1. [Jackson, problem 9.2] A radiating quadrupole consists of a square of side a with charges $\pm q$ at alternate corners. The square rotates with angular velocity ω about an axis normal to the plane of the square and through its center. Calculate the quadrupole moments, the radiation fields, the angular distribution of radiation, and the total radiated power, all in the long-wavelength approximation. What is the frequency of the radiation?

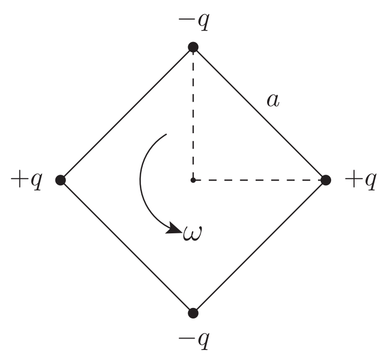

Figure 1: A radiating quadrupole consisting of a square of side a with charges $\pm q$ at alternate corners.

The charge distribution consists of point charges at the four corners of a square in the x–y plane, as depicted in Figure 1. The charges are located at the following positions in Cartesian coordinates,

$$+q\ :\quad \frac{a}{\sqrt{2}}\,(\cos\omega t\,,\,\sin\omega t\,,\,0)\ ,\qquad\qquad +q\ :\quad -\frac{a}{\sqrt{2}}\,(\cos\omega t\,,\,\sin\omega t\,,\,0)\ ,\tag{1}$$

$$-q\ :\quad \frac{a}{\sqrt{2}}\,(\sin\omega t\,,\,-\cos\omega t\,,\,0)\ ,\qquad\qquad -q\ :\quad \frac{a}{\sqrt{2}}\,(-\sin\omega t\,,\,\cos\omega t\,,\,0)\ .\tag{2}$$

The quadrupole moment Cartesian tensor is given by[1]

$$Q_{ij}=\sum_k q_k\left[3x_i^{(k)}x_j^{(k)}-(r^{(k)})^2\delta_{ij}\right]\,,\tag{3}$$

where k labels each charge and i and j label the components of the position vector $\vec{\boldsymbol{x}}$. Note that the charges all lie in the x–y plane (corresponding to $z=0$). Moreover, $r^{(k)}$ is the distance of the kth charge from the origin, located at the center of the square. Hence,

$$(r^{(k)})^2=(x_1^{(k)})^2+(x_2^{(k)})^2+(x_3^{(k)})^2=\tfrac{1}{2}a^2\,,\qquad \text{for all } k\,.$$

This means that

$$\sum_k q_k(r^{(k)})^2=\tfrac{1}{2}a^2\sum_k q_k=0\,,$$

[1] If one uses eq. (4.9) of Jackson, then one should express the charge distribution $\rho(\vec{\boldsymbol{x}},t)$ as a sum of delta functions, whose arguments vanish at the locations of the four charges. Integrating over all space then yields eq. (3).

since there are an equal number of positive and negative charges. Plugging in the location of the four charges in eq. (3), we obtain:

$$Q_{13} = Q_{23} = Q_{33} = 0\,, \qquad Q_{11} = \tfrac{3}{2}\cdot 2\,a^2 q\left[\cos^2\omega t - \sin^2\omega t\right] = 3a^2 q\cos 2\omega t\,,$$
$$Q_{22} = -3a^2 q\cos 2\omega t\,, \qquad Q_{12} = \tfrac{3}{2}\cdot 4\,a^2 q\sin\omega t\cos\omega t = 3a^2 q\sin 2\omega t\,,$$

after employing some well known trigonometric identities. Thus, the electric quadrupole tensor is given by

$$Q_{ij}(t) = 3a^2 q\begin{pmatrix}\cos 2\omega t & \sin 2\omega t & 0\\ \sin 2\omega t & -\cos 2\omega t & 0\\ 0 & 0 & 0\end{pmatrix}\,. \tag{4}$$

In a Cartesian basis, all the elements of the physical multipole tensors are real. We can introduce the complex time-dependent multipole tensor $\widetilde{Q}_{ij}(t)$ by defining

$$\widetilde{Q}_{ij}(t) = 3a^2 q\, e^{-2i\omega t}\begin{pmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\,. \tag{5}$$

One can check that the physical quadrupole Cartesian tensor is given by

$$Q_{ij}(t) = \mathrm{Re}\left[\widetilde{Q}_{ij}(t)\right]\,. \tag{6}$$

To make contact with the convention for harmonic sources employed by eq. (9.1) of Jackson, we note that the complex electric quadrupole tensor is defined to be

$$\widetilde{Q}_{ij}(t) = \int (3x_i x_j - r^2\delta_{ij})\widetilde{\rho}(\vec{\boldsymbol{x}},t)\,d^3x\,,$$

where $\widetilde{\rho}(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})\,e^{-i\omega t}$ and the physical charge density is given by $\rho(\vec{\boldsymbol{x}},t) = \mathrm{Re}\big[\rho(\vec{\boldsymbol{x}})\,e^{-i\omega t}\big]$. However, this would not yield the correct time dependence exhibited in eq. (5). However, the solution is simple—we write:

$$\widetilde{\rho}(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})\,e^{-2i\omega t}\,.$$

That is, one must replace ω with 2ω in the formulae that appear in Chapter 9 of Jackson. Note that this also implies that

$$k = \frac{2\omega}{c}\,. \tag{7}$$

Thus, it follows that

$$\widetilde{Q}_{ij}(t) = Q_{ij}\,e^{-2i\omega t}\,, \tag{8}$$

where

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\vec{\boldsymbol{x}})\,d^3x = 3a^2 q\begin{pmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\,. \tag{9}$$

Using eq. (9), the matrix $\vec{\boldsymbol{Q}}$ whose components are defined by

$$Q_i = \sum_{j=1}^{3} Q_{ij}\hat{\boldsymbol{n}}_j\,, \tag{10}$$

are easily evaluated. Using $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi\,,\ \sin\theta\sin\phi\,,\ \cos\theta)$, we find

$$Q_1 = 3a^2 q \sin\theta\, e^{i\phi}\,, \qquad Q_2 = 3a^2 qi \sin\theta\, e^{i\phi}\,, \qquad Q_3 = 0\,.$$

That is,

$$\vec{\boldsymbol{Q}} = 3a^2 q \sin\theta\, e^{i\phi}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})\,. \tag{11}$$

We can now employ eqs. (7) and (11) in eqs. (9.44), (9.45) and (9.49) of Jackson. First we evaluate

$$\hat{\boldsymbol{n}} \times \vec{\boldsymbol{Q}} = 3a^2 q \sin\theta\, e^{i\phi} \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 1 & i & 0 \end{pmatrix}$$

$$= -3a^2 qi \sin\theta\, e^{i\phi}\left[\cos\theta(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}) - \sin\theta\, e^{i\phi}\hat{\boldsymbol{z}}\right]\,.$$

Therefore, eq. (9.44) of Jackson [in SI units] yields

$$\vec{\boldsymbol{H}} = -\frac{ck^3}{8\pi}\,\frac{e^{ikr}}{r}\, a^2 q \sin\theta\, e^{i\phi}\left[\cos\theta(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}) - \sin\theta\, e^{i\phi}\hat{\boldsymbol{z}}\right]\,.$$

The physical magnetic fields are then given by

$$\begin{aligned}\mathrm{Re}(\vec{\boldsymbol{H}}\, e^{-2i\omega t}) = -\frac{ck^3 a^2 q}{8\pi r}\sin\theta\Big\{&\hat{\boldsymbol{x}}\cos\theta\cos(kr - 2\omega t + \phi) - \hat{\boldsymbol{y}}\cos\theta\sin(kr - 2\omega t + \phi) \\ &-\hat{\boldsymbol{z}}\sin\theta\cos(kr - 2\omega t + 2\phi)\Big\}\,.\end{aligned} \tag{12}$$

The electric fields are obtained by using eq. (9.39) of Jackson. In SI units,

$$\vec{\boldsymbol{E}} = Z_0 \vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}}\,, \tag{13}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Thus,

$$\vec{\boldsymbol{E}} = -\frac{Z_0 k^3}{8\pi}\,\frac{e^{ikr}}{r}\, a^2 q \sin\theta\, e^{i\phi} \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \cos\theta & i\cos\theta & -\sin\theta\, e^{i\phi} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

$$\begin{aligned}= -\frac{Z_0 k^3}{8\pi}\,\frac{e^{ikr}}{r}\, a^2 q \sin\theta\, e^{i\phi}\Big\{&\hat{\boldsymbol{x}}(\sin^2\theta\sin\phi\, e^{i\phi} + i\cos^2\theta) \\ &-\hat{\boldsymbol{y}}(\sin^2\theta\cos\phi\, e^{i\phi} + \cos^2\theta) - i\hat{\boldsymbol{z}}\sin\theta\cos\theta\, e^{i\phi}\Big\}\,.\end{aligned}$$

The physical electric fields are then given by

$$\begin{aligned}\mathrm{Re}(\vec{\boldsymbol{E}}\, e^{-2i\omega t}) = -\frac{Z_0 k^3 a^2 q}{8\pi r}\sin\theta\Big\{&\hat{\boldsymbol{x}}\left[\sin^2\theta\,\sin\phi\,\cos(kr - 2\omega t + 2\phi) - \cos^2\theta\,\sin(kr - 2\omega t + \phi)\right] \\ &-\hat{\boldsymbol{y}}\left[\sin^2\theta\,\cos\phi\,\cos(kr - 2\omega t + 2\phi) + \cos^2\theta\,\cos(kr - 2\omega t + \phi)\right] \\ &+\hat{\boldsymbol{z}}\,\sin\theta\,\cos\theta\,\sin(kr - 2\omega t + 2\phi)\Big\}\,.\end{aligned} \tag{14}$$

As a check, it is easy to verify that eq. (13) is also satisfied by the physical fields given in eqs. (12) and (14).

