1. In SI units, the leading $\mathcal{O}(r^{-1})$ contributions to the multipole expansions of the magnetic field, $\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})e^{-i\omega t}$, and the electric field, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t}$, in the far (radiation) zone are given by:

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(-i)^{\ell+1}}{\sqrt{\ell(\ell+1)}}\left\{a_E(\ell,m)\frac{e^{ikr}}{kr}\vec{\boldsymbol{L}}\big[Y_{\ell m}(\theta,\phi)\big]-\frac{i}{k}a_M(\ell,m)\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\Big[\frac{e^{ikr}}{kr}Y_{\ell m}(\theta,\phi)\Big]\right\},$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})=Z_0\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(-i)^{\ell+1}}{\sqrt{\ell(\ell+1)}}\left\{\frac{i}{k}a_E(\ell,m)\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\Big[\frac{e^{ikr}}{kr}Y_{\ell m}(\theta,\phi)\Big]+a_M(\ell,m)\frac{e^{ikr}}{kr}\vec{\boldsymbol{L}}\big[Y_{\ell m}(\theta,\phi)\big]\right\},$$

where $\vec{\boldsymbol{L}}\equiv -i\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}$ is a differential operator that acts on functions of $\vec{\boldsymbol{x}}=(r,\theta,\phi)$, $k=\omega/c$, $r\equiv|\vec{\boldsymbol{x}}|$, Z_0 is the impedance of free space, and $a_E(\ell,m)$ and $a_M(\ell,m)$ are related to the electric and magnetic multipole moment spherical tensors, $Q_{\ell m}$ and $M_{\ell m}$, respectively:

$$a_E(\ell,m)=-\frac{ick^{\ell+2}}{(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}Q_{\ell m}\,,\qquad a_M(\ell,m)=\frac{ik^{\ell+2}}{(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}M_{\ell m}\,,\tag{1}$$

after setting the intrinsic magnetization to zero for simplicity.

(a) Using the multipole expansions above, derive the corresponding multipole expansions for $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})$ and $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$, where $\hat{\boldsymbol{n}}\equiv\vec{\boldsymbol{x}}/r$.

Using the operator identities,

$$\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{L}}=0\,,\qquad \vec{\boldsymbol{x}}\cdot\big(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big)=i\,\vec{\boldsymbol{L}}^{\,2}\,,\tag{2}$$

and $\hat{\boldsymbol{n}}\equiv\vec{\boldsymbol{x}}/r$, it follows that

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(-i)^{\ell+1}}{(kr)^2}\sqrt{\ell(\ell+1)}\,a_M(\ell,m)e^{ikr}Y_{\ell m}(\theta,\phi)\,,\tag{3}$$

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})=-Z_0\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(-i)^{\ell+1}}{(kr)^2}\sqrt{\ell(\ell+1)}\,a_E(\ell,m)e^{ikr}Y_{\ell m}(\theta,\phi)\,,\tag{4}$$

after using

$$\vec{\boldsymbol{L}}^{\,2}\Big[\frac{e^{ikr}}{r}Y_{\ell m}(\theta,\phi)\Big]=\frac{e^{ikr}}{r}\vec{\boldsymbol{L}}^{\,2}Y_{\ell m}(\theta,\phi)=\frac{e^{ikr}}{r}\ell(\ell+1)Y_{\ell m}(\theta,\phi)\,,\tag{5}$$

where we have made use of the fact that $\vec{\boldsymbol{L}}^{\,2}$ is a purely angular differential operator. After making use of eq. (1), we obtain, to leading order in the large r expansion,

$$\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}) = \frac{e^{ikr}}{r^2} \sum_{\ell m} \frac{(-ik)^\ell(\ell+1)}{(2\ell+1)!!} Y_{\ell m}(\theta,\phi) M_{\ell m} + \mathcal{O}\left(\frac{1}{r^3}\right) , \tag{6}$$

$$\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = \frac{e^{ikr}}{\epsilon_0 r^2} \sum_{\ell m} \frac{(-ik)^\ell(\ell+1)}{(2\ell+1)!!} Y_{\ell m}(\theta,\phi) Q_{\ell m} + \mathcal{O}\left(\frac{1}{r^3}\right) , \tag{7}$$

after using the relations $Z_0\sqrt{\mu_0/\epsilon_0}$ and $c = 1/\sqrt{\mu_0\epsilon_0}$, which yield $Z_0 c = 1/\epsilon_0$.

(b) Using the results of part (a), express the first two terms of the multipole expansion of $\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$ in terms of the electric dipole vector $\vec{\boldsymbol{p}}$ defined in eq. (9.17) of Jackson, and the electric quadrupole vector $\vec{\boldsymbol{Q}}$ defined in eq. (9.43) of Jackson.

Consider first the $\ell = 1$ contribution to $\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$.

$$\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})_{\ell=1} = -\frac{2ik}{3\epsilon_0 r^2}\, e^{ikr} \sum_m Y_{1m}(\theta,\phi) Q_{1m} \,. \tag{8}$$

We shall employ eq. (4.5) of Jackson,

$$Q_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\big(p_x \mp i p_y\big) \,, \qquad\qquad Q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z \,, \tag{9}$$

and make use of the expressions for the spherical harmonics given on p. 109 of Jackson,

$$Y_{1,\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r} \,, \qquad\qquad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \,. \tag{10}$$

It then follows that

$$\begin{aligned}\sum_m Y_{1m}(\theta,\phi) Q_{1m} &= \frac{3}{8\pi r}\big[(x+iy)(p_x - ip_y) + (x-iy)(p_x+ip_y) + 2zp_z\big] \\ &= \frac{3}{4\pi r}(xp_x + yp_y + zp_z) = \frac{3\vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{p}}}{4\pi r} = \frac{3}{4\pi}\, \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{p}}. \end{aligned} \tag{11}$$

Hence,

$$\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})_{\ell=1} = -\frac{ik\, e^{ikr}}{2\pi\epsilon_0 r^2}\, \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{p}}. \tag{12}$$

Next, consider the $\ell = 2$ contribution to $\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)$.

$$\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})_{\ell=2} = -\frac{\pi}{5\epsilon_0 r^2}\, e^{ikr} \sum_m Y_{2m}(\theta,\phi) Q_{2m} \,. \tag{13}$$

We shall employ eq. (4.6) of Jackson,

$$Q_{2,\pm 2} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\big(Q_{xx} \mp 2iQ_{xy} - Q_{yy}\big)\,, \qquad Q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,Q_{zz}\,,$$

$$Q_{2,\pm 1} = \mp\frac{1}{3}\sqrt{\frac{15}{8\pi}}\big(Q_{xz} \mp iQ_{yz}\big)\,. \tag{14}$$

and make use of the expressions for the spherical harmonics given on p. 109 of Jackson,

$$Y_{2,\pm 2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\phi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x \pm iy)^2}{r^2}\,, \tag{15}$$

$$Y_{2,\pm 1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} = \mp\sqrt{\frac{15}{8\pi}}\frac{z(x \pm iy)}{r^2}\,, \tag{16}$$

$$Y_{20}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta - 1) = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,\frac{2z^2 - x^2 - y^2}{r^2}\,. \tag{17}$$

