \mathrm{eV}} = 5.22\times 10^8\ \mathrm{eV} = 522\ \mathrm{MeV}\,. \tag{104}$$

6. [Jackson, problem 11.28] Revisit Problems 6.21 and 6.22 of Jackson from the viewpoint of Lorentz transformations. An electric dipole instantaneously at rest at the origin in the frame K' has potentials, $\Phi' = \vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}'/r'^3$ (where $r' \equiv |\vec{\boldsymbol{x}}'|$), and $\vec{\boldsymbol{A}}' = 0$ (and thus only an electric field). The frame K' moves with uniform velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ in the frame K.

(a) Show that in frame K to first order in β, the potentials are

$$\Phi = \frac{\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}}{R^3}\,, \qquad\qquad \vec{\boldsymbol{A}} = \vec{\boldsymbol{\beta}}\,\frac{\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}}{R^3}\,, \tag{105}$$

where $\vec{\boldsymbol{R}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_0(t)$, with $\vec{\boldsymbol{v}} = d\vec{\boldsymbol{x}}_0/dt$ at time t.

Here, we shall follow the analysis of the class handout entitled *The electromagnetic fields of a uniformly moving charge*. Consider an electric dipole with dipole moment vector $\vec{\boldsymbol{p}}$ moving at constant velocity $\vec{\boldsymbol{v}}$ with respect to the laboratory frame K. The rest frame of the electric dipole will be denoted by K'. In particular, we define the origin of K' to be the location of the charge. A laboratory observer is located at the point $\vec{\boldsymbol{x}} = (x, y, z)$, which denotes the vector that points from the origin of the laboratory frame to the observer. As seen in the rest frame of the electric dipole, the observer is located at the point $\vec{\boldsymbol{x}}\,' = (x', y', z')$, which denotes the vector that points from the origin of K' to the observer.

At time $t = 0$, the electric dipole is located at the origin of the laboratory frame. After a time t has elapsed (as measured in frame K), the electric dipole is located at the point $\vec{\boldsymbol{v}}t$ in the laboratory frame. It is convenient to define the axes of the K' coordinate system such that the K and K' coordinate systems (and their origins) coincide at $t = t' = 0$. As usual we define $x_0 \equiv ct$ and $x_0' \equiv ct'$. The relation between $(x_0\,;\,\vec{\boldsymbol{x}})$ and $(x_0'\,;\,\vec{\boldsymbol{x}}\,')$ is given by

$$x_0' = \gamma(x_0 - \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\,, \tag{106}$$

$$\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{x}} + \frac{(\gamma-1)}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}x_0\,, \tag{107}$$

where

$$\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{v}}/c\,, \qquad \beta \equiv |\vec{\boldsymbol{\beta}}|\,, \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}\,.$$

Since the scalar potential Φ and the vector potential $\vec{\boldsymbol{A}}$ make up a four vector $A^\mu = (\Phi\,;\,\vec{\boldsymbol{A}})$, the corresponding transformation laws are the same as those for $x^\mu = (x^0\,;\,\vec{\boldsymbol{x}})$. Hence,

$$\Phi'(\vec{\boldsymbol{x}}\,') = \gamma\big(\Phi(\vec{\boldsymbol{x}},t) - \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)\big)\,, \tag{108}$$

$$\vec{\boldsymbol{A}}\,'(\vec{\boldsymbol{x}}\,',t') = \vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t) + \frac{(\gamma-1)}{\beta^2}\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)\big)\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}\Phi(\vec{\boldsymbol{x}},t)\,, \tag{109}$$

We can invert these transformation laws by interchanging primed and unprimed variables and taking $\vec{\boldsymbol{\beta}} \to -\vec{\boldsymbol{\beta}}$. Thus,

$$\Phi(\vec{\boldsymbol{x}},t) = \gamma\big(\Phi'(\vec{\boldsymbol{x}}\,',t') + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}\,'(\vec{\boldsymbol{x}}\,',t')\big)\,, \tag{110}$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{A}}\,'(\vec{\boldsymbol{x}}\,',t') + \frac{(\gamma-1)}{\beta^2}\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}\,'(\vec{\boldsymbol{x}}\,',t')\big)\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}\Phi'\,, \tag{111}$$

Let us compare the views from reference frames K and K'. The moving electric dipole as seen from the laboratory frame K is shown in Fig. 2. In addition, we define $\vec{\boldsymbol{R}}$ to be the vector in frame K that points from the location of the electric dipole at time t to the location of the observer. It follows that $\vec{\boldsymbol{x}}_0 = \vec{\boldsymbol{v}}t$. Hence, we can identify:

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{x}} - c\,\vec{\boldsymbol{\beta}}\,t\,. \tag{112}$$

The rest frame K' of the moving electric dipole is depicted in Fig. 3. In this frame, the vector that points from the origin of frame K to the location of the electric dipole is $\vec{\boldsymbol{v}}t'$, where t' is

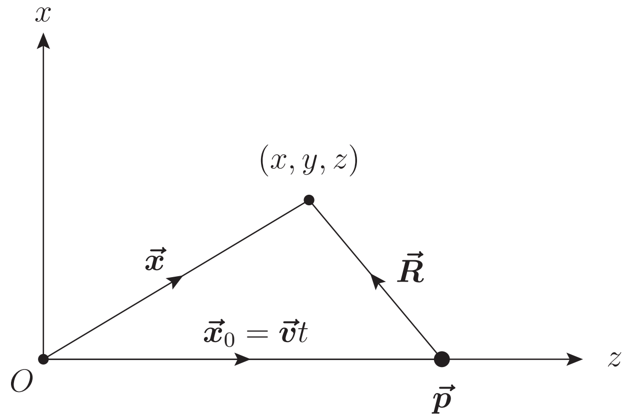

Figure 2: An electric dipole $\vec{\boldsymbol{p}}$ moving at constant velocity $\vec{\boldsymbol{v}}$ in the z-direction as seen from reference frame K. The origin of the laboratory frame K is denoted by O, and $\vec{\boldsymbol{x}}_0$ is the location of the electric dipole at time t.