Next, we compute the time-averaged power radiated per unit solid angle. Using eqs. (9.45) and (9.46) of Jackson,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}\, k^6 \left[\vec{\boldsymbol{Q}}^{\,*}\boldsymbol{\cdot}\vec{\boldsymbol{Q}} - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{Q}}|^2\right] , \tag{15}$$

Using eq. (11), we compute:

$$\vec{\boldsymbol{Q}}^{\,*}\boldsymbol{\cdot}\vec{\boldsymbol{Q}} = 18a^4q^2\sin^2\theta\,, \qquad\qquad |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{Q}}|^2 = |3a^2q\sin^2\theta\, e^{2i\phi}|^2 = 9a^4q^2\sin^2\theta\,.$$

Hence,

$$\vec{\boldsymbol{Q}}^{\,*}\boldsymbol{\cdot}\vec{\boldsymbol{Q}} - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{Q}}|^2 = 9a^4q^2\sin^2\theta(2-\sin^2\theta) = 9a^4q^2\sin^2\theta(1+\cos^2\theta)\,.$$

Eq. (15) then yields

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 a^4 q^2 k^6}{128\pi^2}\sin^2\theta(1+\cos^2\theta)\,.$$

Using $k = 2\omega/c$ [cf. eq. (7)], we obtain

$$\frac{dP}{d\Omega} = \frac{Z_0 a^4 q^2 \omega^6}{2\pi^2 c^4}\sin^2\theta(1+\cos^2\theta)\,. \tag{16}$$

The total radiated power is obtained by integrating over solid angles. Using

$$\int d\Omega\sin^2\theta(1+\cos^2\theta) = 2\pi\int_{-1}^{1}(1-\cos^4\theta)\, d\cos\theta = \frac{16\pi}{5}\,,$$

we end up with

$$P = \frac{8Z_0 a^4 q^2\omega^6}{5\pi c^4}\,. \tag{17}$$

As a check, we can use eq. (9.49) of Jackson,

$$P = \frac{c^2 Z_0 k^6}{1440\pi}\sum_{i,j}|Q_{ij}|^2\,. \tag{18}$$

Using eq. (9)

$$\sum_{i,j}|Q_{ij}|^2 = 36a^4q^2\,.$$

Inserting this back into eq. (18) along with $k = 2\omega/c$, we recover eq. (17) as expected.

ALTERNATIVE SOLUTION:

In class, I showed that for general time dependent charges and current, the physical (real) magnetic and electric fields of E2 radiation are given by

$$\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}},t) = -\frac{1}{24\pi c^2 r}\,\hat{\boldsymbol{n}}\times\frac{\partial^3}{\partial t^3}\vec{\boldsymbol{Q}}\left(t-\frac{r}{c}\right) , \tag{19}$$

$$\vec{\boldsymbol{E}}_{\rm E2}(\vec{\boldsymbol{x}},t) = Z_0\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}},t)\times\hat{\boldsymbol{n}}\,, \tag{20}$$

after converting the formulae given in class from gaussian to SI units. The time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = r^2 \vec{\boldsymbol{S}} \boldsymbol{\cdot} \hat{\boldsymbol{n}}\,, \tag{21}$$

where the Poynting vector is given by eq. (6.109) of Jackson, $\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}$. Using eqs. (19)–(21),

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{Z_0}{576\pi^2 c^4} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \left\{ \left[\hat{\boldsymbol{n}} \times \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right)\right] \times \hat{\boldsymbol{n}} \right\} \times \left[\hat{\boldsymbol{n}} \times \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right)\right] \\
&= -\frac{Z_0}{576\pi^2 c^4} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \left\{ \hat{\boldsymbol{n}} \frac{\partial^3}{\partial t^3} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right) - \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right)\right\} \times \left[\hat{\boldsymbol{n}} \times \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right)\right] \\
&= \frac{Z_0}{576\pi^2 c^4} \left\{ \left| \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right)\right|^2 - \left[\frac{\partial^3}{\partial t^3} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right)\right]^2 \right\}.
\end{aligned} \tag{22}$$

In the above formula, the components of the real vector $\vec{\boldsymbol{Q}}(t)$ are given by eq. (10), where $Q_{ij}(t)$ is given by eq. (4). Using spherical coordinates, $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and

$$\begin{aligned}
Q_1(t) &= Q_{11} n_1 + Q_{12} n_2 + n_3 Q_{13} n_3 = 3a^2 q \sin\theta\big(\cos\phi\cos 2\omega t + \sin\phi \sin 2\omega t\big) \\
&= 3a^2 q \sin\theta \cos(2\omega t - \phi)\,, \\
Q_2(t) &= Q_{21} n_1 + Q_{22} n_2 + Q_{23} n_3 = 3a^2 q \sin\theta\big(\cos\phi \sin 2\omega t - \sin\phi\cos 2\omega t\big) \\
&= 3a^2 q \sin\theta \sin(2\omega t - \phi)\,, \\
Q_3(t) &= Q_{31} n_1 + Q_{32} n_2 + Q_{33} n_3 = 0\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}}(t) &= 3a^2 q \sin^2\theta\big[\cos\phi\cos(2\omega t - \phi) + \sin\phi\sin(2\omega t - \phi)\big] = 3a^2 q \sin^2\theta \cos\big(2(\omega t - \phi)\big)\,, \\
\frac{\partial^3}{\partial t^3} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}}(t) &= 24\omega^3 a^2 q \sin^2\theta \sin\big(2(\omega t - \phi)\big)\,, \\
\frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}}(t) &= 24\omega^3 a^2 q \sin\theta\big(\sin(2\omega t - \phi)\,,\, -\cos(2\omega t - \phi)\,,\, 0\big)\,.
\end{aligned}$$

It then follows that

$$\left| \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right)\right|^2 - \left[\frac{\partial^3}{\partial t^3} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}}\left(t - \frac{r}{c}\right)\right]^2 = 576\omega^6 a^4 q^2 \sin^2\theta \left\{ 1 - \sin^2\theta \cos^2\left(2\left[\omega\left(t - \frac{r}{c}\right) - \phi\right]\right)\right\}. \tag{23}$$

Averaging over one cycle, we obtain,

$$\left\langle \cos^2\left(2\left[\omega\left(t - \frac{r}{c}\right) - \phi\right]\right)\right\rangle = \frac{1}{2}\,. \tag{24}$$

Hence,

$$\frac{d\langle P\rangle}{d\Omega} = \frac{Z_0 \omega^6 a^4 q^2 \sin^2\theta\big(1 - \frac{1}{2}\sin^2\theta\big)}{\pi^2 c^4} = \frac{Z_0 \omega^6 a^4 q^2 \sin^2\theta\big(1 + \cos^2\theta\big)}{2\pi^2 c^4}\,, \tag{25}$$

which reproduces the result previously obtained in eq. (16). Note that in our first derivation above, by using the complex version of $Q_{ij}(t)$ given in eq. (5) along with the complex Poynting vector, one automatically obtains the time-averaged power using eq. (15).

2. [Jackson, problem 9.3]

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

The long wavelength limit is equivalent to the low frequency limit. Thus, one can treat this as an approximate electrostatics problem. Consider the scalar potential Φ of a spherical metallic shell of radius R and infinite conductivity whose upper and lower hemispheres have complex potentials $\pm Ve^{-i\omega t}$ [with the corresponding physical potential given by the real part]. Due to the azimuthal symmetry, the potential is independent of the azimuthal angle ϕ. Thus, in spherical coordinates, we can expand the potential in Legendre polynomials [cf. Section 3.3 of Jackson],

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta)\,, \quad \text{for } r \geq R. \tag{26}$$

The coefficients B_ℓ are obtained by setting $r = R$, multiplying eq. (26) by $P_m(\cos\theta)$, and integrating from $\cos\theta = -1$ to $\cos\theta = 1$. Using the orthogonality relation,

$$\int_{-1}^{1} P_\ell(\cos\theta)P_m(\cos\theta)\,d\cos\theta = \frac{2}{2\ell+1}\,\delta_{\ell m} \tag{27}$$

it follows that

$$B_\ell = \tfrac{1}{2}(2\ell+1)\int_{-1}^{1}\Phi(R,\theta)P_\ell(\cos\theta)\,d\cos\theta\,. \tag{28}$$

Inserting

$$\Phi(R,\theta) = \begin{cases} Ve^{-i\omega t}\,, & \text{for } \cos\theta > 0\,, \\ -Ve^{-i\omega t}\,, & \text{for } \cos\theta < 0\,, \end{cases} \tag{29}$$

we obtain,

$$\begin{aligned} B_\ell &= \tfrac{1}{2}(2\ell+1)Ve^{-i\omega t}\left\{\int_0^1 P_\ell(\cos\theta)\,d\cos\theta - \int_{-1}^{0} P_\ell(\cos\theta)\,d\cos\theta\right\} \\ &= \tfrac{1}{2}(2\ell+1)Ve^{-i\omega t}\int_0^1 \big[P_\ell(\cos\theta) - P_\ell(-\cos\theta)\big]\,d\cos\theta \\ &= \tfrac{1}{2}(2\ell+1)Ve^{-i\omega t}\big[1-(-1)^\ell\big]\int_0^1 P_\ell(\cos\theta)\,d\cos\theta\,, \end{aligned} \tag{30}$$

after making use of the relation $P_\ell(-\cos\theta) = (-1)^\ell P_\ell(-\cos\theta)$.