It then follows that

$$\begin{aligned}
\sum_m Y_{2m}(\theta,\phi)Q_{2m} &= \frac{5}{32\pi r^2}\big[(Q_{xx} - 2iQ_{xy} - Q_{yy})(x+iy)^2 + 4(Q_{xz} - iQ_{yz})z(x+iy) + \text{c.c.}\big] \\
&\qquad + \frac{5}{16\pi r^2}Q_{zz}(2z^2 - x^2 - y^2) \\
&= \frac{5}{16\pi r^2}\Big[Q_{xx}(x^2 - y^2) - Q_{yy}(x^2 - y^2) + 4(xyQ_{xy} + xzQ_{xz} + yzQ_{yz}) \\
&\qquad + 2Q_{zz}z^2 + (Q_{xx} + Q_{yy})(x^2 + y^2)\Big]\,,
\end{aligned} \tag{18}$$

where c.c. stands for complex conjugate of the previous expression, and we have made use of the traceless electric quadrupole tensor to write $Q_{zz} = -Q_{xx} - Q_{yy}$ in the last line of eq. (18). The end result is

$$\begin{aligned}
\sum_m Y_{2m}(\theta,\phi)Q_{2m} &= \frac{5}{8\pi r^2}\big[x^2Q_{xx} + y^2Q_{yy}z^2 + Q_{zz} + 2(xyQ_{xy} + xzQ_{xz} + yzQ_{yz}\big) \\
&= \frac{5}{8\pi}\sum_{ij} Q_{ij}\hat{n}_i\hat{n}_j = \frac{5}{8\pi}\boldsymbol{\hat{n}}\boldsymbol{\cdot}\vec{\boldsymbol{Q}}\,.
\end{aligned} \tag{19}$$

Inserting this result back into eq. (13) yields

$$\boldsymbol{\hat{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})_{\ell=2} = -\frac{k^2e^{ikr}}{8\pi\epsilon_0 r^2}\boldsymbol{\hat{n}}\boldsymbol{\cdot}\vec{\boldsymbol{Q}}\,. \tag{20}$$

Combining the results of eqs. (12) and (20), we obtain:

$$\boldsymbol{\hat{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = -\frac{ike^{ikr}}{2\pi\epsilon_0 r^2}\Big[\boldsymbol{\hat{n}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} - \tfrac{1}{4}ik\,\boldsymbol{\hat{n}}\boldsymbol{\cdot}\vec{\boldsymbol{Q}}\Big] + \mathcal{O}\left(\frac{1}{r^3}\right)\,. \tag{21}$$

Note that the contribution of the electric dipole moment above coincides with the result that was used in solving problem 5 [Jackson, problem 9.8] on Problem Set 3.

2. [Jackson, problem 9.12] An almost spherical surface is defined by

$$R(\theta) = R_0 \left[1 + \beta\, P_2(\cos\theta)\right] \tag{22}$$

has inside of it a uniform volume distribution of charge totaling Q. The small parameter β varies harmonically in time at frequency ω. This corresponds to surface waves on the sphere. Keeping only lowest order terms in β and making the long-wavelength approximation, calculate the nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated.

First, we need to evaluate the charge density $\rho(\vec{\boldsymbol{x}}, t)$. It is a constant ρ_0 for $r \leq R(\theta)$ and zero otherwise. Since the total charge Q is conserved (and hence time independent),

$$Q = \int d^3x\, \rho(\vec{\boldsymbol{x}}, t) = \rho_0 \int r^2\, dr\, d\cos\theta\, d\phi\, \Theta(R(\theta) - r)\,,$$

where the step function is defined as,

$$\Theta(x) = \begin{cases} 1\,, & \text{for } x > 0\,, \\ 0\,, & \text{for } x < 0\,. \end{cases}$$

Thus,

$$Q = 2\pi\rho_0 \int_{-1}^{1} d\cos\theta \int_0^{R(\theta)} r^2\, dr = \frac{2\pi R_0^3 \rho_0}{3} \int_{-1}^{1} d\cos\theta\, \left[1 + \beta\, P_2(\cos\theta)\right]^3\,.$$

Assuming that $|\beta| \ll 1$ and dropping terms of $\mathcal{O}(\beta^2)$, it follows that

$$Q = \frac{2\pi R_0^3 \rho_0}{3} \int_{-1}^{1} d\cos\theta\, \left[1 + 3\beta\, P_2(\cos\theta) + \mathcal{O}(\beta^2)\right] = \frac{4\pi R_0^3 \rho_0}{3} \left[1 + \mathcal{O}(\beta^2)\right]\,,$$

after using the orthogonality relation,

$$\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\, \delta_{\ell\ell'}\,.$$

The parameter β varies harmonically with time. Using complex notation,

$$\beta = \beta_0 e^{-i\omega t}$$

Hence, including all terms up to and including $\mathcal{O}(\beta_0)$,

$$\rho(\vec{\boldsymbol{x}}, t) = \frac{3Q}{4\pi R_0^3}\, \Theta(R_0 + R_0\beta_0 P_2(\cos\theta) e^{-i\omega t} - r)\,. \tag{23}$$

Next, we compute the elements of the multipole tensor in the spherical basis,

$$\begin{aligned} Q_{\ell m}(t) &= \int d^3x\, r^\ell\, Y^*_{\ell m}(\theta, \phi)\, \rho(\vec{\boldsymbol{x}}, t) \\ &= \frac{3Q}{4\pi R_0^3} \int d\Omega\, Y^*_{\ell m}(\theta, \phi) \int_0^{R(\theta)} r^{\ell+2}\, dr \\ &= \frac{3QR_0^\ell}{4\pi(\ell+3)} \int d\Omega\, Y^*_{\ell m}(\theta, \phi)\, \left[1 + \beta_0 P_\ell(\cos\theta) e^{-i\omega t}\right]^{\ell+3}\,, \end{aligned}$$

where $\ell = 1, 2, 3, \ldots$ and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$. Note that the $\ell = m = 0$ moment does not enter the multipole expansion of the radiation fields. Working to first order in β_0, we can approximate

$$\left[1+\beta_0 P_\ell(\cos\theta)e^{-i\omega t}\right]^{\ell+3} = 1+(\ell+3)\beta_0 P_\ell(\cos\theta)e^{-i\omega t}+\mathcal{O}(\beta_0^2)\,.$$

Writing

$$P_\ell(\cos\theta)=\sqrt{\frac{4\pi}{2\ell+1}}\,Y_{\ell 0}(\theta,\phi)\,,$$

it follows that for $\ell\neq 0$,

$$\begin{aligned} Q_{\ell m}(t) &= \frac{3QR_0^\ell}{4\pi(\ell+3)}\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\left[1+(\ell+3)\beta_0\, e^{-i\omega t}\sqrt{\frac{4\pi}{5}}Y_{20}(\theta,\phi)\right] \\ &= \frac{3QR_0^\ell}{(\ell+3)\sqrt{4\pi}}\left[\frac{(\ell+3)\beta_0\, e^{-i\omega t}}{\sqrt{5}}\delta_{\ell 2}\,\delta_{m0}\right]\,, \end{aligned} \qquad (24)$$

after employing the orthogonality relation of the spherical harmonics [cf. eq. (3.55) of Jackson],

$$\int Y^*_{\ell m}(\theta,\phi)\,Y_{\ell' m'}(\theta,\phi)\,d\Omega=\delta_{\ell\ell'}\,\delta_{mm'}\,.$$

Writing $Q_{\ell m}(t)=Q_{\ell m}e^{-i\omega t}$, it follows that

$$Q_{\ell m}=\frac{3Q\beta_0 R_0^2}{\sqrt{20\pi}}\,\delta_{\ell,2}\,\delta_{m,0}\,.$$

That is, the only non-zero electric multipole moment is

$$Q_{20}=\frac{3Q\beta_0 R_0^2}{\sqrt{20\pi}}\,. \qquad (25)$$

An alternative derivation of eq. (25)