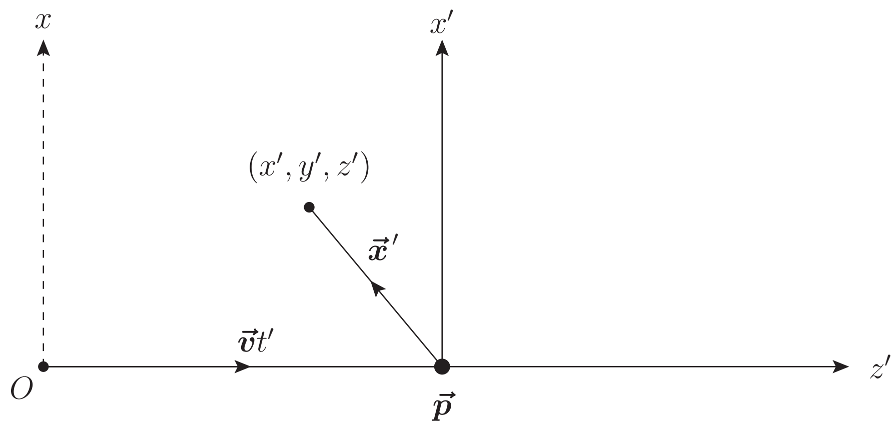

Figure 3: An electric dipole $\vec{\boldsymbol{p}}$ moving at constant velocity $\vec{\boldsymbol{v}}$ in the z'-direction as seen from reference frame K'. The origin of the laboratory frame K is denoted by O. The x-axis of frame K is indicated by a dashed line.

the time elapsed as measured in frame K' (where $t = t' = 0$ marks the time when the frames K and K' coincided). In particular, note that eq. (107) can be rewritten in the following equivalent form,

$$\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{R}} + \frac{(\gamma - 1)}{\beta^2}(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}}\,, \tag{113}$$

after noting that $\vec{\boldsymbol{\beta}} x_0 = \vec{\boldsymbol{v}}t$.

In the rest frame K' of the electric dipole, the scalar and vector potentials are time-independent and are given by

$$\Phi'(\vec{\boldsymbol{x}}\,', t') = \frac{\vec{\boldsymbol{p}} \boldsymbol{\cdot} \vec{\boldsymbol{x}}\,'}{r'^3}\,, \qquad\qquad \vec{\boldsymbol{A}}\,'(\vec{\boldsymbol{x}}\,', t') = 0\,. \tag{114}$$

where $r' \equiv |\vec{\boldsymbol{x}}\,'|$. After making use of eq. (113), and using $\gamma^2 - 1 = \beta^2\gamma^2$ to simplify the resulting expression, we end up with

$$r'^2 \equiv |\vec{\boldsymbol{x}}\,'|^2 = R^2 + \gamma^2(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{R}})^2\,, \tag{115}$$

where $R \equiv |\vec{\boldsymbol{R}}|$. Hence, using eqs. (107) and (114) with $\vec{\boldsymbol{x}}\,'$ and r' given by eqs. (113) and (115), respectively, we obtain:

$$\Phi(\vec{\boldsymbol{x}},t)=\gamma\Phi'(\vec{\boldsymbol{x}}',t')=\frac{\gamma\,\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}}+\gamma(\gamma-1)\big(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}}\big)\big(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\beta}}\big)/\beta^2}{\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})^2\big]^{3/2}}\,, \tag{116}$$

Likewise, setting $\vec{\boldsymbol{A}}'=0$ in eqs. (110) and (111) yields

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{\beta}}\,\Phi(\vec{\boldsymbol{x}},t)\,. \tag{117}$$

Since Jackson only asks for the expressions for Φ and $\vec{\boldsymbol{A}}$ to first order in β, the above results simplify greatly. In particular, $\gamma=(1-\beta^2)^{-1/2}=1+\mathcal{O}(\beta^2)$. Hence, eqs. (116) and (117) yield

$$\Phi(\vec{\boldsymbol{x}},t)=\frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}}}{R^3}+\mathcal{O}(\beta^2)\,,\qquad\qquad \vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{\beta}}\,\frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}}}{R^3}+\mathcal{O}(\beta^2)\,, \tag{118}$$

where $\vec{\boldsymbol{R}}\equiv\vec{\boldsymbol{x}}-c\vec{\boldsymbol{\beta}}t$, in agreement with the results quoted in eq. (105).

Note that $\vec{\boldsymbol{p}}$ is the electric dipole moment vector in the rest frame of the electric dipole. It is an intrinsic property of the particle (like the mass). So, there is no problem in using this quantity in the expressions for the scalar and vector potential in the laboratory frame K.

(b) Show explicitly that the potentials in K satisfy the Lorenz condition.

The Lorenz condition,

$$\partial_\mu A^\mu=\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0\,, \tag{119}$$

is a Lorentz-invariant condition that is trivially satisfied in the reference frame K', where $\vec{\boldsymbol{A}}'=0$ and $\Phi'=\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}'/r'^3$ is time independent. Thus, eq. (119) is also satisfied in reference frame K' since it must be satisfied in any inertial reference frame.

One can also verify explicitly that eq. (119) is satisfied in reference frame K. First, note that $\vec{\boldsymbol{\nabla}}=\vec{\boldsymbol{\nabla}}_R$, where $\vec{\boldsymbol{\nabla}}=(\partial/\partial x^1\,,\,\partial/\partial x^2\,,\,\partial/\partial x^3)$, and $\vec{\boldsymbol{\nabla}}_R=\big(\partial/\partial R^1\,,\,\partial/\partial R^2\,,\,\partial/\partial R^3\big)$, where $R^i=x^i-c\beta t$ $(i=1,2,3)$ are the components of the vector $\vec{\boldsymbol{R}}$. Moreover, the time-dependence of $\Phi(\vec{\boldsymbol{x}},t)$ is due to the time dependence of $\vec{\boldsymbol{R}}=\vec{\boldsymbol{x}}-c\beta t$ [cf. eq. (116)]. Hence, using the chain rule,

$$\frac{1}{c}\frac{\partial\Phi}{\partial t}=\frac{1}{c}\frac{\partial\vec{\boldsymbol{R}}}{\partial t}\cdot\vec{\boldsymbol{\nabla}}_R\Phi=-\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\nabla}}_R\Phi\,. \tag{120}$$

Finally, since $\vec{\boldsymbol{A}}=\vec{\boldsymbol{\beta}}\Phi$ [cf. eq. (117)],

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}=\vec{\boldsymbol{\nabla}}_R\cdot\big(\vec{\boldsymbol{\beta}}\Phi\big)=\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\nabla}}_R\Phi+\Phi\vec{\boldsymbol{\nabla}}_R\cdot\vec{\boldsymbol{\beta}}=\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\nabla}}_R\Phi\,, \tag{121}$$

since $\vec{\boldsymbol{\beta}}$ is fixed (and thus independent of $\vec{\boldsymbol{R}}$). Adding eqs. (120) and (121) yields

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0\,, \tag{122}$$

and the Lorenz condition is established (without any approximations).