Thus, only odd vales of ℓ contribute to the sum in eq. (26). In the electric dipole approximation, we only need to retain the $\ell = 1$ term in the series. Hence,

$$B_1 = 3V^{-i\omega t}\int_0^1 \cos\theta\,d\cos\theta = \tfrac{3}{2}V^{-i\omega t}\,, \tag{31}$$

and

$$\Phi(r,\theta) = \frac{3R^2 V}{2r^2}\cos\theta\,, \quad \text{for } r \geq R. \tag{32}$$

Let us compare eq. (32) with the scalar potential of an electric dipole $\vec{\boldsymbol{p}}$ [cf. eq.(4.10) of Jackson],

$$\Phi(\vec{\boldsymbol{x}}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{|\vec{\boldsymbol{p}}|\cos\theta}{r^2}\,, \tag{33}$$

where θ is the angle between the vectors $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{x}}$ and $\cos\theta = \vec{\boldsymbol{x}}\cdot\hat{\boldsymbol{z}}$. Thus, we can identify

$$\vec{\boldsymbol{p}}(t) = 6\pi\epsilon_0 R^2 V e^{-i\omega t}\hat{\boldsymbol{z}} = 6\pi\epsilon_0 R^2 V e^{-i\omega t}(\hat{\boldsymbol{n}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\,. \tag{34}$$

Since $\vec{\boldsymbol{p}}(t) = \vec{\boldsymbol{p}}e^{-i\omega t}$, we can use the results of eqs. (9.19)–(9.24) of Jackson. The radiation fields are given by

$$\vec{\boldsymbol{H}} = \frac{ck^2}{4\pi}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\frac{e^{ikr}}{r}\,, \tag{35}$$

$$\vec{\boldsymbol{E}} = Z_0\vec{\boldsymbol{H}}\times\hat{\boldsymbol{n}}\,. \tag{36}$$

The angular distribution of the time-averaged radiated power is

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|\vec{\boldsymbol{p}}|^2\sin^2\theta\,. \tag{37}$$

Integrating over the solid angle yields,

$$P = \frac{c^2 Z_0 k^4}{12\pi}|\vec{\boldsymbol{p}}|^2\,. \tag{38}$$

Note that

$$\hat{\boldsymbol{n}}\times\hat{\boldsymbol{z}} = \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta) = -\hat{\boldsymbol{\phi}}\sin\theta\,, \tag{39}$$

$$(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{z}})\times\hat{\boldsymbol{n}} = -\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{n}}\sin\theta = -\hat{\boldsymbol{\theta}}\sin\theta\,. \tag{40}$$

Inserting $\vec{\boldsymbol{p}} = 6\pi\epsilon_0 R^2 V(\hat{\boldsymbol{n}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)$ in eqs. (35)–(38), and using $Z_0 = \sqrt{\mu_0/\epsilon_0}$ and $c = 1/\sqrt{\mu_0\epsilon_0}$ (which implies that $Z_0^{-1} = \epsilon_0 c$), we end up with

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}) = -\tfrac{3}{2}Z_0^{-1}k^2R^2V\frac{e^{ikr}}{r}\sin\theta\,\hat{\boldsymbol{\phi}}, \qquad \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = -\tfrac{3}{2}k^2R^2V\frac{e^{ikr}}{r}\sin\theta\,\hat{\boldsymbol{\theta}}\,. \tag{41}$$

The corresponding real physical fields are given by $\mathrm{Re}\,\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})e^{-i\omega t}$ and $\mathrm{Re}\,\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t}$,

$$\mathrm{Re}\big[\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})e^{-i\omega t}\big] = -\tfrac{3}{2}Z_0^{-1}k^2R^2V\frac{\sin\theta\cos(kr-\omega t)}{r}\,\hat{\boldsymbol{\phi}}, \tag{42}$$

$$\mathrm{Re}\big[\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t}\big] = -\tfrac{3}{2}k^2R^2V\frac{\sin\theta\cos(kr-\omega t)}{r}\,\hat{\boldsymbol{\theta}}\,. \tag{43}$$

Finally, the angular distribution of the time-averaged radiated power and the total integrated power are given by

$$\frac{dP}{d\Omega} = \tfrac{9}{8}Z_0^{-1}(kR)^4V^2\sin^2\theta\,, \qquad P = 3\pi Z_0^{-1}(kR)^4V^2\,. \tag{44}$$

3. [Jackson, problem 9.6]

(a) Starting from the general expression given by Jackson eq. (9.2) for $\vec{\boldsymbol{A}}$ and the corresponding expression for Φ, expand both $R = \left|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'\right|$ and $t' = t - R/c$ to first order in $\left|\vec{\boldsymbol{x}}'\right|/r$ to obtain the electric dipole potentials for arbitrary time variation

$$\Phi(\vec{\boldsymbol{x}}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}_{\rm ret} + \frac{1}{cr}\hat{\boldsymbol{n}}\boldsymbol{\cdot}\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right], \tag{45}$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}, t) = \frac{\mu_0}{4\pi r}\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}, \tag{46}$$

where $\vec{\boldsymbol{p}}_{\rm ret} = \vec{\boldsymbol{p}}(t' = t - r/c)$ is the dipole moment evaluated at the retarded time measured from the origin.

We begin with Jackson eq. (9.2),

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}, t) = \frac{\mu_0}{4\pi}\int d^3x'\int dt'\,\frac{\vec{\boldsymbol{J}}\left(\vec{\boldsymbol{x}}', t'\right)}{\left|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'\right|}\delta\left(t' + \frac{\left|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'\right|}{c} - t\right). \tag{47}$$

Integrating over t' yields

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}, t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec{\boldsymbol{J}}\left(\vec{\boldsymbol{x}}', t - \left|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'\right|/c\right)}{\left|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'\right|}. \tag{48}$$

Using eq. (9.7) of Jackson,

$$R \equiv |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'| = r - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}' + \mathcal{O}\left(\frac{1}{r}\right), \tag{49}$$

where $r \equiv |\vec{\boldsymbol{x}}|$ and $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$. Therefore, it follows that,

$$t - \frac{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|}{c} = t - \frac{r}{c} + \frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}'}{c} + \mathcal{O}\left(\frac{1}{r}\right). \tag{50}$$

Hence, the leading order contribution in the limit of large r is given by

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}, t) = \frac{\mu_0}{4\pi r}\int d^3x'\,\vec{\boldsymbol{J}}\left(\vec{\boldsymbol{x}}', t - r/c\right). \tag{51}$$

In the same spirit of Jackson eq. (9.14), we can employ an integration by parts by writing

$$J_i = \partial'_k(J_k x'_i) - x'_i(\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}), \tag{52}$$

where there is an implicit sum over the repeated index k. Hence,

$$\int d^3x'\,\vec{\boldsymbol{J}} = -\int d^3x'\,\vec{\boldsymbol{x}}'(\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}), \tag{53}$$

under the assumption that the current is localized (so that the surface term at infinity vanishes). We can now use the continuity equation,

$$\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}',t') + \frac{\partial\rho(\vec{\boldsymbol{x}}',t')}{\partial t'} = 0\,. \tag{54}$$

Hence, eqs. (53) and (54) yield,

$$\int d^3x'\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}',t-r/c) = \frac{\partial}{\partial t}\int d^3x'\,\vec{\boldsymbol{x}}'\rho(\vec{\boldsymbol{x}}',t-r/c) = \frac{\partial\vec{\boldsymbol{p}}(t-r/c)}{\partial t}\,, \tag{55}$$

after using the definition of the electric dipole vector $\vec{\boldsymbol{p}}$ [Jackson, eq. (9.17)], generalized to the case of a time-dependent charge density. Note that since the second argument of $\rho(\vec{\boldsymbol{x}}',t')$ is $t'\simeq t-r/c$, we were able to replace $\partial/\partial t'$ with $\partial/\partial t$ (via the chain rule), which can then be pulled outside the integral in eq. (55).