Since we are working to first order in β_0, it is convenient to expand the Θ-function that appears in eq. (23) using the fact that $\delta(x)=d\Theta(x)/dx$. Thus, to $\mathcal{O}(\beta_0)$,

$$\rho(\vec{\boldsymbol{x}},t)=\frac{3Q}{4\pi R_0^3}\left[\Theta(R_0-r)+R_0\beta_0 P_2(\cos\theta)\,e^{-i\omega t}\,\delta(R_0-r)\right]\,. \qquad (26)$$

Using eq. (9.170) of Jackson, we can evaluate the multipole moments for $\ell\neq 0$,

$$\begin{aligned} Q_{\ell m}(t) &= \int d^3x\, r^\ell\, Y^*_{\ell m}(\theta,\phi)\,\rho(\vec{\boldsymbol{x}},t) \\ &= \frac{3Q}{4\pi R_0^3}\int d^3x\, r^\ell\, Y^*_{\ell m}(\theta,\phi)\,\left[\Theta(R_0-r)+R_0\beta_0 P_2(\cos\theta)\,e^{-i\omega t}\,\delta(R_0-r)\right] \\ &= \frac{3QR_0}{4\pi R_0^3}\left[\sqrt{\frac{4\pi}{5}}\,R_0^{\ell+3}\beta_0\, e^{-i\omega t}\,\delta_{\ell 2}\,\delta_{m0}\right]\,, \end{aligned} \qquad (27)$$

which reproduces eq. (24).

As for the other possible multipole moments, we first note that there is no magnetization in this problem so that $Q'_{\ell m} = M'_{\ell m} = 0$. [cf. eqs.(9.170) and (9.172) of Jackson]. However, there is a non-zero harmonic current density due to motion of electric charges. The azimuthal symmetry of the problem implies that $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)$, when written in spherical coordinates, has no $\hat{\boldsymbol{\phi}}$ component and is independent of ϕ.[1] That is,

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = \Big[J_r(r,\theta)\,\hat{\boldsymbol{n}} + J_\theta(r,\theta)\,\hat{\boldsymbol{\theta}}\Big]\; e^{-i\omega t}\,,$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$ is the unit vector in the radial direction. Using eq. (9.172) of Jackson (in SI units), with $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\,e^{-i\omega t}$,

$$M_{\ell m} = -\frac{1}{\ell+1}\int d^3x\, r^\ell\, Y^*_{\ell m}(\theta,\phi)\,\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}))\,. \tag{28}$$

Using,

$$\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = r\,\hat{\boldsymbol{n}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = r\,J_\theta(r,\theta)\,\hat{\boldsymbol{\phi}}\,,$$

we conclude that

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})) = \frac{1}{\sin\theta}\,\frac{\partial J_\theta}{d\phi} = 0\,.$$

Hence, it follows from eq. (28) that

$$M_{\ell m} = 0\,.$$

The angular distribution of the radiated power can be obtained from eqs. (9.151) and (9.169) of Jackson,

$$\frac{dP}{d\Omega} = \tfrac{1}{2}Z_0c^2k^{2\ell+2}\,\frac{\ell+1}{\ell[(2\ell+1)!!]^2}\,|Q_{\ell m}|^2\,|\vec{\boldsymbol{X}}_{\ell m}|^2\,, \tag{29}$$

where

$$\vec{\boldsymbol{X}}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{L}}\,Y_{\ell m}(\theta,\phi)\,,$$

is a vector spherical harmonic. Integrating over solid angles is trivial since the $\vec{\boldsymbol{X}}_{\ell m}$ are normalized to unity. Thus,

$$P = \tfrac{1}{2}Z_0c^2k^{2\ell+2}\,\frac{\ell+1}{\ell[(2\ell+1)!!]^2}\,|Q_{\ell m}|^2\,. \tag{30}$$

Inserting the value for Q_{20} obtained in eq. (25) into the above formulae, and noting that

$$|\vec{\boldsymbol{X}}_{20}|^2 = \frac{15}{8\pi}\,\sin^2\theta\cos^2\theta\,,$$

according to Table 9.1 on p. 437 of Jackson, it follows that

$$\frac{dP}{d\Omega} = \frac{3Z_0c^2Q^2\beta_0^2R_0^4k^6}{2000\pi}\,\frac{15}{8\pi}\,\sin^2\theta\cos^2\theta\,,$$

and

$$P = \frac{3Z_0c^2Q^2\beta_0^2R_0^4k^6}{2000\pi}\,.$$

[1]An explicit expression for $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)$ will be given in an added note following this solution.

ADDED NOTE:

In this added note, we shall obtain an explicit form for $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)$ which is valid to first order in β. As noted previously, the azimuthal symmetry of the problem implies that $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)$ has no $\hat{\boldsymbol{\phi}}$ component and is independent of the azimuthal angle ϕ. That is,

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \Big[J_r(r,\theta)\hat{\boldsymbol{n}} + J_\theta(r,\theta)\hat{\boldsymbol{\theta}}\Big]\, e^{-i\omega t}\,,$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$ is the unit vector in the radial direction. Using the continuity equation,

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} + \frac{d\rho}{dt} = 0\,,$$

we can compute $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}}$ using the result for ρ obtained in eq. (26). Hence,

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} = -\frac{\partial\rho}{\partial t} = \frac{3i\omega Q\beta_0}{4\pi R_0^2}\, P_2(\cos\theta)\, e^{-i\omega t}\,\delta(R_0-r)\,. \tag{31}$$

In spherical coordinates, we have

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} = \frac{1}{r^2\sin\theta}\left[\sin\theta\,\frac{\partial}{\partial r}\left(r^2 J_r\right) + r\,\frac{\partial}{\partial\theta}\left(\sin\theta\, J_\theta\right) + r\frac{\partial J_\phi}{\partial\phi}\right]\,.$$

Since $\vec{\boldsymbol{J}}$ arises due to charges in motion, it must be proportional to β. Thus, to leading order in β, it must also be true that $\vec{\boldsymbol{J}}$ is proportional to $\Theta(R_0-r)$ since we can drop any β-dependence in the argument of the Θ-function (as the dropped terms will only contribute at higher order in β). Hence, as $J_\phi = 0$, we can write:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \beta_0\, e^{-i\omega t}\,\Theta(R_0-r)\left[J_1\hat{\boldsymbol{n}} + J_2\hat{\boldsymbol{\theta}}\right]\,. \tag{32}$$

Using eq. (32), we compute

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} &= \beta_0\, e^{-i\omega t}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\Theta(R_0-r)J_1\right) + \frac{1}{r\sin\theta}\,\Theta(R_0-r)\frac{\partial}{\partial\theta}\left(\sin\theta\, J_2\right)\right\}\\ &= \beta_0\, e^{-i\omega t}\left\{-J_1\,\delta(R_0-r) + \Theta(R_0-r)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 J_1\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, J_2\right)\right]\right\}\,. \end{aligned}\tag{33}$$

Comparing this result with eq. (31), we can immediately equate the coefficients of the delta function, which yields

$$J_1 = -\frac{3i\omega Q\beta_0}{4\pi R_0^2}\, P_2(\cos\theta)\,. \tag{34}$$

Inserting this result back into eq. (33), and comparing again with eq. (31), we conclude that the overall coefficient of the step function must vanish. That is,

$$-\frac{3i\omega Q\beta_0}{4\pi R_0^2}\, P_2(\cos\theta)\,\frac{2}{r} + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\, J_2\right) = 0\,.$$

Using $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$, we obtain

$$-\frac{3i\omega Q\beta_0}{4\pi R_0^2}\sin\theta\left(3\cos^2\theta - 1\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\, J_2\right) = 0\,.$$