(c) Show that to first order in β, the electric field $\vec{\boldsymbol{E}}$ in K is just the electric dipole field (centered at $\vec{\boldsymbol{x}}_0$), or a dipole field plus time-dependent higher multipoles, if viewed from a fixed origin, and the magnetic field is $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}$. Where is the effective magnetic dipole moment of Jackson, Problem 6.21 or Problem 11.27 part (a)?

In reference frame K', the electric and magnetic fields due to an electric dipole $\vec{\boldsymbol{p}}$ is given by:

$$\vec{\boldsymbol{E}}'(\vec{\boldsymbol{x}}') = -\vec{\boldsymbol{\nabla}}'\Phi' - \frac{1}{c}\frac{\partial \vec{\boldsymbol{A}}'}{dt'} = -\vec{\boldsymbol{\nabla}}'\left(\frac{\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}'}{r'^3}\right) = \frac{3\vec{\boldsymbol{x}}'(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}')}{r'^5} - \frac{\vec{\boldsymbol{p}}}{r'^3}\,, \tag{123}$$

$$\vec{\boldsymbol{B}}'(\vec{\boldsymbol{x}}') = \vec{\boldsymbol{\nabla}}' \times \vec{\boldsymbol{A}}' = 0\,, \tag{124}$$

under the assumption that $r' \equiv |\vec{\boldsymbol{x}}'| \neq 0$.[13]

The electric and magnetic fields in frame K' are related to the corresponding fields in frame K by eq. (11.149) of Jackson. To obtain the electric and magnetic fields in frame K in terms of the corresponding fields in frame K', one simply changes $\vec{\boldsymbol{\beta}} \to -\vec{\boldsymbol{\beta}}$. That is,

$$\vec{\boldsymbol{E}} = \gamma(\vec{\boldsymbol{E}}' - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}') - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}')\,, \tag{125}$$

$$\vec{\boldsymbol{B}} = \gamma(\vec{\boldsymbol{B}}' + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}') - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}')\,. \tag{126}$$

Since $\vec{\boldsymbol{B}}' = 0$, these equations simplify to

$$\vec{\boldsymbol{E}} = \gamma\vec{\boldsymbol{E}}' - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}')\,, \tag{127}$$

$$\vec{\boldsymbol{B}} = \gamma\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'\,. \tag{128}$$

Noting that $\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}} = \gamma\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'$, it follows that

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}\,. \tag{129}$$

Neglecting terms of $\mathcal{O}(\beta^2)$, it follows that $\gamma = 1 + \mathcal{O}(\beta^2)$ and

$$\frac{\gamma^2\beta^2}{\gamma+1} = \gamma - 1 = \mathcal{O}(\beta^2)\,. \tag{130}$$

In this approximation, eq. (127) reduces to

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}' + \mathcal{O}(\beta^2) = \frac{3\vec{\boldsymbol{x}}'(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}')}{r'^5} - \frac{\vec{\boldsymbol{p}}}{r'^3} + \mathcal{O}(\beta^2)\,, \tag{131}$$

after making use of eq. (123). Finally, we note that eqs. (113) and (115) yield, $\vec{\boldsymbol{x}}' = \vec{\boldsymbol{R}} + \mathcal{O}(\beta^2)$ and $r' = R + \mathcal{O}(\beta^2)$. Inserting these results into eq. (131), we end up with

$$\vec{\boldsymbol{E}} = \frac{3\vec{\boldsymbol{R}}(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})}{R^5} - \frac{\vec{\boldsymbol{p}}}{R^3} + \mathcal{O}(\beta^2)\,, \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}\,. \tag{132}$$

[13]By assuming that $r' \neq 0$, we explicitly exclude the delta function singularity at $\vec{\boldsymbol{x}}' = 0$ in the expression for $\vec{\boldsymbol{E}}'(\vec{\boldsymbol{x}}')$, which is exhibited in eq. (4.20) of Jackson.

Recall that $\vec{\boldsymbol{R}} \equiv \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_0$, where $\vec{\boldsymbol{x}}_0 = \vec{\boldsymbol{v}}t$. Hence, to first order in β, the electric field $\vec{\boldsymbol{E}}$ in K is just the electric dipole field centered at $\vec{\boldsymbol{x}}_0$. If the electric dipole is viewed from a fixed origin, then we must express the electric field as a function of $\vec{\boldsymbol{x}}$. Then, if we denote $r \equiv |\vec{\boldsymbol{x}}|$ and $r_0 \equiv |\vec{\boldsymbol{x}}_0| = c\beta t$, then

$$R = |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_0| = \left[r^2 + r_0^2 - 2r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}_0\right]^{1/2} = r\left[1 + \frac{r_0^2 - 2r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}_0}{r^2}\right]^{1/2} = r - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}} + \mathcal{O}\left(\frac{1}{r}\right) . \quad (133)$$

where we have introduced the unit vector $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$, and

$$\hat{\boldsymbol{R}} \equiv \frac{\vec{\boldsymbol{R}}}{R} = \hat{\boldsymbol{n}} - \frac{\vec{\boldsymbol{x}}_0}{r} - \frac{\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}_0)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) . \quad (134)$$

Then, eq. (132) yields

$$\vec{\boldsymbol{E}} = \frac{3\hat{\boldsymbol{n}}(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{n}}) - \vec{\boldsymbol{p}}}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) , \quad (135)$$

The $\mathcal{O}(1/r^4)$ terms can only arise from higher multipole moments. For example, carrying out the next term in the expansion, one finds

$$\vec{\boldsymbol{E}} = \frac{3\hat{\boldsymbol{n}}(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{n}}) - \vec{\boldsymbol{p}}}{r^3} + \frac{15(\vec{\boldsymbol{x}}_0\cdot\hat{\boldsymbol{n}})(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} - 3\left[(\vec{\boldsymbol{x}}_0\cdot\vec{\boldsymbol{p}})\hat{\boldsymbol{n}} + (\vec{\boldsymbol{x}}_0\cdot\hat{\boldsymbol{n}})\vec{\boldsymbol{p}} + (\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{n}})\vec{\boldsymbol{x}}_0\right]}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) , \quad (136)$$

where the $\mathcal{O}(1/r^4)$ term is recognized as an electric quadrupole field [cf. Jackson problem 6.21 part (c)]. Thus, if the electric dipole is viewed from a fixed origin, it would correspond to an electric dipole moment plus time-dependent higher multipole terms.