Plugging eq. (55) back into eq. (51), we confirm eq. (46),

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t) = \frac{\mu_0}{4\pi r}\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\,, \tag{56}$$

where $\vec{\boldsymbol{p}}_{\rm ret}\equiv\vec{\boldsymbol{p}}\,(t'=t-r/c)$.

Next, we consider the expression for Φ given on p. 410 of Jackson,

$$\Phi(\vec{\boldsymbol{x}},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\int dt'\,\frac{\rho\left(\vec{\boldsymbol{x}}',t'\right)}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}\,\delta\left(t'+\frac{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}{c}-t\right)\,. \tag{57}$$

Integrating over t' yields

$$\Phi(\vec{\boldsymbol{x}},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho\left(\vec{\boldsymbol{x}}',t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|/c\right)}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}\,. \tag{58}$$

We again make use of eqs. (49) and (50). However, in contrast with the calculation of $\vec{\boldsymbol{A}}$ above, here we shall keep the first subleading term in the large r expansion. That is, we shall employ:

$$\frac{1}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|} \simeq \frac{1}{r}+\frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}'}{r^2}\,, \tag{59}$$

$$\rho\left(\vec{\boldsymbol{x}}',t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|/c\right) \simeq \rho\left(\vec{\boldsymbol{x}}',t-r/c\right)+\frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}'}{c}\frac{\partial\rho\left(\vec{\boldsymbol{x}}',t-r/c\right)}{\partial t}\,. \tag{60}$$

Inserting the above results into eq. (58), we obtain

$$\begin{aligned}\Phi(\vec{\boldsymbol{x}},t) &= \frac{1}{4\pi r\epsilon_0}\int d^3x'\left[\rho\left(\vec{\boldsymbol{x}}',t-r/c\right)+\frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}'}{c}\frac{\partial\rho\left(\vec{\boldsymbol{x}}',t-r/c\right)}{\partial t}+\frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}'}{r}\rho\left(\vec{\boldsymbol{x}}',t-r/c\right)\right]\\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r}+\frac{1}{cr}\hat{\boldsymbol{n}}\boldsymbol{\cdot}\frac{\partial\vec{\boldsymbol{p}}(t-r/c)}{\partial t}+\frac{1}{r^2}\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}(t-r/c)\right]\end{aligned} \tag{61}$$

where $Q \equiv \int d^3x' \rho\left(\vec{\boldsymbol{x}}', t - r/c\right)$ is the total charge of the sources, which must be time-independent due to the conservation of charge. Thus, the first term in eq. (61) is static and does not contribute to the radiation (see p. 410 of Jackson). The remaining two terms in eq. (61) correspond to the electric dipole potentials,

$$\Phi(\vec{\boldsymbol{x}}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}_{\rm ret} + \frac{1}{cr}\hat{\boldsymbol{n}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right], \tag{62}$$

which confirms the results given in eq. (45).

(b) Calculate the dipole electric and magnetic fields directly from these potentials and show that

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}, t) = \frac{\mu_0}{4\pi}\left[-\frac{1}{r^2}\hat{\boldsymbol{n}}\times\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t} - \frac{1}{cr}\hat{\boldsymbol{n}}\times\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right]$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \frac{1}{4\pi\epsilon_0}\left\{\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}_{\rm ret}\right) - \vec{\boldsymbol{p}}_{\rm ret}}{r^3}\right] + \frac{1}{c^2 r}\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)\right\}$$

We can now use eqs. (45) and (46) to compute the electric and magnetic fields. In light of Jackson eqs. (6.7) and (6.9),

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\Phi - \frac{\partial\vec{\boldsymbol{A}}}{\partial t}, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}. \tag{63}$$

To obtain an expression for $\vec{\boldsymbol{E}}$, we must evaluate:

$$\vec{\boldsymbol{\nabla}}\left(\frac{1}{r^3}\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{p}}_{\rm ret} + \frac{1}{cr^2}\vec{\boldsymbol{x}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right), \tag{64}$$

where we have put $\hat{\boldsymbol{n}} = \vec{\boldsymbol{x}}/r$ and $r \equiv |\vec{\boldsymbol{x}}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. We shall make use of the chain rules discussed in class.[2] In particular, if $w = f(x, t')$ and t' is a function of t, then

$$\left(\frac{\partial w}{\partial x}\right)_t = \left(\frac{\partial w}{\partial x}\right)_{t'} + \left(\frac{\partial w}{\partial t'}\right)_x\left(\frac{\partial t'}{\partial x}\right)_t, \tag{65}$$

$$\left(\frac{\partial w}{\partial t}\right)_x = \left(\frac{\partial w}{\partial t'}\right)_x\left(\frac{\partial t'}{\partial t}\right)_x, \tag{66}$$

where the subscripts to the right of the right parentheses indicate the variable that is held fixed when performing the partial differentiation. In the applications below, $w = \vec{\boldsymbol{p}}_{\rm ret}(t')$ where $t' \equiv t - r/c$. It then follows that

$$\left(\frac{\partial w}{\partial x}\right)_{t'} = 0, \qquad\qquad \left(\frac{\partial t'}{\partial t}\right)_x = 1. \tag{67}$$

[2]See Section 2 of the class handout entitled: *The equations of Jefimenko, Panofsky, and Phillips.*

In evaluating eq. (64), we first note that

$$\vec{\boldsymbol{\nabla}}\left(\frac{1}{r^n}\right)=\vec{\boldsymbol{\nabla}}\left(\frac{1}{(x_1^2+x_2^2+x_3^2)^{n/2}}\right)=-\frac{n\vec{\boldsymbol{x}}}{(x_1^2+x_2^2+x_3^2)^{(n+2)/2}}=-\frac{n\vec{\boldsymbol{x}}}{r^{n+2}}\,. \tag{68}$$

Second, eqs. (65) and (66) yield:

$$\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{p}}_{\rm ret})=\vec{\boldsymbol{p}}_{\rm ret}+\left(\vec{\boldsymbol{x}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t'}\right)\vec{\boldsymbol{\nabla}}t'\,,\qquad\qquad \frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}=\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t'}\frac{\partial t'}{\partial t}=\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t'}\,, \tag{69}$$

in light of eq. (67). Note further that

$$\vec{\boldsymbol{\nabla}}t'=-\frac{1}{c}\vec{\boldsymbol{\nabla}}r=-\frac{\vec{\boldsymbol{x}}}{cr}=-\frac{\hat{\boldsymbol{n}}}{c}\,, \tag{70}$$

after putting $n=-1$ in eq. (68). Hence, it follows that

$$\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{p}}_{\rm ret})=\vec{\boldsymbol{p}}_{\rm ret}-\frac{r}{c}\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right)\,. \tag{71}$$

Using the results obtained above,

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\left(\frac{1}{r^3}\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{p}}_{\rm ret}+\frac{1}{cr^2}\vec{\boldsymbol{x}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right)&=\frac{\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{p}}_{\rm ret})-3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}_{\rm ret})}{r^3}+\frac{\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{x}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right)-2\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right)}{cr^2}\\
&=\frac{\vec{\boldsymbol{p}}_{\rm ret}-3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}_{\rm ret})}{r^3}-\frac{1}{cr^2}\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right)\\
&\quad+\frac{\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}-2\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right)}{cr^2}-\frac{1}{c^2r}\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)\\
&=\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left(\frac{\vec{\boldsymbol{p}}_{\rm ret}-3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}_{\rm ret})}{r^3}\right)-\frac{1}{c^2r}\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)\,.\end{aligned} \tag{72}$$

Thus, we have obtained

$$-\vec{\boldsymbol{\nabla}}\Phi=\frac{1}{4\pi\epsilon_0}\left\{\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\hat{\boldsymbol{n}}\,(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}_{\rm ret})-\vec{\boldsymbol{p}}_{\rm ret}}{r^3}\right]+\frac{1}{c^2r}\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)\right\}\,. \tag{73}$$

Next, we use eq. (46) to obtain

$$-\frac{\partial\vec{\boldsymbol{A}}}{\partial t}=-\frac{\mu_0}{4\pi r}\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}=-\frac{1}{4\pi\epsilon_0c^2r}\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\,, \tag{74}$$

after making use of $\mu_0\epsilon_0=1/c^2$. Adding eqs. (73) and (74) and noting that

$$\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)=\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)-\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\,, \tag{75}$$

we end up with

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = -\vec{\boldsymbol{\nabla}}\Phi - \frac{\partial\vec{\boldsymbol{A}}}{\partial t} = \frac{1}{4\pi\epsilon_0}\left\{\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\hat{\boldsymbol{n}}\,(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}_{\rm ret}) - \vec{\boldsymbol{p}}_{\rm ret}}{r^3}\right] + \frac{1}{c^2 r}\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)\right\}, \tag{76}$$

which confirms the result for the electric field given by Jackson.