Hence,

$$J_2\sin\theta = -\frac{3i\omega Q\beta_0}{4\pi R_0^2}\int\left(3\cos^2\theta - 1\right)\,d\cos\theta = \frac{3i\omega Q}{4\pi R_0^2}\cos\theta(1-\cos^2\theta)\,,$$

or equivalently,[2]

$$J_2 = \frac{3i\omega Q\beta_0}{8\pi R_0^2}\sin 2\theta\,. \tag{35}$$

Inserting eqs. (34) and (35) back into eq. (32) yields the final result, which is valid at first order in β_0,

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = -\frac{3i\omega Q\beta_0}{8\pi R_0^2}\,e^{-i\omega t}\,\Theta(R_0 - r)\left[(3\cos^2\theta - 1)\hat{\boldsymbol{n}} - \sin 2\theta\,\hat{\boldsymbol{\theta}}\right]\,.$$

3. [Jackson, problem 9.15] Two fixed electric dipoles of dipole moment p, with their axes parallel but their moments directed oppositely, are located in the x–y plane a distance $2a$ apart. Both dipole moment axes are perpendicular to the plane (i.e., they point parallel and antiparallel to the z-axis, respectively). The dipoles rotate with constant angular speed ω about a z axis located halfway between them. The motion is nonrelativistic ($\omega a/c \ll 1$).

(a) Find the lowest nonvanishing multipole moments. [Hint: Problem 6.21 is relevant.]

Using the results of Jackson problem 6.21, it then follows that

$$\rho(\vec{\boldsymbol{x}},t) = -(\vec{\boldsymbol{p}}_1\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})\delta^3\big(\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}_1(t)\big) - (\vec{\boldsymbol{p}}_2\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})\delta^3\big(\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}_2(t)\big)\,, \tag{36}$$

where $\vec{\boldsymbol{p}}_1 = -\vec{\boldsymbol{p}}_2 = p\hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{r}}_1(t) = -\vec{\boldsymbol{r}}_2(t) = a\big(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t\big)$. Similarly,

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = -\vec{\boldsymbol{v}}_1(\vec{\boldsymbol{p}}_1\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})\delta^3\big(\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}_1(t)\big) - \vec{\boldsymbol{v}}_2(\vec{\boldsymbol{p}}_2\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})\delta^3\big(\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}_2(t)\big)\,, \tag{37}$$

where $\vec{\boldsymbol{v}}_i = d\vec{\boldsymbol{r}}_i/dt$. That is, $\vec{\boldsymbol{v}}_1 = -\vec{\boldsymbol{v}}_2 = -\omega a\big(\hat{\boldsymbol{x}}\sin\omega t - \hat{\boldsymbol{y}}\cos\omega t\big)$.

First we compute the electric dipole moment. Note that $\hat{\boldsymbol{z}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}} = \partial/\partial z$, which acts on the delta function in eq. (36). Hence,

$$\begin{aligned}\vec{\boldsymbol{p}}(t) &= \int d^3x\,\vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}},t)\\ &= -p\int d^3x\,\vec{\boldsymbol{x}}\,\frac{\partial}{\partial z}\Big[\delta^3\big(\vec{\boldsymbol{x}} - a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big) - \delta^3\big(\vec{\boldsymbol{x}} + a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big)\Big]\,.\end{aligned}$$

[2] In evaluation of the indefinite integral, the constant of integration must be set to zero, since J_2 must be non-singular at $\theta = 0$ and at $\theta = \pi$.

Next, we integrate by parts. The surface term at infinity vanishes since the charges are localized. Noting that $\vec{\boldsymbol{x}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, it follows that

$$\frac{\partial}{\partial z}\vec{\boldsymbol{x}} = \hat{\boldsymbol{z}}\,.$$

Hence,

$$\vec{\boldsymbol{p}}(t) = p\hat{\boldsymbol{z}}\int d^3x\,\Big[\delta^3\big(\vec{\boldsymbol{x}} - a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big) - \delta^3\big(\vec{\boldsymbol{x}} + a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big)\Big] = 0\,, \quad (38)$$

since each delta function integrates to unity.

Second, we compute the magnetic dipole moment.

$$\vec{\boldsymbol{m}}(t) = \frac{1}{2}\int d^3x\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)\,.$$

In terms of the components,

$$m_i(t) = \frac{1}{2}\epsilon_{ijk}\int d^3x\,x_j J_k(\vec{\boldsymbol{x}}, t)\,,$$

where there is an implicit sum over the two pairs of repeated indices. Using eq. (37) and identifying $x_3 = z$, it follows that

$$\begin{aligned} m_i(t) &= \tfrac{1}{2}\omega pa\big(\hat{\boldsymbol{x}}\sin\omega t - \hat{\boldsymbol{y}}\cos\omega t\big)_k\epsilon_{ijk} \\ &\qquad\times\int d^3x\,x_j\,\frac{\partial}{\partial x_3}\Big[\delta^3\big(\vec{\boldsymbol{x}} - a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big) + \delta^3\big(\vec{\boldsymbol{x}} + a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big)\Big] \\ &= -\tfrac{1}{2}\omega pa\big(\hat{\boldsymbol{x}}\sin\omega t - \hat{\boldsymbol{y}}\cos\omega t\big)_k\epsilon_{ijk}\delta_{j3} \\ &\qquad\times\int d^3x\,\Big[\delta^3\big(\vec{\boldsymbol{x}} - a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big) + \delta^3\big(\vec{\boldsymbol{x}} + a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big)\Big] \\ &= -\omega pa\big(\hat{\boldsymbol{x}}\sin\omega t - \hat{\boldsymbol{y}}\cos\omega t\big)_k\epsilon_{i3k}. \end{aligned} \quad (39)$$

after integrating by parts and employing $\partial x_j/\partial x_3 = \delta_{j3}$. Note that by definition, $\big(\hat{\boldsymbol{x}}\sin\omega t - \hat{\boldsymbol{y}}\cos\omega t\big)_1 = \sin\omega t$ and $\big(\hat{\boldsymbol{x}}\sin\omega t - \hat{\boldsymbol{y}}\cos\omega t\big)_2 = -\cos\omega t$. Hence,

$$m_1(t) = -\omega pa\cos\omega t\,, \qquad m_2(t) = -\omega pa\sin\omega t\,, \qquad m_3(t) = 0\,. \quad (40)$$

We can rewrite this result as $\vec{\boldsymbol{m}}(t) = \mathrm{Re}(\vec{\boldsymbol{m}}e^{-i\omega t})$, where

$$\vec{\boldsymbol{m}} = -\omega pa(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})\,. \quad (41)$$

As discussed in section 9.3 of Jackson, the magnetic dipole and electric quadrupole radiation enter at the same order (as a consequence of $\omega a/c \ll 1$). Thus, the contribution of the electric quadrupole cannot be neglected in this problem. We therefore compute

$$Q_{ij}(t) = \int\big[3x_ix_j - r^2\delta_{ij}\big]\rho(\vec{\boldsymbol{x}}, t)\,,$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$. Using eq. (36) and identifying $x_3 = z$,

$$\begin{aligned} Q_{ij}(t) = -p \int d^3x \left[3x_i x_j - (x_1^2 + x_2^2 + x_3^2)\delta_{ij}\right] \\ \times \frac{\partial}{\partial x_3}\Big[\delta^3\big(\vec{\boldsymbol{x}} - a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big) - \delta^3\big(\vec{\boldsymbol{x}} + a(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\big)\Big]\end{aligned}$$