A moving electric dipole moment acquires a magnetic dipole moment, but this effect is quite subtle. It turns out that the $\vec{\boldsymbol{B}}$ field obtained in eq. (132) has two separate sources. One source can be attributed to a magnetic dipole moment

$$\vec{\boldsymbol{m}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{\beta}} . \quad (137)$$

while the second source arises from the electric polarization current that is due to the polarization produced by a moving electric dipole. Details can be found in V. Hnizdo, *Magnetic dipole moment of a moving electric dipole*, Am. J. Phys. **80**, 645 (2012), with further discussion in V. Hnizdo and Kirk T. McDonald, in a set of notes entitled, *Fields and Moments of a Moving Electric Dipole*.[14] If $\vec{\boldsymbol{m}}$ is given by eq. (137), then the corresponding vector potential given by Jackson eq. (5.55) in gaussian units, in the frame K, is given by

$$\vec{\boldsymbol{A}} = \frac{\vec{\boldsymbol{m}} \times \vec{\boldsymbol{R}}}{R^3} = \frac{\vec{\boldsymbol{R}} \times (\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{p}})}{R^3} = \frac{(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}} - (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{p}}}{R^3} . \quad (138)$$

Comparing this result with eq. (118), we see that there is an extra term in eq. (138). Hnizdo and McDonald argue that the total vector potential of a moving electric dipole in frame K is actually

[14] V. Hnizdo and Kirk T. McDonald, *Fields and Moments of a Moving Electric Dipole*, is available as a download from Kirk T. McDonald's websitre: `http://kirkmcd.princeton.edu/examples/movingdipole.pdf`.

due to two contributions: the so-called electric-polarization current density and the magnetic-polarization current density. Hence, in the reference frame K, they write :

$$\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}_p + \vec{\boldsymbol{A}}_m\,, \tag{139}$$

where

$$\vec{\boldsymbol{A}}_p = \frac{(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{p}}}{R^3}\,, \qquad\qquad \vec{\boldsymbol{A}}_m = \frac{(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}} - (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{p}}}{R^3}\,. \tag{140}$$

Thus, $\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}_p + \vec{\boldsymbol{A}}_m$ coincides with the result obtained in eq. (118) to leading order in β, whereas the magnetic dipole moment is extracted from $\vec{\boldsymbol{A}}_m$. However, Hnizdo and McDonald note that one can decompose $\vec{\boldsymbol{A}}$ in a different way, $\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}_s + \vec{\boldsymbol{A}}_a$, where the corresponding symmetric and antisymmetric combinations are defined as

$$\vec{\boldsymbol{A}}_s = \frac{(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}} + (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{p}}}{2R^3}\,, \qquad\qquad \vec{\boldsymbol{A}}_a = \frac{(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}} - (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{p}}}{2R^3} = \tfrac{1}{2}\vec{\boldsymbol{A}}_m\,. \tag{141}$$

This is the choice that Jackson has made implicitly in problems 6.21 and 11.27(a). Because of the extra factor of 1/2, Jackson concludes that $\vec{\boldsymbol{m}} = \frac{1}{2}\vec{\boldsymbol{p}}\times\vec{\boldsymbol{\beta}}$, in contrast to eq. (137). Hnizdo and McDonald thus argue that the identification of the magnetic dipole moment $\vec{\boldsymbol{m}}$ of a moving electric dipole moment is a convention and depends on the choice made in the decomposition of $\vec{\boldsymbol{A}}$. Whether one decomposition is preferred over another is still a matter of debate in the literature.

<u>BONUS MATERIAL</u>

One can perform the computation of part (c) without any approximations by following the method outlined in the class handout entitled, *The electromagnetic fields of a uniformly moving charge*. First, we note that the corresponding electromagnetic fields in the rest frame K' of the charge are given (in gaussian units) by:

$$\vec{\boldsymbol{E}}'(\vec{\boldsymbol{x}}') = \frac{3\vec{\boldsymbol{x}}'(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}')}{r'^5} - \frac{\vec{\boldsymbol{p}}}{r'^3}\,, \qquad\qquad \vec{\boldsymbol{B}}'(\vec{\boldsymbol{x}}') = 0\,. \tag{142}$$

after setting $\beta = 0$ in eq. (131). We now resolve the vectors above into components parallel and perpendicular to the velocity vector. In particular,

$$\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}}'_\parallel + \vec{\boldsymbol{x}}'_\perp\,,$$

and

$$r'^2 = \vec{\boldsymbol{x}}'\cdot\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}}'_\parallel\cdot\vec{\boldsymbol{x}}'_\parallel + \vec{\boldsymbol{x}}'_\perp\cdot\vec{\boldsymbol{x}}'_\perp\,, \tag{143}$$

since $\vec{\boldsymbol{x}}_\parallel\cdot\vec{\boldsymbol{x}}_\perp = 0$. Likewise, we can identify the longitudinal and transverse electric fields in frame K',

$$\vec{\boldsymbol{E}}'_\parallel = \frac{3\vec{\boldsymbol{x}}'_\parallel\big[\vec{\boldsymbol{p}}\cdot\big(\vec{\boldsymbol{x}}'_\parallel + \vec{\boldsymbol{x}}'_\perp\big)\big]}{r'^5} - \frac{\vec{\boldsymbol{p}}_\parallel}{r'^3}\,, \tag{144}$$

$$, \qquad \vec{\boldsymbol{E}}'_\perp = \frac{3\vec{\boldsymbol{x}}'_\perp\big[\vec{\boldsymbol{p}}\cdot\big(\vec{\boldsymbol{x}}'_\parallel + \vec{\boldsymbol{x}}'_\perp\big)\big]}{r'^5} - \frac{\vec{\boldsymbol{p}}_\perp}{r'^3}\,. \tag{145}$$

We shall make use of the transformation law for the electric and magnetic fields under a Lorentz boost:

$$\vec{\boldsymbol{E}}'_{\parallel} = \vec{\boldsymbol{E}}_{\parallel}\,, \qquad\qquad \vec{\boldsymbol{E}}'_{\perp} = \gamma(\vec{\boldsymbol{E}}_{\perp} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}_{\perp})\,, \tag{146}$$

$$\vec{\boldsymbol{B}}'_{\parallel} = \vec{\boldsymbol{B}}_{\parallel}\,, \qquad\qquad \vec{\boldsymbol{B}}'_{\perp} = \gamma(\vec{\boldsymbol{B}}_{\perp} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}_{\perp})\,. \tag{147}$$