Next, we compute $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}$ using eq. (46). Using a vector identity that can be found on the inside cover of Jackson,

$$\vec{\boldsymbol{\nabla}}\times\left(\frac{1}{r}\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right) = \frac{1}{r}\vec{\boldsymbol{\nabla}}\times\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t} + \vec{\boldsymbol{\nabla}}\left(\frac{1}{r}\right)\times\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}. \tag{77}$$

In light of eq. (68),

$$\vec{\boldsymbol{\nabla}}\left(\frac{1}{r}\right) = -\frac{\vec{\boldsymbol{x}}}{r^3} = -\frac{\hat{\boldsymbol{n}}}{r^2}\,. \tag{78}$$

Finally, we make use of eqs. (65) and (70) to obtain

$$\left(\vec{\boldsymbol{\nabla}}\times\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right)_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t}\right)_k = \epsilon_{ijk}\left(\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)_k\frac{\partial t'}{\partial x_j} = -\epsilon_{ijk}\left(\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right)_k\frac{\hat{\boldsymbol{n}}_j}{c}\,, \tag{79}$$

where there are implicit sums over the repeated index pairs j and k, respectively. That is,

$$\vec{\boldsymbol{\nabla}}\times\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t} = -\frac{1}{c}\,\hat{\boldsymbol{n}}\times\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\,. \tag{80}$$

Hence, it follows that

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = -\frac{\mu_0}{4\pi}\left[\frac{1}{r^2}\,\hat{\boldsymbol{n}}\times\frac{\partial\vec{\boldsymbol{p}}_{\rm ret}}{\partial t} + \frac{1}{cr}\hat{\boldsymbol{n}}\times\frac{\partial^2\vec{\boldsymbol{p}}_{\rm ret}}{\partial t^2}\right], \tag{81}$$

which confirms the result for the magnetic field given by Jackson.

(c) Show explicitly how you can go back and forth between these results and the harmonic fields of Jackson eq. (9.18) by the substitutions $-i\omega \leftrightarrow \partial/\partial t$ and $\vec{\boldsymbol{p}}\,e^{ikr-i\omega t} \leftrightarrow \vec{\boldsymbol{p}}_{\rm ret}\,(t')$.

Assuming that the sources and fields are harmonic, one can write

$$\vec{\boldsymbol{p}}_{\rm ret}(t') = \vec{\boldsymbol{p}}\,e^{-i\omega t'} = \vec{\boldsymbol{p}}\,e^{-i(kr-\omega t)}\,, \tag{82}$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t}\,, \tag{83}$$

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})e^{-i\omega t}\,, \tag{84}$$

where we have used $t' \equiv t - r/c$ in obtaining eq. (82). Inserting eqs. (82)–(84) into eqs. (76) and (81), and making use of $\vec{\boldsymbol{B}} = \mu_0\vec{\boldsymbol{H}}$ and $\omega = ck$,

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}) = \frac{ck^2}{4\pi}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\,, \tag{85}$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = \frac{1}{4\pi\epsilon_0}\left\{\left[3\hat{\boldsymbol{n}}\,(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}) - \vec{\boldsymbol{p}}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr} - k^2\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\frac{e^{ikr}}{r}\right\}\,, \tag{86}$$

which reproduces eq. (9.18) of Jackson after writing $\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}}) = -(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\times\hat{\boldsymbol{n}}$.

After multiplying both sides of eqs. (86) and (85) by $e^{-i\omega t}$ and writing $k = \omega/c$, one can then obtain eqs. (76) and (81) from eqs. (86) and (85) by the substitutions $\vec{\boldsymbol{p}}\,e^{ikr-i\omega t} \to \vec{\boldsymbol{p}}_{\rm ret}\,(t')$, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t} \to \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$, $\mu_0\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})e^{-i\omega t} \to \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)$, $-i\omega \to \partial/\partial t$, and $-\omega^2 \to \partial^2/\partial t^2$. Likewise, reversing the arrows permits one to obtain eqs. (85) and (86) from eqs. (81) and (76).

4. [Jackson, problem 9.7, part (a)]

Show for a real electric dipole $\vec{\boldsymbol{p}}(t)$ that the instantaneous radiated power per unit solid angle at a distance r from the dipole in a direction $\hat{\boldsymbol{n}}$ is

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^2}\left|\left[\hat{\boldsymbol{n}}\times\frac{d^2\vec{\boldsymbol{p}}}{dt'^2}(t')\right]\times\hat{\boldsymbol{n}}\right|^2\,, \tag{87}$$

where $t' \equiv t - r/c$ is the retarded time. For a magnetic dipole moment $\vec{\boldsymbol{m}}(t)$, substitute $(1/c)\ddot{\vec{\boldsymbol{m}}}\times\hat{\boldsymbol{n}}$ for $(\hat{\boldsymbol{n}}\times\ddot{\vec{\boldsymbol{p}}})\times\hat{\boldsymbol{n}}$.

In light of the second equation of eq. (69), $d\vec{\boldsymbol{p}}_{\rm ret}/dt = d\vec{\boldsymbol{p}}(t')/dt = d\vec{\boldsymbol{p}}(t')/dt'$. Thus, we shall simply denote $d^2\vec{\boldsymbol{p}}_{\rm ret}/dt^2 = \ddot{\vec{\boldsymbol{p}}}$ in what follows. It is sufficient to only retain terms of $\mathcal{O}(1/r)$ in eqs. (76) and (81). Hence,

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}},t) = -\frac{1}{4\pi cr}\hat{\boldsymbol{n}}\times\ddot{\vec{\boldsymbol{p}}}, \tag{88}$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \frac{1}{4\pi\epsilon_0 c^2 r}\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\ddot{\vec{\boldsymbol{p}}}\right), \tag{89}$$

after using $\vec{\boldsymbol{B}} = \mu_0\vec{\boldsymbol{H}}$. It follows that

$$\vec{\boldsymbol{E}} = Z_0\vec{\boldsymbol{H}}\times\hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{90}$$

where $Z_0 \equiv \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space, after noting that $\epsilon_0 c = 1/Z_0$ (which is a consequence of $\epsilon_0\mu_0 = 1/c^2$).

For real (time-dependent) fields, the equivalent of eq. (9.21) of Jackson is:[3]

$$\frac{dP(t)}{d\Omega} = \lim_{r\to\infty}\left(r^2\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}\right). \tag{91}$$

In particular, note that to leading order in $1/r$,

$$\hat{\boldsymbol{n}}\cdot\left[(\vec{\boldsymbol{H}}\times\hat{\boldsymbol{n}})\times\vec{\boldsymbol{H}}\right] = -\hat{\boldsymbol{n}}\cdot\left[\vec{\boldsymbol{H}}(\vec{\boldsymbol{H}}\cdot\hat{\boldsymbol{n}}) - \hat{\boldsymbol{n}}|\vec{\boldsymbol{H}}|^2\right] = |\vec{\boldsymbol{H}}|^2 - (\vec{\boldsymbol{H}}\cdot\hat{\boldsymbol{n}})^2 = |\vec{\boldsymbol{H}}\times\hat{\boldsymbol{n}}|^2\,.$$

Hence, eq. (91) yields

$$\frac{dP(t)}{d\Omega} = \lim_{r\to\infty} Z_0 r^2|\vec{\boldsymbol{H}}\times\hat{\boldsymbol{n}}|^2\,. \tag{92}$$

Note that only the $\mathcal{O}(1/r)$ terms of $\vec{\boldsymbol{H}}$ contribute in the $r\to\infty$ limit.

[3]Here, we use eq. (6.109) of Jackson, which gives $\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}$.

Finally, we make use of eq. (88) to obtain

$$\vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}} = -\frac{1}{4\pi cr}\big(\hat{\boldsymbol{n}} \times \ddot{\vec{\boldsymbol{p}}}\big) \times \hat{\boldsymbol{n}}\,. \tag{93}$$

Hence, eq. (92) yields

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^2}\big|\big(\hat{\boldsymbol{n}} \times \ddot{\vec{\boldsymbol{p}}}\big) \times \hat{\boldsymbol{n}}\big|^2\,, \tag{94}$$

in agreement with eq. (87).