Integrating by parts and using $\delta^3(\vec{\boldsymbol{v}}) = \delta(v_1)\delta(v_2)\delta(v_3)$, we end up with

$$\begin{aligned} Q_{ij}(t) = p \int d^3x \left[3x_i\delta_{j3} + 3x_j\delta_{3i} - 2x_3\delta_{ij}\right]\delta(x_3) \\ \times\Big[\delta(x_1 - a\cos\omega t)\delta(x_2 - a\sin\omega t) - \delta(x_1 + a\cos\omega t)\delta(x_2 + a\sin\omega t)\Big]. \end{aligned} \quad (42)$$

Using $x_3\delta(x_3) = 0$, we see that the only nonvanishing components of Q_{ij} are

$$Q_{13}(t) = Q_{31}(t) = 6pa\cos\omega t = 6pa\,\mathrm{Re}(e^{-i\omega t})\,, \quad (43)$$
$$Q_{23}(t) = Q_{32}(t) = 6pa\sin\omega t = 6pa\,\mathrm{Re}(ie^{-i\omega t})\,. \quad (44)$$

We can rewrite this result as $Q_{ij}(t) = \mathrm{Re}(Q_{ij}e^{-i\omega t})$, where

$$Q_{ij} = 6pa\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0\end{pmatrix}\,. \quad (45)$$

Thus, the lowest nonvanishing multipole moments are the magnetic dipole moment and electric quadrupole moment given in eqs. (41) and (45), respectively.

(b) Show that the magnetic field in the radiation zone is, apart from an overall phase factor,

$$\vec{\boldsymbol{H}} = \frac{cpa}{2\pi}k^3\left[(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})\cos\theta - \hat{\boldsymbol{z}}\sin\theta e^{i\phi}\right]\cos\theta\frac{e^{ikr}}{r} \quad (46)$$

For harmonic sources, the physical magnetic field vector is given by $\vec{\boldsymbol{H}}(t) = \mathrm{Re}(\vec{\boldsymbol{H}}e^{-i\omega t})$, where $\vec{\boldsymbol{H}}$ is the complex magnetic field vector. Using eq. (9.35) of Jackson, the complex magnetic field vector in the radiation zone from magnetic dipole (M1) radiation is given by

$$\vec{\boldsymbol{H}}_{M1} = \frac{k^2}{4\pi}\left\{(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{m}})\times\hat{\boldsymbol{n}}\,\frac{e^{ikr}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right\}\,, \quad (47)$$

where $k = \omega/c$ and the unit vector $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta$. Using eq. (41),

$$\begin{aligned}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{m}})\times\hat{\boldsymbol{n}} &= \vec{\boldsymbol{m}} - (\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{m}})\hat{\boldsymbol{n}} \\ &= -\omega pa\Big[\hat{\boldsymbol{x}}(1 - \sin^2\theta\cos\phi\,e^{i\phi}) + i\hat{\boldsymbol{y}}(1 + i\sin^2\theta\sin\phi\,e^{i\phi}) - \hat{\boldsymbol{z}}\sin\theta\cos\theta\,e^{i\phi}\Big].\end{aligned}$$

Writing $\omega = ck$, we end up with

$$\vec{\boldsymbol{H}}_{M1} = -\frac{cpak^3}{4\pi}\,\frac{e^{ikr}}{r}\Big[\hat{\boldsymbol{x}}(1 - \sin^2\theta\cos\phi\,e^{i\phi}) + i\hat{\boldsymbol{y}}(1 + i\sin^2\theta\sin\phi\,e^{i\phi}) - \hat{\boldsymbol{z}}\sin\theta\cos\theta\,e^{i\phi}\Big]. \quad (48)$$

Using eq. (9.44) of Jackson, the complex magnetic field vector in the radiation zone from electric quadrupole (E2) radiation is given by

$$\vec{\boldsymbol{H}}_{E2} = -\frac{ick^3}{24\pi}\,\frac{e^{ikr}}{r}\,\hat{\boldsymbol{n}}\times\vec{\boldsymbol{Q}} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{49}$$

where the components of vector $\vec{\boldsymbol{Q}}$ are given by $Q_i = \sum_{j=1}^{3} Q_{ij}n_j$ [cf. eq. (9.43) of Jackson]. Using eq. (45), we then obtain

$$Q_1 = Q_{13}n_3 = 6pa\cos\theta\,, \tag{50}$$
$$Q_2 = Q_{23}n_3 = 6ipa\cos\theta\,, \tag{51}$$
$$Q_3 = Q_{31}n_1 + Q_{32}n_2 = 6pa\sin\theta\, e^{i\phi}\,. \tag{52}$$

It then follows that

$$\begin{aligned}
\vec{\boldsymbol{H}}_{E2} &= -\frac{icpak^3}{4\pi}\,\frac{e^{ikr}}{r}\det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta & i\cos\theta & \sin\theta\, e^{i\phi}\end{pmatrix} \\
&= \frac{cpak^3}{4\pi}\,\frac{e^{ikr}}{r}\Big\{-\hat{\boldsymbol{x}}\big[i\sin^2\theta\sin\phi\, e^{i\phi} + \cos^2\theta\big] + i\hat{\boldsymbol{y}}\big[\sin^2\theta\cos\phi\, e^{i\phi} - \cos^2\theta\big] \\
&\qquad\qquad + \hat{\boldsymbol{z}}\big[\sin\theta\cos\theta\, e^{i\phi}\big]\Big\}.
\end{aligned} \tag{53}$$

The total complex magnetic field vector is thus given by

$$\vec{\boldsymbol{H}} = \vec{\boldsymbol{H}}_{M1} + \vec{\boldsymbol{H}}_{E2} = -\frac{cpak^3}{2\pi}\,\frac{e^{ikr}}{r}\,\big[(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})\cos\theta - \hat{\boldsymbol{z}}\sin\theta\, e^{i\phi}\big]\cos\theta\,, \tag{54}$$

which coincides with Jackson's result given in eq. (46) up to an overall sign.

<u>*REMARKS*</u>:

In eqs. (36) and (37), we assumed that $\vec{\boldsymbol{r}}_1(t) = -\vec{\boldsymbol{r}}_2(t) = a\big(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t\big)$, which fixes the location of the electric dipoles in the x–y plane at $t=0$. More generally, we could have chosen

$$\vec{\boldsymbol{r}}_1(t) = -\vec{\boldsymbol{r}}_2(t) = a\big[\hat{\boldsymbol{x}}\cos(\omega t+\varphi) + \hat{\boldsymbol{y}}\sin(\omega t+\varphi)\big]\,,$$

which corresponds to a different initial condition for the electric dipoles. In this case, the complex magnetic dipole moment would be modified by $\vec{\boldsymbol{m}} \to \vec{\boldsymbol{m}}e^{-i\varphi}$, which implies [cf. eq. (47)] that the complex magnetic field vector obtained in eq. (46) is modified by an overall phase factor. Likewise, the complex electric quadrupole moment tensor would also be modified by $Q_{ij} \to Q_{ij}e^{-i\varphi}$. This is why Jackson states that the expression for the magnetic field in eq. (46) is given "apart from an overall phase factor." Indeed, by choosing $\varphi = \pi$, eq. (54) would be multiplied by -1, which would then coincide precisely with the result given in eq. (46). In part (c), we see that the time-averaged power depends on the magnitude of the complex magnetic field vector and thus is independent of the choice of the overall phase factor.