In analyzing the uniformly moving electric dipole, $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ are known, so we have to invert eqs. (109) and (110) to obtain the electromagnetic fields in frame K. This is easily done by interchanging the primed and unprimed fields while reversing the sign of $\vec{\boldsymbol{\beta}}$. That is,

$$\vec{\boldsymbol{E}}_{\perp} = \gamma(\vec{\boldsymbol{E}}'_{\perp} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}'_{\perp})\,, \qquad\qquad \vec{\boldsymbol{B}}_{\perp} = \gamma(\vec{\boldsymbol{B}}'_{\perp} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'_{\perp})\,. \tag{148}$$

We are now ready to evaluate the electromagnetic fields in frame K. First, we employ eqs. (146)–(148) to obtain

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_{\parallel} + \vec{\boldsymbol{E}}_{\perp} = \vec{\boldsymbol{E}}'_{\parallel} + \gamma\vec{\boldsymbol{E}}'_{\perp}\,, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_{\parallel} + \vec{\boldsymbol{B}}_{\perp} = \gamma\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'_{\perp}\,. \tag{149}$$

Using eqs. (129) and (144), it follows that

$$\vec{\boldsymbol{E}} = \frac{3\big(\vec{\boldsymbol{x}}'_{\parallel} + \gamma\vec{\boldsymbol{x}}'_{\perp}\big)\big[\vec{\boldsymbol{p}}\cdot\big(\vec{\boldsymbol{x}}'_{\parallel} + \vec{\boldsymbol{x}}'_{\perp}\big)\big]}{r'^{\,5}} - \frac{\vec{\boldsymbol{p}}_{\parallel} + \gamma\vec{\boldsymbol{p}}_{\perp}}{r'^{\,3}}\,, \tag{150}$$

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}\,. \tag{151}$$

Next, we need to convert the primed coordinate into the unprimed coordinates. In light of eq. (113),

$$\vec{\boldsymbol{x}}'_{\parallel} = \gamma\vec{\boldsymbol{R}}_{\parallel}\,, \qquad\qquad \vec{\boldsymbol{x}}'_{\perp} = \vec{\boldsymbol{R}}_{\perp}\,, \tag{152}$$

after noting that $\vec{\boldsymbol{R}}_{\parallel} = (\boldsymbol{\beta}\cdot\vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}}/\beta^2$. Hence,

$$\vec{\boldsymbol{x}}'_{\parallel} + \gamma\vec{\boldsymbol{x}}'_{\perp} = \gamma\vec{\boldsymbol{R}}\,, \qquad\qquad r'^2 = \gamma^2 R_{\parallel}^2 + R_{\perp}^2 = R^2 + \gamma^2(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})^2\,, \tag{153}$$

in light of eq. (115). Furthermore, since $\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{p}}_{\parallel} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{R}}_{\parallel} = 0$, it follows that $\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{p}}_{\perp} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{R}}_{\perp} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{R}}$.

Finally, we shall make use of

$$\vec{\boldsymbol{p}}_{\parallel} + \gamma\vec{\boldsymbol{p}}_{\perp} = \vec{\boldsymbol{p}}_{\parallel} + \gamma(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}_{\parallel}) = \frac{(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{p}})\vec{\boldsymbol{\beta}}}{\beta^2} + \gamma\left(\vec{\boldsymbol{p}} - \frac{(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{p}})\vec{\boldsymbol{\beta}}}{\beta^2}\right) = \gamma\vec{\boldsymbol{p}} - \frac{\gamma-1}{\beta^2}\big(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{p}}\big)\vec{\boldsymbol{\beta}}\,, \tag{154}$$

$$\gamma\vec{\boldsymbol{R}}_{\parallel} + \vec{\boldsymbol{R}}_{\perp} = \vec{\boldsymbol{R}} + (\gamma-1)\vec{\boldsymbol{R}}_{\parallel} = \vec{\boldsymbol{R}} + \frac{\gamma-1}{\beta^2}\big(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}}\big)\vec{\boldsymbol{\beta}}\,. \tag{155}$$

Hence, we end up with:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \frac{3\gamma\vec{\boldsymbol{R}}\left[\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}} + (\gamma-1)\big(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}}\big)\big(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\beta}}\big)/\beta^2\right]}{\left[R^2 + \gamma^2(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})^2\right]^{5/2}} - \frac{\gamma\vec{\boldsymbol{p}} - (\gamma-1)\vec{\boldsymbol{\beta}}\big(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{p}}\big)/\beta^2}{\left[R^2 + \gamma^2(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})^2\right]^{3/2}}\,, \tag{156}$$

Likewise, we use eq. (151) to obtain:

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)=\frac{3\gamma\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{R}}\left[\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}+(\gamma-1)\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}\big)\big(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}\big)/\beta^2\right]}{\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{5/2}}-\frac{\gamma\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{p}}}{\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{3/2}}\,. \tag{157}$$

The time-dependence of the electric and magnetic fields in frame K arise through $\vec{\boldsymbol{R}}\equiv\vec{\boldsymbol{x}}-c\vec{\boldsymbol{\beta}}t$. One can verify that if terms of $\mathcal{O}(\beta^2)$ are neglected, then the above equations reduce to those exhibited in eq. (132).

As a further check of eq. (156), we present below an independent computation of the electric field starting from scalar and vector potential given in eqs. (116) and (117). The electric field is then obtained by evaluating

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=-\vec{\boldsymbol{\nabla}}\Phi-\frac{1}{c}\frac{\partial\vec{\boldsymbol{A}}}{\partial t}=-\vec{\boldsymbol{\nabla}}\Phi-\frac{\vec{\boldsymbol{\beta}}}{c}\frac{\partial\Phi}{\partial t}\,. \tag{158}$$

Recall that $\vec{\boldsymbol{\nabla}}=\vec{\boldsymbol{\nabla}}_R$, where $\vec{\boldsymbol{\nabla}}=(\partial/\partial x^1\,,\,\partial/\partial x^2\,,\,\partial/\partial x^3)$, and $\vec{\boldsymbol{\nabla}}_R=\big(\partial/\partial R^1\,,\,\partial/\partial R^2\,,\,\partial/\partial R^3\big)$, where $R^i=x^i-c\beta t$ ($i=1,2,3$) are the components of the vector $\vec{\boldsymbol{R}}$. Next, the time-dependence of $\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)$ is due to the time dependence of $\vec{\boldsymbol{R}}=\vec{\boldsymbol{x}}-c\beta t$. Therefore, using the chain rule [or employing the result of eq. (120)],

$$\frac{\vec{\boldsymbol{\beta}}}{c}\frac{\partial\Phi}{\partial t}=\frac{\vec{\boldsymbol{\beta}}}{c}\frac{\partial\vec{\boldsymbol{R}}}{\partial t}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}_R\Phi=-\vec{\boldsymbol{\beta}}\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}_R\Phi\big)\,. \tag{159}$$