In the case of magnetic dipole radiation, Jackson notes on the top of p. 414 that one can simply make use of the corresponding results for electric dipole radiation with the interchange of $\vec{\boldsymbol{E}} \to Z_0\vec{\boldsymbol{H}}$, $Z_0\vec{\boldsymbol{H}} \to -\vec{\boldsymbol{E}}$, and $\vec{\boldsymbol{p}} \to \vec{\boldsymbol{m}}/c$. In light of eqs. (88) and (89), the magnetic dipole radiation fields at $\mathcal{O}(1/r)$ are given by

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \frac{Z_0}{4\pi c^2 r}\hat{\boldsymbol{n}} \times \ddot{\vec{\boldsymbol{m}}}\,, \tag{95}$$

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \frac{1}{4\pi c^2 r}\hat{\boldsymbol{n}} \times \big(\hat{\boldsymbol{n}} \times \ddot{\vec{\boldsymbol{m}}}\big)\,. \tag{96}$$

One can check that eq. (90) is still valid. Therefore, eqs. (92) and (95) yield

$$\frac{dP(t)}{d\Omega} = \lim_{r\to\infty}\frac{r^2}{Z_0}|\vec{\boldsymbol{E}}|^2 = \frac{Z_0}{16\pi^2 c^4}|\hat{\boldsymbol{n}} \times \ddot{\vec{\boldsymbol{m}}}|^2\,. \tag{97}$$

This result confirms Jackson's claim that for a magnetic dipole moment $\vec{\boldsymbol{m}}(t)$, substitute $(1/c)\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{n}}$ for $(\hat{\boldsymbol{n}} \times \ddot{\vec{\boldsymbol{p}}}) \times \hat{\boldsymbol{n}}$ in eq. (87) to obtain the instantaneous radiated power per unit solid angle at a distance r from the magnetic dipole in a direction $\hat{\boldsymbol{n}}$

An alternative method for solving problem 9.7(a) via Fourier transforms

In the analysis of sections 2 and 3 in Chapter 9 of Jackson, the physical electric and magnetic fields are given by

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \mathrm{Re}\big[\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, \omega)e^{-i\omega t}\big]\,, \qquad \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \mathrm{Re}\big[\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, \omega)e^{-i\omega t}\big]\,.$$

The complex fields $\vec{\boldsymbol{E}} \equiv \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, \omega)$ and $\vec{\boldsymbol{H}} \equiv \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, \omega)$ are then employed to compute the differential power distribution via eq. (9.21) of Jackson.

To generalize to the case of arbitrary time dependence, we introduce the Fourier transforms,

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \int_{-\infty}^{\infty}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, \omega)e^{-i\omega t}\,d\omega\,, \qquad \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \int_{-\infty}^{\infty}\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, \omega)e^{-i\omega t}\,d\omega\,.$$

Since $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)$ and $\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t)$ are the physical fields, they must be real fields. This requirement imposes reality conditions on the Fourier coefficients,

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, -\omega) = \vec{\boldsymbol{E}}^{\,*}(\vec{\boldsymbol{x}}, \omega)\,, \qquad \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, -\omega) = \vec{\boldsymbol{H}}^{\,*}(\vec{\boldsymbol{x}}, \omega)\,.$$

Consider first the case of electric dipole radiation. The $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ fields given in eq. (9.19) of Jackson are in fact the Fourier coefficients. Since $k = \omega/c$, it follows that

$$\vec{\boldsymbol{H}}_{\rm E1}(\vec{\boldsymbol{x}},\omega) = \frac{\omega^2}{4\pi c}\big[\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}}(\omega)\big]\frac{e^{i\omega r/c}}{r}\,, \qquad\qquad \vec{\boldsymbol{E}}_{\rm E1}(\vec{\boldsymbol{x}},\omega) = Z_0\,\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}},\omega)\times\hat{\boldsymbol{n}}\,,$$

where the time-dependent electric dipole moment is obtained via the Fourier transform,

$$\vec{\boldsymbol{p}}(t) = \int_{-\infty}^{\infty}\vec{\boldsymbol{p}}(\omega)\,e^{-i\omega t}\,d\omega\,. \tag{98}$$

Hence,

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\rm E1}(\vec{\boldsymbol{x}},t) &= \frac{1}{4\pi cr}\,\hat{\boldsymbol{n}}\times\int_{-\infty}^{\infty}\omega^2\,\vec{\boldsymbol{p}}\,(\omega)e^{-i\omega(t-r/c)}\,d\omega \\
&= -\frac{1}{4\pi cr}\,\hat{\boldsymbol{n}}\times\frac{d^2}{dt^2}\int_{-\infty}^{\infty}\vec{\boldsymbol{p}}(\omega)\,e^{-i\omega(t-r/c)}\,d\omega \\
&= -\frac{1}{4\pi cr}\,\hat{\boldsymbol{n}}\times\frac{d^2\vec{\boldsymbol{p}}}{dt_0^2}(t')\,,
\end{aligned}$$

where $t'\equiv t-r/c$. Thus, we have established eq. (88). From this, we may obtain the power distribution given in eq. (94) as before.

5. [Jackson, problem 9.8]

(a) Show that a classical oscillating electric dipole $\vec{\boldsymbol{p}}$ with fields given by eq. (9.18) of Jackson radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \frac{k^3}{12\pi\epsilon_0}\,{\rm Im}\left[\vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}}\right]\,.$$

In class, we derived the following result for the radiated angular momentum per unit time in gaussian units (which was denoted by $\vec{\boldsymbol{\tau}}$):

$$\vec{\boldsymbol{\tau}} = -\frac{r^3}{8\pi}\,{\rm Re}\int\big[(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^*)(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) + (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{B}})(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}^*)\big]\,d\Omega\,, \tag{99}$$

where $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are the complex electric and magnetic field vectors (after removing the harmonic $e^{-i\omega t}$ factor). To rewrite this in SI units, we must replace $\vec{\boldsymbol{E}}\to\sqrt{4\pi\epsilon_0}\,\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}\to\sqrt{4\pi\mu_0}\,\vec{\boldsymbol{H}}$, where $c = 1/\sqrt{\epsilon_0\mu_0}$. Note that we must also replace $\rho\to\rho/\sqrt{4\pi\epsilon_0}$ and $\vec{\boldsymbol{J}}\to\vec{\boldsymbol{J}}/\sqrt{4\pi\epsilon_0}$, which means that $\vec{\boldsymbol{p}}\to\vec{\boldsymbol{p}}/\sqrt{4\pi\epsilon_0}$ [cf. Table 3 on p. 782 of Jackson]. Thus, in SI units, we have

$$\vec{\boldsymbol{\tau}} = -\tfrac{1}{2}r^3\,{\rm Re}\int\big[\epsilon_0(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^*)(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) + \mu_0(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{H}})(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}^*)\big]\,d\Omega\,, \tag{100}$$

We now make use of the electric dipole fields given by eq. (9.18) of Jackson,

$$\vec{\boldsymbol{H}} = \frac{ck^2}{4\pi}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\,\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\,,$$

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\times\hat{\boldsymbol{n}}\,\frac{e^{ikr}}{r}+[3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}})-\vec{\boldsymbol{p}}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}.$$

Note that $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}=0$ and

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\left\{2\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} = -\frac{ik}{2\pi\epsilon_0 r^2}\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}\,e^{ikr}+\mathcal{O}\left(\frac{1}{r^3}\right).$$

Thus, we only need to keep the $\mathcal{O}(1/r)$ terms in $\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^*$. Using the vector identity,

$$\hat{\boldsymbol{n}}\times\big\{(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\times\hat{\boldsymbol{n}}\big\} = \hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}},$$

for a unit vector $\hat{\boldsymbol{n}}$, it follows that,

$$\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^* = \frac{k^2}{4\pi\epsilon_0}\,\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}}^*\,\frac{e^{-ikr}}{r}+\mathcal{O}\left(\frac{1}{r}\right).$$

Hence, it follows that

$$\vec{\boldsymbol{\tau}} = \text{Re}\,\frac{ik^3}{16\pi^2\epsilon_0}\int d\Omega\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}\,(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}}^*)\,,$$

where we have dropped terms that vanish in the limit of $r\to\infty$. In component form,

$$\tau_i = \text{Re}\,\frac{ik^3}{16\pi^2\epsilon_0}\epsilon_{ijk}p_\ell p_k^*\int d\Omega\,n_j n_\ell\,, \qquad (101)$$

where there is an implicit sum over the repeated indices j, k, and ℓ. Using eq. (9.47) of Jackson,

$$\int d\Omega\,n_j n_\ell = \frac{4\pi}{3}\,\delta_{j\ell}\,.$$

Inserting this result into eq. (101) yields

$$\tau_i = \text{Re}\,\frac{ik^3}{12\pi\epsilon_0}(\vec{\boldsymbol{p}}\times\vec{\boldsymbol{p}}^*)_i\,.$$

Finally, noting that $\text{Re}(iz) = -\,\text{Im}\,z$ for any complex number z, and $\vec{\boldsymbol{p}}\times\vec{\boldsymbol{p}}^* = -\vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}}$, we end up with[4]

$$\vec{\boldsymbol{\tau}} = \frac{d\vec{\boldsymbol{L}}}{dt} = \frac{k^3}{12\pi\epsilon_0}\,\text{Im}(\vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}})\,. \qquad (102)$$

[4] In class, I wrote $\vec{\boldsymbol{\tau}} = -d\vec{\boldsymbol{L}}/dt$, where $-d\vec{\boldsymbol{L}}/dt$ denotes the rate of angular momentum *lost* by the radiating sources, which is equal to the rate of angular momentum transported to infinity. Jackson denotes this quantity $d\vec{\boldsymbol{L}}/dt$ without the explicit minus sign. Thus, I have adopted Jackson's convention in obtaining eq. (102).

(b) What is the ratio of angular momentum radiated to energy radiated? Interpret.