(c) Show that the angular distribution of the radiation is proportional to $(\cos^2\theta + \cos^4\theta)$ and the total time-averaged power radiated is

$$P = \frac{4}{15\pi\epsilon_0} ck^6 p^2 a^2 \tag{55}$$

The time-averaged power per unit solid angle is given by eq. (9.21) of Jackson,

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} \tfrac{1}{2}\,\mathrm{Re}\big[r^2\,\hat{\boldsymbol{n}}\cdot(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*)\big]\,, \tag{56}$$

where $\vec{\boldsymbol{E}} = Z_0\vec{\boldsymbol{H}}\times\hat{\boldsymbol{n}}$ in the radiation zone [cf. eq.(9.19) of Jackson]. In eq. (56), harmonic sources have been assumed, and $\vec{\boldsymbol{E}}(t) = \mathrm{Re}(\vec{\boldsymbol{E}}e^{-i\omega t})$ and $\vec{\boldsymbol{H}}(t) = \mathrm{Re}(\vec{\boldsymbol{H}}e^{-i\omega t})$ define the complex electric and magnetic field vectors $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$, respectively. In particular,

$$\hat{\boldsymbol{n}}\cdot(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*) = Z_0\,\hat{\boldsymbol{n}}\cdot[(\vec{\boldsymbol{H}}\times\hat{\boldsymbol{n}})\times\vec{\boldsymbol{H}}^*] = Z_0\big[|\vec{\boldsymbol{H}}|^2 - |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}|^2\big]\,.$$

In light of eqs. (47) and (49), $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}} = 0$. Hence,

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} \tfrac{1}{2} Z_0 r^2 |\vec{\boldsymbol{H}}|^2\,. \tag{57}$$

Using eq. (54),

$$\frac{dP}{d\Omega} = \frac{Z_0 c^2 p^2 a^2 k^6}{8\pi^2}\cos^2\theta(2\cos^2\theta + \sin^2\theta)\,. \tag{58}$$

Noting that $2\cos^2\theta + \sin^2\theta = 1 + \cos^2\theta$, we end up with

$$\frac{dP}{d\Omega} = \frac{Z_0 c^2 p^2 a^2 k^6}{8\pi^2}\cos^2\theta(1 + \cos^2\theta)\,, \tag{59}$$

which corresponds to an angular distribution of the radiation that is proportional to $(\cos^2\theta + \cos^4\theta)$.

Integrating over solid angles,

$$\int d\Omega(\cos^2\theta + \cos^4\theta) = 2\pi\int_{-1}^{1}(\cos^2\theta + \cos^4\theta)\,d\cos\theta = 4\pi\left(\tfrac{1}{3} + \tfrac{1}{5}\right) = \frac{32\pi}{15}\,. \tag{60}$$

Hence,

$$P = \frac{Z_0 c^2 p^2 a^2 k^6}{8\pi^2}\left(\frac{32\pi}{15}\right) = \frac{4 Z_0 c^2 p^2 a^2 k^6}{15\pi}\,. \tag{61}$$

Finally, using $Z_0 \equiv \sqrt{\mu_0/\epsilon_0}$ and $c = 1/\sqrt{\epsilon_0\mu_0}$ [cf. Section 7.1 of Jackson], it follows that $Z_0 c = 1/\epsilon_0$. Hence, eq. (61) yields

$$P = \frac{4cp^2a^2k^6}{15\pi\epsilon_0}\,, \tag{62}$$

in agreement with eq. (55).

4. [Jackson, problem 9.17] Treat the linear antenna of Jackson, problem 9.16 (on Problem Set 3) by the multipole expansion method.

(a) Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly and in the long-wavelengths approximation.

As in Jackson, problem 9.16, we shall choose the z-axis to lie along the antenna, and let $z = 0$ correspond to the center of the antenna. Then, $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\, e^{-i\omega t}$, where

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = I \sin\left(\frac{2\pi z}{d}\right) \delta(x)\, \delta(y)\, \hat{\boldsymbol{z}}\,, \qquad \text{for } |z| \leq \tfrac{1}{2}d\,, \tag{63}$$

where d is the length of the antenna. It is convenient to rewrite this in spherical coordinates. Note that $\hat{\boldsymbol{z}} = \hat{n}$ for $\cos\theta = 1$ (i.e., $\theta = 0$) and $\hat{\boldsymbol{z}} = -\hat{\boldsymbol{n}}$ for $\cos\theta = -1$ (i.e., $\theta = \pi$), where $\hat{\boldsymbol{n}}$ is a unit vector pointing in the radial direction, and $r \equiv |\vec{\boldsymbol{x}}|$ is the radial coordinate. Hence, we may write[3]

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = \frac{I}{2\pi r^2} \sin\left(\frac{2\pi r}{d}\right) \big[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\big]\Theta(\tfrac{1}{2}d - r)\hat{\boldsymbol{n}}\,, \tag{64}$$

where I have inserted the Heavyside step function since the current $I(z,t) = 0$ for $|z| > \frac{1}{2}d$. In obtaining eq. (64), I used the fact that

$$\sin\left(\frac{2\pi z}{d}\right) = \sin\left(\frac{2\pi r\,\mathrm{sgn}(z)}{d}\right) = \mathrm{sgn}(z) \sin\left(\frac{2\pi r}{d}\right)\,,$$

where the sign function $\mathrm{sgn}(z)$ is defined as

$$\mathrm{sgn}(z) = \begin{cases} +1\,, & \text{for } z > 0, \\ -1\,, & \text{for } z < 0. \end{cases}$$

Finally, we note that $\hat{\boldsymbol{n}} = \mathrm{sgn}(z)\hat{\boldsymbol{z}}$ along the z-axis.

We shall make use of eqs. (9.167) and (9.168) of Jackson for the electric and magnetic multipole coefficients. In the absence of magnetization, in SI units,

$$a_E(\ell, m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}} \int Y^*_{\ell m}(\theta,\phi) \left\{ c\rho(\vec{\boldsymbol{x}}) \frac{\partial}{\partial r}\big[r j_\ell(kr)\big] + ik\, \vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) j_\ell(kr) \right\} d^3x\,, \tag{65}$$

$$a_M(\ell, m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}} \int Y^*_{\ell m}(\theta,\phi) \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \big(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\big) j_\ell(kr) d^3x\,. \tag{66}$$

It is convenient to integrate by parts in evaluating the first term of the integrand in eq. (65). The surface term can be dropped, since the charge density is localized. Since $d^3x = r^2\, dr\, d\Omega$, after integrating by parts, one obtains

$$-\frac{\partial}{\partial r}\big[r^2 \rho(\vec{\boldsymbol{x}})\big]\, dr = -\left(\frac{\partial\rho(\vec{\boldsymbol{x}})}{\partial r} + \frac{2}{r}\rho(\vec{\boldsymbol{x}})\right) r^2\, dr\,.$$

[3] Note that this differs from eq. (9.179) of Jackson by a relative sign. This difference is due to the fact that for the antenna showed in Figure 9.6 of Jackson, we have $I(-z) = I(z)$. In contrast, in this problem, eq. (63) yields $I(-z) = -I(z)$.

It then follows that

$$a_E(\ell,m)=\frac{k^2}{i\sqrt{\ell(\ell+1)}}\int Y^*_{\ell m}(\theta,\phi)j_\ell(kr)\biggl\{-c\left(2+r\frac{\partial}{\partial r}\right)\rho(\vec{\boldsymbol{x}})+ik\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\biggr\}d^3x\,, \qquad (67)$$

which is the version obtained in class.[4]

First consider the computation of $a_M(\ell,m)$. Using the vector identity,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\bigl(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}\bigr)=\vec{\boldsymbol{J}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{x}})-\vec{\boldsymbol{x}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{J}})=-\vec{\boldsymbol{x}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{J}})\,,$$

after using $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{x}}=0$. However, the current density given in eq. (64) is purely radial, which implies that $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{J}}=0$. Therefore, we conclude that $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\bigl(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}\bigr)=0$, which implies that

$$a_M(\ell,m)=0\,. \qquad (68)$$

That is, all the magnetic multipole coefficients vanish.