Hence, eq. (158) yields

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=-\vec{\boldsymbol{\nabla}}_R\Phi+\vec{\boldsymbol{\beta}}\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}_R\Phi\big)\,. \tag{160}$$

In the evaluation of $\vec{\boldsymbol{\nabla}}_R\Phi$, note that

$$\vec{\boldsymbol{\nabla}}_R(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})=\vec{\boldsymbol{p}}\,,\qquad\qquad \vec{\boldsymbol{\nabla}}_R(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})=\vec{\boldsymbol{\beta}}\,, \tag{161}$$

and

$$\vec{\boldsymbol{\nabla}}_R\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{-3/2}=-\tfrac{3}{2}\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{-5/2}\,2\big[\vec{\boldsymbol{R}}+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}}\big]\,. \tag{162}$$

Therefore,

$$\begin{aligned}\vec{\boldsymbol{\nabla}}_R\Phi&=\gamma\left\{\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}+\frac{\gamma-1}{\beta^2}\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}\big)\big(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}\big)\right\}\vec{\boldsymbol{\nabla}}_R\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{-3/2}+\frac{\gamma\left(\vec{\boldsymbol{p}}+\dfrac{\gamma-1}{\beta^2}(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})\vec{\boldsymbol{\beta}}\right)}{\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{3/2}}\\&=-\frac{3\gamma\big[\vec{\boldsymbol{R}}+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}}\big]}{\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{5/2}}\left[\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}+\frac{\gamma-1}{\beta^2}\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}\big)\big(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}\big)\right]+\frac{\gamma\left(\vec{\boldsymbol{p}}+\dfrac{\gamma-1}{\beta^2}(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})\vec{\boldsymbol{\beta}}\right)}{\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{3/2}}\end{aligned} \tag{163}$$

and

$$\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}_R\Phi)=-\frac{3\gamma^3\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}\big)\vec{\boldsymbol{\beta}}}{\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{5/2}}\left[\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}+\frac{\gamma-1}{\beta^2}\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}}\big)\big(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}\big)\right]+\frac{\gamma^2\big(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}\big)\vec{\boldsymbol{\beta}}}{\big[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\big]^{3/2}}\,, \tag{164}$$

after using $1+\gamma^2\beta^2=\gamma^2$ in the first numerator on the right hand side of eq. (164) above. Inserting the results of eqs. (163) and (164) back into eq. (160), we end up with:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=\frac{3\gamma\vec{\boldsymbol{R}}}{\left[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})^2\right]^{5/2}}\left[\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{R}}+\frac{\gamma-1}{\beta^2}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\beta}})\right]-\frac{\gamma\vec{\boldsymbol{p}}-\dfrac{\gamma-1}{\beta^2}(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\beta}})\vec{\boldsymbol{\beta}}}{\left[R^2+\gamma^2(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})^2\right]^{3/2}}\,, \tag{165}$$

after making use of the identity $\gamma^2/(\gamma+1)=(\gamma-1)/\beta^2$. Thus, we have confirmed the result previously obtained in eq. (156).

APPENDIX: An alternative solution to Jackson, problem 11.15

In frame K, we have

$$\vec{\boldsymbol{E}}=E_0\,\hat{\boldsymbol{x}}\,,\qquad \vec{\boldsymbol{B}}=B_x\,\hat{\boldsymbol{x}}+B_y\,\hat{\boldsymbol{y}}\,, \tag{166}$$

with

$$\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{B}}=|\vec{\boldsymbol{E}}|\,|\vec{\boldsymbol{B}}|\cos\theta=E_0B_0\cos\theta=2E_0^2\cos\theta\,, \tag{167}$$

after writing $|\vec{\boldsymbol{E}}|=E_0$ and $|\vec{\boldsymbol{B}}|=B_0=2E_0$. It follows that

$$B_x=2E_0\cos\theta\,,\qquad B_y=2E_0\sin\theta\,. \tag{168}$$

The electric and magnetic fields are parallel in a reference frame K' which is moving at a velocity $\vec{\boldsymbol{v}}\equiv c\vec{\boldsymbol{\beta}}$ with respect to reference frame K. That is, the fields in K' satisfy,

$$\vec{\boldsymbol{E}}\,'\times\vec{\boldsymbol{B}}\,'=0\,. \tag{169}$$

The electric and magnetic fields in frame K' are related to the corresponding fields in frame K by eq. (11.149) of Jackson,

$$\vec{\boldsymbol{E}}\,'=\gamma(\vec{\boldsymbol{E}}+\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})-\frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}})\,, \tag{170}$$

$$\vec{\boldsymbol{B}}\,'=\gamma(\vec{\boldsymbol{B}}-\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}})-\frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{B}})\,. \tag{171}$$

Plugging the results for the electric and magnetic fields in reference frame K' given by eqs. (170) and (171) into eq. (169), one can work out the following expressions. First,

$$(\vec{\boldsymbol{E}}+\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})\times(\vec{\boldsymbol{B}}-\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}})=\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}-\vec{\boldsymbol{\beta}}\left[E^2+B^2-\vec{\boldsymbol{\beta}}\cdot(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}})\right]+\vec{\boldsymbol{E}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}})+\vec{\boldsymbol{B}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{B}})\,, \tag{172}$$

where $E\equiv|\vec{\boldsymbol{E}}|$ and $B\equiv|\vec{\boldsymbol{B}}|$. Second,

$$(\vec{\boldsymbol{E}}+\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})\times\vec{\boldsymbol{\beta}}=-\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}-\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{B}})+\beta^2\vec{\boldsymbol{B}}\,, \tag{173}$$

$$\vec{\boldsymbol{\beta}}\times(\vec{\boldsymbol{B}}-\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}})=\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}-\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}})+\beta^2\vec{\boldsymbol{E}}\,. \tag{174}$$

Hence, we obtain,

$$\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = \gamma^2 \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \biggl\{ \gamma^2 [E^2 + B^2 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}})] - \frac{\gamma^3}{\gamma+1} \bigl[(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{E}})^2 + (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{B}})^2 \bigr] \biggr\}$$
$$+\gamma^2 \bigl[\vec{\boldsymbol{E}} (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}) + \vec{\boldsymbol{B}} (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{B}}) \bigr] \left[1 - \frac{\gamma \beta^2}{\gamma+1} \right] - \frac{\gamma^3}{\gamma+1} \biggl\{ (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}) \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}} - (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{B}}) \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}} \biggr\} . \tag{175}$$