Eq. (9.24) of Jackson states that the total power radiated is given by

$$P = \frac{c^2 Z_0 k^4}{12\pi}\,|\vec{\boldsymbol{p}}|^2\,, \tag{103}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. In particular, note that $c\,\epsilon_0 Z_0 = 1$. Thus, using $\omega = kc$ it follows that

$$\frac{\vec{\boldsymbol{\tau}}}{P} = \frac{\mathrm{Im}(\vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}})}{\omega|\vec{\boldsymbol{p}}|^2}\,. \tag{104}$$

To interpret eq. (104), consider the case where the electric dipole moment possesses a definite value of m in the spherical basis. Recall that

$$q_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}(p_x \mp ip_y)\,, \qquad\qquad q_{10} = \sqrt{\frac{3}{4\pi}}\,p_z\,.$$

Consider three cases:

1. If $q_{11} \neq 0$ and $q_{10} = q_{1,-1} = 0$, then

$$\vec{\boldsymbol{p}} = \frac{p}{\sqrt{2}}(-1\,,\,-i\,,\,0) \quad\Longrightarrow\quad \vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}} = i|\vec{\boldsymbol{p}}|^2\hat{\boldsymbol{z}}\,;$$

2. If $q_{10} \neq 0$ and $q_{11} = q_{1,-1} = 0$, then

$$\vec{\boldsymbol{p}} = p(0\,,\,0\,,\,1) \qquad\Longrightarrow\qquad \vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}} = 0\,;$$

3. If $q_{1,-1} \neq 0$ and $q_{11} = q_{10} = 0$, then

$$\vec{\boldsymbol{p}} = \frac{p}{\sqrt{2}}(1\,,\,-i\,,\,0) \qquad\Longrightarrow\qquad \vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}} = -i|\vec{\boldsymbol{p}}|^2\hat{\boldsymbol{z}}\,,$$

where $p \equiv |\vec{\boldsymbol{p}}|$. That is,

$$\vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}} = im|\vec{\boldsymbol{p}}|^2\hat{\boldsymbol{z}}\,, \qquad \text{for } m = -1, 0, +1\,.$$

Inserting this result into eq. (104) yields

$$\frac{\tau_z}{P} = \frac{dL_z/dt}{dU/dt} = \frac{m}{\omega}\,.$$

In the quantum mechanics of electromagnetic radiation, photons possess an energy $U = \hbar\omega$ and a spin angular momentum $S_z = m\hbar$, so that $S_z/U = m/\omega$. The analogy is quite striking!

(c) For a charge e rotating in the x–y plane at radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $dL_z/dt = e^2k^3a^2/(6\pi\epsilon_0)$. What about a charge oscillating along the z axis?

For a charge e rotating in the x–y plane at radius a and angular speed ω, the components of the electric dipole vector are given by, $\vec{\boldsymbol{p}} = ea(\cos\omega t\,,\,\sin\omega t\,,\,0)$. This result can be rewritten as

$$\vec{\boldsymbol{p}} = \mathrm{Re}\Big\{ea\,e^{-i\omega t}\,(1\,,\,i\,,\,0)\Big\}\,.$$

Thus, we may define a *complex* electric dipole vector,

$$\vec{\boldsymbol{p}}(t) = \vec{\boldsymbol{p}}\,e^{-i\omega t}\,, \qquad \text{where } \vec{\boldsymbol{p}} = ea\,(1\,,\,i\,,\,0)\,.$$

It then follows that

$$\vec{\boldsymbol{p}}^{\,*}\times\vec{\boldsymbol{p}} = \det\begin{pmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ ea & -iea & 0\\ ea & iea & 0\end{pmatrix} = 2ie^2a^2\,\hat{\boldsymbol{z}}\,.$$

Hence, $\mathrm{Im}(\vec{\boldsymbol{p}}^{\,*}\times\vec{\boldsymbol{p}}) = 2e^2a^2\,\hat{\boldsymbol{z}}$. Inserting this result into eq. (102) yields

$$\tau_z = \frac{dL_z}{dt} = \frac{e^2k^3a^2}{6\pi\epsilon_0}\,.$$

For a charge oscillating along the z-axis, the *real* physical charge density is

$$\rho(\vec{\boldsymbol{x}},t) = q\delta(x)\delta(y)\delta(z - z_0\cos\omega t)\,.$$

Hence, $\vec{\boldsymbol{p}} = \hat{\boldsymbol{z}}\,qz_0\cos\omega t = \hat{\boldsymbol{z}}\,qz_0\,\mathrm{Re}\,e^{-i\omega t}$. Thus, we identify the corresponding complex electric dipole moment vector (with the harmonic factor stripped off) as

$$\vec{\boldsymbol{p}} = \hat{\boldsymbol{z}}\,qz_0\,.$$

Note that this is in fact a *real* vector, in which case $\vec{\boldsymbol{p}}^{\,*}\times\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}\times\vec{\boldsymbol{p}} = 0$. Hence, for this case, $\vec{\boldsymbol{\tau}} = 0$ and no angular momentum is radiated.

The above two cases correspond to $m = 1$ and $m = 0$, respectively, which were treated explicitly at the end of part (b).

(d) What are the results corresponding to parts (a) and (b) for magnetic dipole radiation?

For magnetic dipole radiation, we use eqs. (9.35) and (9.36) of Jackson,

$$\vec{\boldsymbol{E}} = -\frac{Z_0k^2}{4\pi}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{m}}\,)\,\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\,,$$

$$\vec{\boldsymbol{H}} = \frac{1}{4\pi}\left\{k^2(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{m}}\,)\times\hat{\boldsymbol{n}}\,\frac{e^{ikr}}{r} + [3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{m}}\,) - \vec{\boldsymbol{m}}\,]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}\,.$$

As noted by Jackson at the top of p. 414, one can obtain results for magnetic dipole radiation from that of electric dipole radiation by the following set of interchanges,

$$\vec{\boldsymbol{E}} \to Z_0\,\vec{\boldsymbol{H}}\,, \qquad Z_0\vec{\boldsymbol{H}} \to -\vec{\boldsymbol{E}}\,, \qquad \vec{\boldsymbol{p}} \to \vec{\boldsymbol{m}}/c\,.$$

Applying these interchanges on the results obtained in eqs. (102) and (103) yields

$$\vec{\boldsymbol{\tau}} = \frac{\mu_0 k^3}{12\pi}\,\mathrm{Im}(\vec{\boldsymbol{m}}^{\,*}\times\vec{\boldsymbol{m}})\,, \tag{105}$$

$$P = \frac{\mu_0 c k^4}{12\pi}\,|\vec{\boldsymbol{m}}|^2\,. \tag{106}$$

The corresponding ratio of these two quantities is:

$$\frac{\vec{\boldsymbol{\tau}}}{P} = \frac{\mathrm{Im}(\vec{\boldsymbol{m}}^{\,*}\times\vec{\boldsymbol{m}})}{\omega|\vec{\boldsymbol{m}}|^2}\,. \tag{107}$$

The analysis of part (b) is nearly identical, with the magnetic dipole moment in the spherical basis replacing the electric dipole moment. Thus, again, we conclude that

$$\vec{\boldsymbol{m}}^{\,*}\times\vec{\boldsymbol{m}} = im|\vec{\boldsymbol{m}}|^2\hat{\boldsymbol{z}}\,,\qquad \text{for } m=-1,0,+1\,.$$

which again yields

$$\frac{\tau_z}{P} = \frac{dL_z/dt}{dU/dt} = \frac{m}{\omega}\,.$$

6. [Jackson, problem 9.16] A thin linear antenna of length d is excited in such a way that a sinusoidal current makes a full wavelength of oscillation as shown in the figure below.

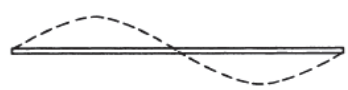

Figure 2: A thin linear antenna with a sinusoidal current that makes a full wavelength of oscillation.