To evaluate the electric multipole coefficients, $a_E(\ell,m)$, we can either use eq. (65) or eq. (67). We shall first employ eq. (65), and then in an addendum we will provide details of the calculation that makes use of eq. (67).

For harmonic sources [cf. eq. (9.15) of Jackson], $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}=i\omega\rho$. Using eq. (64), we see that $\vec{\boldsymbol{J}}$ is purely radial, $\vec{\boldsymbol{J}}=J_r\hat{\boldsymbol{n}}$, and

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}=\frac{1}{r^2}\frac{\partial}{\partial r}\bigl(r^2J_r\bigr)&=\frac{I}{r^2}\bigl[\delta(\cos\theta-1)+\delta(\cos\theta+1)\bigr]\\ &\qquad\times\left\{\frac{1}{d}\cos\left(\frac{2\pi r}{d}\right)\Theta(\tfrac{1}{2}d-r)-\frac{1}{2\pi}\sin\left(\frac{2\pi r}{d}\right)\delta(\tfrac{1}{2}d-r)\right\}.\end{aligned}$$

Noting that $\sin(2\pi r/d)\delta(\frac{1}{2}d-r)=\sin\pi\,\delta(\frac{1}{2}d-r)=0$, we can drop the delta function in the previous equation. We conclude that

$$\rho(\vec{\boldsymbol{x}})=\frac{1}{ikc}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})=\frac{I}{ikcr^2d}\cos\left(\frac{2\pi r}{d}\right)\bigl[\delta(\cos\theta-1)+\delta(\cos\theta+1)\bigr]\Theta(\tfrac{1}{2}d-r)\,, \qquad (69)$$

after making use of $\omega=kc$. We also note that eq. (64) yields

$$\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})=\frac{I}{2\pi r}\sin\left(\frac{2\pi r}{d}\right)\bigl[\delta(\cos\theta-1)+\delta(\cos\theta+1)\bigr]\Theta(\tfrac{1}{2}d-r)\,. \qquad (70)$$

after using $\vec{\boldsymbol{x}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}=r$.

Plugging eqs. (69) and (70) into eq. (65), and evaluating the integral using spherical coordinates, $d^3x=r^2\,dr\,d\Omega$,

$$\begin{aligned}a_E(\ell,m)=\frac{Ik}{d\sqrt{\ell(\ell+1)}}&\int Y^*_{\ell m}(\theta,\phi)\bigl[\delta(\cos\theta-1)+\delta(\cos\theta+1)\bigr]d\Omega\\ &\times\int_0^{d/2}r\,dr\biggl\{-\cos\left(\frac{2\pi r}{d}\right)\frac{1}{r}\frac{\partial}{\partial r}\bigl[rj_\ell(kr)\bigr]+\frac{k^2}{2\pi}\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)\biggr\}. \qquad (71)\end{aligned}$$

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[4]One small advantage of using eq. (65) instead of eq. (67) is that no delta functions arise in the computation [cf. eq. (87) below]. By employing eq. (65), Jackson can simply set the limits of the radial integration to $0\leq r\leq\frac{1}{2}d$, and otherwise ignore the implicit Heavyside step function in his analysis of the linear, centerfed antenna on pp. 445–446.

We first evaluate the angular integral above. Writing $d\Omega = d\cos\theta\, d\phi$, consider the integral

$$\frac{1}{2\pi}\int Y^*_{\ell m}(\theta,\phi)\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]\,d\cos\theta\,d\phi\,.$$

Since $Y^*_{\ell m}(\theta,\phi)\propto e^{-im\phi}$, the ϕ integral yields

$$\int_0^{2\pi} e^{-im\phi}d\phi = \delta_{m0}\,.$$

Thus, in light of eq. (3.53) of Jackson and the following properties of the associated Legendre polynomials, $P_\ell^m(1)=\delta_{m0}$ and $P_\ell^m(-1)=(-1)^\ell\delta_{m0}$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int Y^*_{\ell m}(\theta,\phi)\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]\,d\cos\theta\,d\phi \\
&\qquad=\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\int_{-1}^1 P_\ell^m(\cos\theta)\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]\,d\cos\theta\,\frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\,d\phi \\
&\qquad=\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\,\big[P_\ell^m(1)+P_\ell^m(-1)\big]\frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\,d\phi \\
&\qquad=\delta_{m0}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\big[1+(-1)^\ell\big]\,.
\end{aligned}\tag{72}$$

Plugging eq. (72) back into eq. (71),

$$a_E(\ell,m)=\frac{Ik}{d}\sqrt{\frac{\pi(2\ell+1)}{\ell(\ell+1)}}\,\delta_{m0}\big[1+(-1)^\ell\big]\int_0^{d/2}\left\{-\cos\left(\frac{2\pi r}{d}\right)\frac{\partial}{\partial r}\big[rj_\ell(kr)\big]+\frac{k^2rd}{2\pi}\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)\right\}dr\,.$$

The first term of the integrand above can be rewritten using an integration by parts,

$$\begin{aligned}
\int_0^{d/2}\cos\left(\frac{2\pi r}{d}\right)\frac{\partial}{\partial r}\big[rj_\ell(kr)\big] &= \cos\left(\frac{2\pi r}{d}\right)rj_\ell(kr)\bigg|_0^{d/2}+\frac{2\pi}{d}\int_0^{d/2} r\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)dr \\
&= -\tfrac{1}{2}d\,j_\ell\big(\tfrac{1}{2}kd\big)+\frac{2\pi}{d}\int_0^{d/2} r\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)dr\,.
\end{aligned}$$

We then end up with

$$a_E(\ell,m)=\frac{Ik}{2}\sqrt{\frac{2\ell+1}{\pi\ell(\ell+1)}}\,\delta_{m0}\big[1+(-1)^\ell\big]\left\{\pi j_\ell\big(\tfrac{1}{2}kd\big)+\left[k^2-\left(\frac{2\pi}{d}\right)^2\right]\int_0^{d/2}\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)\,r\,dr\right\}\,. \tag{73}$$

By assumption, the sinusoidal current makes a full wavelength, which implies that

$$k=\frac{2\pi}{d}\,. \tag{74}$$

Hence, after setting $kd = 2\pi$ in eq. (73), we arrive at the final result,

$$a_E(\ell, m) = \frac{Ik}{2}\sqrt{\frac{\pi(2\ell+1)}{\ell(\ell+1)}}\delta_{m0}\big[1 + (-1)^\ell\big]j_\ell(\pi)\,. \tag{75}$$

We now consider the long wavelength approximation, $kd \ll 1$. We will do the computation in two ways. First we will start with eq. (73) and use the small argument approximation for the spherical Bessel function,

$$j_\ell(kr) \simeq \frac{(kr)^\ell}{(2\ell+1)!!}\,.$$

Changing variables to $x \equiv 2r/d$,

$$a_E(\ell, m) \simeq \frac{Ik}{2(2\ell+1)!!}\sqrt{\frac{\pi(2\ell+1)}{\ell(\ell+1)}}\delta_{m0}\big[1 + (-1)^\ell\big]\left(\frac{kd}{2}\right)^\ell\left\{1 - \pi\int_0^1 x^{\ell+1}\sin(\pi x)dx\right\}\,,$$

after dropping the $\mathcal{O}(k^2)$ term in the factor that multiplies the integral in eq. (73). Integrating by parts yields

$$\pi\int_0^1 x^{\ell+1}\sin(\pi x)dx = 1 + (\ell+1)\int_0^1 x^\ell\cos(\pi x)dx\,.$$