We can simplify the above expression by using $\gamma^2 = (1-\beta^2)^{-1}$, which yields $\beta^2 = (\gamma^2 - 1)/\gamma^2$. Hence,

$$1 - \frac{\gamma\beta^2}{\gamma+1} = 1 - \frac{\gamma-1}{\gamma} = \frac{1}{\gamma} . \tag{176}$$

We then end up with,

$$\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = \gamma^2 \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} + \gamma \bigl[\vec{\boldsymbol{E}} (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}) + \vec{\boldsymbol{B}} (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{B}}) \bigr] - \gamma^2 h \vec{\boldsymbol{\beta}} - \frac{\gamma^3}{\gamma+1} \biggl\{ (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}) \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}} - (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{B}}) \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}} \biggr\} , \tag{177}$$

where

$$h \equiv E^2 + B^2 - k \bigl[E^2 B^2 - (\vec{\boldsymbol{E}} \boldsymbol{\cdot} \vec{\boldsymbol{B}})^2 \bigr] - \frac{\gamma}{\gamma+1} \biggl\{ \bigl[k_1 E^2 + k_2 (\vec{\boldsymbol{E}} \boldsymbol{\cdot} \vec{\boldsymbol{B}}) \bigr]^2 + \bigl[k_1 (\vec{\boldsymbol{E}} \boldsymbol{\cdot} \vec{\boldsymbol{B}}) + k_2 B^2 \bigr]^2 \biggr\} . \tag{178}$$

The only way to satisfy $\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = 0$ is if the right hand side of eq. (177) is proportional to $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$.[15]

One way of ensuring that the right hand side of eq. (177) is proportional to $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ is to take $\vec{\boldsymbol{\beta}}$ to be parallel to $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$. That is, there exists a nonzero constant k such that

$$\vec{\boldsymbol{\beta}} = k \, \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} . \tag{179}$$

Note that eq. (179) implies that $\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{E}} = \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{B}} = 0$. Hence, eq. (177) simplifies to,

$$\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = \gamma^2 \vec{\boldsymbol{\beta}} \left[\frac{1+\beta^2}{k} - E^2 - B^2 \right] . \tag{180}$$

It then follows that

$$\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = 0 \quad \Longrightarrow \quad \frac{1+\beta^2}{k} = E^2 + B^2 . \tag{181}$$

Using eqs. (166) and (168), $E^2 + B^2 = 5E_0^2$ and

$$\vec{\boldsymbol{\beta}} = k \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} = \frac{2E_0^2 k \sin\theta}{\beta} \vec{\boldsymbol{\beta}} . \tag{182}$$

Thus, one can identify,

$$k = \frac{\beta}{2E_0^2 \sin\theta} . \tag{183}$$

[15] Recall that if $\{\vec{\boldsymbol{v}}_i\}$ is a set of linearly independent vectors, then the only solution to $\sum_i c_i \vec{\boldsymbol{v}}_i = 0$ is $c_i = 0$ for all i.

Plugging this result into eq. (181) yields,

$$2\sin\theta(1+\beta^2) = 5\beta\,, \tag{184}$$

which reproduces the result previously obtained in eq. (37).

As a check of our calculation, let us verify explicitly that $\vec{\boldsymbol{E}}\,'$ is parallel to $\vec{\boldsymbol{B}}\,'$. Inserting eq. (179) into eqs. (170) and (171) yields,

$$\vec{\boldsymbol{E}}\,' = \gamma\vec{\boldsymbol{E}}\left(1-kB^2\right) + \gamma k\vec{\boldsymbol{B}}(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\,, \tag{185}$$

$$\vec{\boldsymbol{B}}\,' = \gamma\vec{\boldsymbol{B}}\left(1-kE^2\right) + \gamma k\vec{\boldsymbol{E}}(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\,. \tag{186}$$

In light of eqs. (166)–(168) and eq. (183),

$$\vec{\boldsymbol{E}}\,' = \gamma E_0\big[(1-2\beta\sin\theta)\hat{\boldsymbol{x}} + 2\beta\cos\theta\,\hat{\boldsymbol{y}}\big]\,, \tag{187}$$

$$\vec{\boldsymbol{B}}\,' = \gamma E_0\big[2\cos\theta\,\hat{\boldsymbol{x}} + (2\sin\theta-\beta)\hat{\boldsymbol{y}}\big]\,. \tag{188}$$

We can now check that

$$\vec{\boldsymbol{E}}\,'\times\vec{\boldsymbol{B}}\,' = [2\sin\theta(1+\beta^2)-5\beta]\hat{\boldsymbol{z}} = 0\,, \tag{189}$$

after employing eq. (184), which completes the check of the calculation.

The two limiting cases are now easily analyzed. In the case of $\theta \ll 1$, we can work to first order in θ. As noted below eq. (38), $\beta \simeq \frac{2}{5}\theta$ and $\gamma = (1-\beta^2)^{-1/2} \simeq 1 + \mathcal{O}(\beta^2)$. Since we are working to first order in θ, we also must work to first order in β. In particular we can neglect terms such as $\beta\theta$. Hence, in this limiting case, eqs. (187) and (188) yield

$$\vec{\boldsymbol{E}}\,' = \tfrac{1}{2}\vec{\boldsymbol{B}}\,' = E_0(\hat{\boldsymbol{x}} + 2\beta\hat{\boldsymbol{y}})\,, \qquad \text{for } \beta \simeq \tfrac{2}{5}\theta \ll 1\,, \tag{190}$$

where we have neglected terms that are second order (or higher) in β. Finally, in the limit of $\theta \to \frac{1}{2}\pi$, eq. (38) yields $\beta = \frac{1}{2}$. Then $\gamma = 2/\sqrt{3}$, and eqs. (187) and (188) yield

$$\vec{\boldsymbol{E}}\,' = 0\,, \qquad \vec{\boldsymbol{B}}\,' = \sqrt{3}E_0\hat{\boldsymbol{y}}\,, \qquad \text{for } \theta = \tfrac{1}{2}\pi\,. \tag{191}$$

Thus, we have reproduced the results of eqs. (39) and (40).

<u>REMARK</u>:

Another strategy to find all possible boosts that result in parallel electric and magnetic fields is to start with eqs. (170) and (171) and impose the condition $\vec{\boldsymbol{E}}\,'\times\vec{\boldsymbol{B}}\,' = 0$ to determine the most general form for the boost. We again denote the boost parameter by $\vec{\boldsymbol{\beta}}$.