(a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

Choose the z-axis to lie along the antenna, and let $z=0$ correspond to the center of the antenna. Then, $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\,e^{-i\omega t}$, where

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = I\sin\left(\frac{2\pi z}{d}\right)\delta(x)\,\delta(y)\,\hat{\boldsymbol{z}}\,,\qquad \text{for } |z|\leq \tfrac{1}{2}d\,, \tag{108}$$

where d is the length of the antenna. In class, we derived the following results for the complex magnetic and electric fields (assumed to be harmonic) in gaussian units,

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{i\omega}{c^2 r}\,e^{i(kr-\omega t)}\,\hat{\boldsymbol{n}}\times\int d^3x'\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\,e^{-ik\vec{\boldsymbol{x}}'\cdot\hat{\boldsymbol{n}}} + \mathcal{O}\left(\frac{1}{r^2}\right)\,,$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \boldsymbol{B}(\vec{\boldsymbol{x}},t)\times\hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right)\,,$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$ and $r \equiv |\vec{\boldsymbol{x}}|$. In SI units,[5] the above results take the following form,

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \frac{i\omega}{4\pi c\, r}\, e^{i(kr-\omega t)}\, \hat{\boldsymbol{n}} \times \int d^3x'\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\, e^{-ik\vec{\boldsymbol{x}}'\cdot\hat{\boldsymbol{n}}} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{109}$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = Z_0 \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) \times \hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{110}$$

Using eq. (9.21) of Jackson, the time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \tfrac{1}{2}\,\mathrm{Re}\left[r^2\, \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}^*\right].$$

In light of eq. (110), we compute

$$(\vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}}) \times \vec{\boldsymbol{H}}^* = -\vec{\boldsymbol{H}}^* \times (\vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}}(\vec{\boldsymbol{H}} \cdot \vec{\boldsymbol{H}}^*) - \vec{\boldsymbol{H}}(\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{H}}^*) = \hat{\boldsymbol{n}}|\vec{\boldsymbol{H}}|^2,$$

where at the last step we $\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{H}}^* = 0$, which is a consequence of eq. (109). Hence,

$$\frac{dP}{d\Omega} = \tfrac{1}{2} Z_0 r^2 |\vec{\boldsymbol{H}}|^2. \tag{111}$$

Thus, our task is to compute the integral,

$$\int d^3x'\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\, e^{-ik\vec{\boldsymbol{x}}'\cdot\hat{\boldsymbol{n}}}.$$

By assumption, the sinusoidal current makes a full wavelength, which implies that

$$k = \frac{2\pi}{d}. \tag{112}$$

Inserting eq. (108) into the integral above and employing rectangular coordinates,

$$\int d^3x'\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\, e^{-ik\vec{\boldsymbol{x}}'\cdot\hat{\boldsymbol{n}}} = \hat{\boldsymbol{z}}\, I \int_{-d/2}^{d/2} \sin kz\, e^{-ikz\cos\theta}\, dz,$$

where θ is the angle between $\hat{\boldsymbol{n}}$ and the positive z-axis (which corresponds to the usual polar angle of spherical coordinates). The following indefinite integral appears in many integration tables,

$$\int e^{az} \sin kz\, dz = \frac{e^{az}(a \sin kz - k\cos kz)}{a^2 + k^2}.$$

Using eq. (112), the limits of integration are $|z| \leq \pi/k$,

$$\begin{aligned}
\int_{-\pi/k}^{\pi/k} \sin kz\, e^{-ikz\cos\theta}\, dz &= \frac{e^{-ikz\cos\theta}(-ik\cos\theta \sin kz - k\cos kz)}{(-ik\cos\theta)^2 + k^2}\Bigg|_{-\pi/k}^{\pi/k} \\
&= \frac{e^{-i\pi\cos\theta} - e^{i\pi\cos\theta}}{k\sin^2\theta} \\
&= -\frac{2i\sin(\pi\cos\theta)}{k\sin^2\theta}.
\end{aligned}$$

[5]To convert from gaussian to SI units, we must replace the fields $\vec{\boldsymbol{E}} \to \sqrt{4\pi\epsilon_0}\,\vec{\boldsymbol{E}}$, and $\vec{\boldsymbol{B}} \to \sqrt{4\pi\mu_0}\,\vec{\boldsymbol{H}}$, where $c = 1/\sqrt{\epsilon_0\mu_0}$ and the current $\vec{\boldsymbol{J}} \to \vec{\boldsymbol{J}}/\sqrt{4\pi\epsilon_0}$ [cf. Table 3 on p. 782 of Jackson].

Using $\omega = kc$ and $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}} = -\sin\theta\,\hat{\boldsymbol{\phi}}$, it follows that

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \frac{I\omega}{2\pi kc\,r}\, e^{i(kr-\omega t)}\, \frac{\sin(\pi\cos\theta)}{\sin^2\theta}\, \hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}} = -\frac{I}{2\pi r}\, e^{i(kr-\omega t)}\, \frac{\sin(\pi\cos\theta)}{\sin\theta}\, \hat{\boldsymbol{\phi}}\,.$$

Plugging this result into eq. (111), we end up with

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]^2 . \tag{113}$$

A plot of the angular distribution is shown in Figure 3.

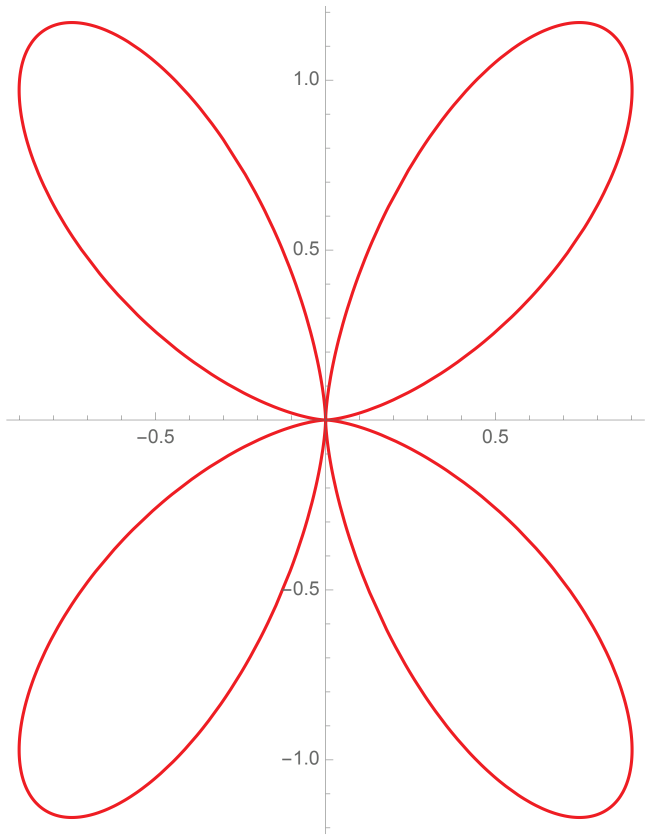

Figure 3: A polar plot of the antenna pattern of a thin linear antenna with a sinusoidal current that makes a full wavelength of oscillation. The angular distribution of the radiated power is given by eq. (113) and is proportional to $\sin^2(\pi\cos\theta)/\sin^2\theta$. Normalization has been chosen such that $Z_0 I^2 = 8\pi^2$. This plot was created with Mathematica software.

(b) Determine the total power radiated and find a numerical value for the radiation resistance.

The total power is

$$P = \int \frac{dP}{d\Omega}\, d\Omega = 2\pi \int_{-1}^{1} \frac{dP}{d\Omega}\, d\cos\theta\,,$$

since the angular distribution obtained in eq. (113) is independent of the azimuthal angle ϕ. Defining $x \equiv \cos\theta$, and employing $\sin^2\theta = 1 - \cos^2\theta$ in the denominator of eq. (113),

$$P = \frac{Z_0 I^2}{4\pi} \int_{-1}^{1} \frac{\sin^2(\pi x)}{1-x^2}\, dx = \frac{Z_0 I^2}{8\pi} \int_{-1}^{1} \frac{1-\cos(2\pi x)}{1-x^2}\, dx\,,$$

after employing a well-known trigonometric identity. We now apply the method of partial fractions to write

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right).$$

The resulting two integrals are equal after making a variable change $x \to -x$ in the first integral. Thus,

$$P = \frac{Z_0 I^2}{8\pi} \int_{-1}^{1} \frac{1-\cos(2\pi x)}{1+x}\, dx\,.$$

Next, we make a change of variables, $t = 2\pi(1+x)$, which converts the above integral into the following form,

$$P = \frac{Z_0 I^2}{8\pi} \int_{0}^{4\pi} \frac{1-\cos t}{t}\, dt\,.$$

This integral can be evaluated in terms of the cosine integral, which is defined as

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\, dt\,.$$

It then follows that:[6]

$$\int_0^x \frac{1-\cos t}{t}\, dt = \gamma + \ln x - \mathrm{Ci}(x)\,, \tag{114}$$

where $\gamma \simeq 0.5772$ is the Euler constant. Thus,

$$P = \frac{Z_0 I^2}{8\pi}\left[\gamma + \ln(4\pi) - \mathrm{Ci}(4\pi)\right].$$

Using the following numerical value, $\mathrm{Ci}(4\pi) = -0.006$,[7] we obtain

$$P = \frac{Z_0 I^2}{8\pi}(3.114)\,.$$

The corresponding radiative resistance (in ohms) is equal to the coefficient of $\frac{1}{2}I^2$ [cf. the text below eq. (9.29) of Jackson]. Thus, using $Z_0 = 376.7$ ohms [given below eq. (7.11)′ of Jackson],

$$R_{\rm rad} = (3.114)\,\frac{Z_0}{4\pi} = 93.3 \text{ ohms}\,. \tag{115}$$

[6]See e.g. formula 8.230 no. 2 on p. 895 of I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (8th edition), edited by Daniel Zwillinger and Victor Moll (Academic Press, Elsevier, Inc., Waltham, MA, 2015). Eq. (114) can also be found on p. 41 [cf. problem 3 on this page] of N.N. Lebedev, *Special Functions and Their Applications* (Dover Publications, Inc., Mineola, NY, 1972).

[7]For example, one can consult Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., Mineola, NY, 1965), which provides numerical tables of the cosine integral. Alternatively, one can use a mathematical program such as Mathematica or Maple to evaluate the cosine integral directly.