Since $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}$ are three linearly independent vectors, $\vec{\boldsymbol{\beta}}$ can be written in the following form,

$$\vec{\boldsymbol{\beta}} = k_1\vec{\boldsymbol{E}} + k_2\vec{\boldsymbol{B}} + k\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}\,, \tag{192}$$

where the constants k_1, k_2 and k are to be determined. It then follows that,

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = E^2k_1 + (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})k_2\,, \qquad \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})k_1 + B^2k_2\,,$$

$$\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}} = -k_2\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} - k\big[(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{E}} - E^2\vec{\boldsymbol{B}}\big]\,, \qquad \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} = k_1\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} + k\big[(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{B}} - B^2\vec{\boldsymbol{E}}\big]\,.$$

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}) = k|\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}|^2 = k\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\,,$$

and $\beta \equiv |\vec{\boldsymbol{\beta}}|$, where

$$\beta = k_1^2 E^2 + 2k_1 k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}) + k_2^2 B^2 + k^2\big[E^2 B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\,. \tag{193}$$

This last equation is needed to obtain an expression for $\gamma \equiv (1-\beta^2)^{-1/2}$. Plugging the above results into eq. (177) yields,

$$\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = c_1\vec{\boldsymbol{E}} + c_2\vec{\boldsymbol{B}} + c_3\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}\,, \tag{194}$$

in reference frame K'', where

$$c_1 = -\gamma^2 k_1 h + \gamma\big[k_1 E^2 + k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\big] + \frac{\gamma^3 k k_1}{\gamma+1}\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\,, \tag{195}$$

$$c_2 = -\gamma^2 k_2 h + \gamma\big[k_1(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}) + k_2 B^2\big] + \frac{\gamma^3 k k_2}{\gamma+1}\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\,, \tag{196}$$

$$c_3 = \gamma^2(1-kh) - \frac{\gamma^3}{\gamma+1}\big[k_1^2E^2 + 2k_1k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}) + k_2^2B^2\big]\,. \tag{197}$$

and h is given by eq. (178), which we rewrite below for the reader's convenience,

$$h \equiv E^2+B^2-k\big[E^2B^2-(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]-\frac{\gamma}{\gamma+1}\left\{\big[k_1E^2+k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\big]^2+\big[k_1(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})+k_2B^2\big]^2\right\}. \tag{198}$$

To find solutions $\{k_1, k_2, k\}$ to the equation $\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = 0$, we set $c_1 = c_2 = c_3 = 0$. This yields three nonlinear equations for the three unknowns, k_1, k_2 and k. The one solution obtained previously with $\vec{\boldsymbol{\beta}} = \beta_0\hat{\boldsymbol{z}}$ corresponds to $k_1 = k_2 = 0$ and $kh = 1$, where k is given by eq. (183). Here, we write β_0 to distinguish this special case from the general case under consideration. In this special case, $c_1 = c_2 = 0$ automatically and $c_3 = 0$ yields $kh = 1$ which implies that

$$k(E^2+B^2) - k^2\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big] = 1\,. \tag{199}$$

Using eqs. (27)–(29), $E^2 + B^2 = 5E_0^2$ and $E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2 = 4E_0^4\sin^2\theta$. Hence,

$$4E_0^4k^2\sin^2\theta - 5E_0^2k + 1 = 0\,. \tag{200}$$

Using eq. (183) to eliminate k (replacing β with β_0 as noted above), eq. (179) is equivalent to

$$\beta_0^2 - \frac{5\beta_0}{2\sin\theta} + 1 = 0\,, \tag{201}$$

which yields eq. (184) for the special case of $\vec{\boldsymbol{\beta}} = \beta_0\hat{\boldsymbol{z}}$, as expected.

More generally, one can verify that eq. (50) provides a family of solutions to eqs. (195)–(197) with $c_1 = c_2 = c_3 = 0$. In light of eqs. (46) and (50), we can identify,

$$k_1 = \frac{\beta'(\sin\theta - 2\beta_0)}{\gamma_0 E_0\sin\theta\sqrt{1-4\beta_0\sin\theta+4\beta_0^2}}\,, \tag{202}$$

$$k_2 = \frac{\beta'\beta_0\cos\theta}{\gamma_0 E_0\sin\theta\sqrt{1-4\beta_0\sin\theta+4\beta_0^2}}\,, \tag{203}$$

$$k = \frac{\beta_0}{2E_0^2\sin\theta}\,, \tag{204}$$

where β_0 is given by eq. (201), $\gamma_0 \equiv (1-\beta_0^2)^{-1/2}$ and β' is an arbitrary number such that $0 \leq \beta' \leq 1$. I have checked using Mathematica that after plugging in eqs. (202)–(204) into eqs. (195)–(198) along with $E^2 = E_0^2$, $B^2 = 4E_0^2$, and $\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{B}} = 2E_0^2 \cos\theta$, the end result is,

$$c_1 = -\frac{2\gamma C E_0 \big[2(\beta_0^2+1)\sin\theta - 5\beta_0\big]}{(1+\gamma)\sin\theta}\left[\gamma + \frac{1-\frac{1}{2}\beta_0\sin\theta - C^2\cos^2\theta}{1-\beta_0^2 - C^2(1-4\beta_0\sin\theta + 4\beta_0^2)}\right], \tag{205}$$

$$c_2 = \frac{\gamma C E_0 \cos\theta \big[2(\beta_0^2+1)\sin\theta - 5\beta_0\big]}{(1+\gamma)\sin\theta}\left[\gamma + \frac{1-C^2(1-2\beta_0\sin\theta)}{1-\beta_0^2 - C^2(1-4\beta_0\sin\theta + 4\beta_0^2}\right], \tag{206}$$

$$c_3 = \frac{\gamma^2 \big[2(\beta_0^2+1)\sin\theta - 5\beta_0\big]\big[1+\gamma-\gamma C^2(1-2\beta_0\sin\theta)\big]}{2(1+\gamma)\sin\theta}, \tag{207}$$

where

$$C \equiv \frac{\beta'}{\gamma_0\sqrt{1-4\beta_0\sin\theta + 4\beta_0^2}}. \tag{208}$$

Indeed, if β_0 satisfies eq. (201) then we find that $c_1 = c_2 = c_3 = 0$. Thus, I have verified that a boost to the frame with boost parameter given by eq. (50) yields $\vec{\boldsymbol{E}}\,' \times \vec{\boldsymbol{B}}\,' = 0$. I believe that $\{k_1, k_2, k\}$ given by eqs. (202)–(204) provides all possible solutions, but I do not have a proof of this statement.