Hence, we obtain a slightly simpler result,

$$a_E(\ell, m) \simeq -\frac{Ik}{2(2\ell+1)!!}\sqrt{\frac{\pi(\ell+1)(2\ell+1)}{\ell}}\delta_{m0}\big[1 + (-1)^\ell\big]\left(\frac{kd}{2}\right)^\ell\int_0^1 x^\ell\cos(\pi x)dx\,. \tag{76}$$

As a check of eq. (76), we can perform the computation using eqs. (9.169)–(9.170) of Jackson (after setting the magnetization to zero),

$$a_E(\ell, m) \simeq \frac{ck^{\ell+2}}{i(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}Q_{\ell m}\,, \tag{77}$$

where

$$Q_{\ell m} = \int r^\ell Y^*_{\ell m}(\theta, \phi)\rho(\vec{\boldsymbol{x}})\,d^3x\,. \tag{78}$$

Inserting eq. (69) into eq. (78), and making use of eq. (72),

$$Q_{\ell m} = \frac{I\sqrt{\pi(2\ell+1)}}{ikcd}\delta_{m0}\big[1 + (-1)^\ell\big]\int_0^{d/2} r^\ell\cos\left(\frac{2\pi r}{d}\right)dr\,,$$

after using $\omega = kc$. Changing variables to $x = 2r/d$,

$$Q_{\ell m} = \frac{I\sqrt{\pi(2\ell+1)}}{2ikc}\delta_{m0}\left(\frac{d}{2}\right)^\ell\big[1 + (-1)^\ell\big]\int_0^1 x^\ell\cos(\pi x)dx\,.$$

Plugging this result into eq. (77) yields

$$a_E(\ell, m) \simeq -\frac{Ik}{2(2\ell+1)!!}\sqrt{\frac{\pi(\ell+1)(2\ell+1)}{\ell}}\delta_{m0}\big[1 + (-1)^\ell\big]\left(\frac{kd}{2}\right)^\ell\int_0^1 x^\ell\cos(\pi x)dx\,,$$

in agreement with eq. (76).

We now evaluate these results explicitly for the electric dipole ($\ell = 1$) and the electric quadrupole ($\ell = 2$). Due to the factor of $1 + (-1)^\ell$, we immediately see that only even ℓ multipoles survive. Hence, the electric dipole coefficient vanishes. Thus, we henceforth focus on the electric quadrupole coefficient. First, we use the exact result given in eq. (75). Using

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x\,,$$

it follows that $j_2(\pi) = 3/\pi^2$. Hence, for $kd = 2\pi$ and $\ell = 2$, eq. (75) yields

$$a_E(2,0) = Ik\sqrt{\frac{15}{2\pi^3}}\,. \tag{79}$$

Let us compare this result with eq. (76), which was obtained in the long wavelength approximation.

$$a_E(2,0) \simeq -Ik\sqrt{\frac{\pi}{30}}\left(\frac{kd}{2}\right)^2\int_0^1 x^2\cos(\pi x)dx\,.$$

Performing the integral,

$$\int_0^1 x^2\cos(\pi x)dx = \frac{1}{\pi^3}\bigl[2\pi x\cos(\pi x) + (\pi^2x^2 - 2)\sin(\pi x)\bigr]\Big|_0^1 = -\frac{2}{\pi^2}\,,$$

we end up with

$$a_E(2,0) \simeq Ik\sqrt{\frac{2}{15\pi^3}}\left(\frac{kd}{2}\right)^2\,.$$

This result should only be valid for $kd \ll 1$. Nevertheless, to compare with eq. (79), we bravely put $kd = 2\pi$ to obtain

$$a_E(2,0) \simeq Ik\sqrt{\frac{2\pi}{15}}\,, \tag{80}$$

which is larger than the exact result given in eq. (79) by a factor of $2\pi^2/15 \simeq 1.316$. Not too bad!

Note that in light of eq. (77), it follows from eq. (80) that

$$Q_{20} = \frac{2iI\sqrt{5\pi}}{ck^3}\,. \tag{81}$$

This can be converted into a Cartesian tensor using eq. (4.6) of Jackson. Since $Q_{2m} = Q_{20}\delta_{m0}$, it follows that $Q_{xx} = Q_{yy}$, $Q_{xy} = Q_{xz} = Q_{yz} = 0$, and

$$Q_{zz} = \frac{4Q_{20}}{\sqrt{5}} = \frac{8\pi iI}{ck^3}\,, \tag{82}$$

Since Q_{ij} is traceless, $Q_{xx} + Q_{yy} + Q_{zz} = 0$. Hence,

$$Q_{ij} = \frac{4\pi iI}{ck^3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\,. \tag{83}$$

We can verify the result of eq. (83) with a direct computation. Starting from

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\vec{\boldsymbol{x}})\, d^3x\,, \tag{84}$$

we insert the expression for $\rho(\vec{\boldsymbol{x}})$ given in eq. (69) after setting $d = 2\pi/k$,

$$\begin{aligned} Q_{ij} = \frac{I}{2\pi i c}\int_0^\infty \cos(kr)\Theta\left(\frac{\pi}{k} - r\right)\, dr \int_{-1}^1 \big[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\big]\, d\cos\theta \\ \times \int_0^{2\pi} (3x_i x_j - r^2\delta_{ij})\rho(\vec{\boldsymbol{x}})\, d\phi\,, \end{aligned} \tag{85}$$

where $\vec{\boldsymbol{x}} = r(\sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}})$. By inspection, the integral over ϕ vanishes for $i \neq j$. Moreover, eq. (85) immediately yields $Q_{xx} = Q_{yy}$ and $Q_{xx} + Q_{yy} + Q_{zz} = 0$. Thus, it is sufficient to evaluate Q_{zz}.

$$\begin{aligned} Q_{zz} &= -\frac{iI}{c}\int_0^{\pi/k} r^2\cos(kr)\, dr \int_{-1}^1 \big[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\big](3\cos^2\theta - 1)d\cos\theta \\ &= -\frac{4iI}{c}\int_0^{\pi/k} r^2\cos(kr)\, dr = \frac{8\pi i I}{ck^3}\,, \end{aligned} \tag{86}$$

which confirms the result previously obtained in eq. (82).

<u>*ADDENDUM*</u>:

As promised, we exhibit the necessary calculations to obtain $a_E(\ell, m)$ starting from eq. (67). In this method, one needs to keep track of the Heavyside step function, since it will generate a delta function when computing $\partial\rho/\partial r$ that cannot be ignored, as noted in footnote 4.

In this method, we use eq. (69) to compute

$$-\left(2 + r\frac{\partial}{\partial r}\right)\rho(\vec{\boldsymbol{x}}) = \big[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\big]\frac{I}{ickrd}\left\{\frac{2\pi}{d}\sin\left(\frac{2\pi r}{d}\right)\Theta(\tfrac{1}{2}d - r) + \cos\left(\frac{2\pi r}{d}\right)\delta(r - \tfrac{1}{2}d)\right\}.$$

The delta function piece can be simplified by using $\cos(2\pi r/d)\delta(r - \frac{1}{2}d) = \cos\pi\,\delta(r - \frac{1}{2}d) = -\delta(r - \frac{1}{2}d)$. Hence,

$$-\left(2 + r\frac{\partial}{\partial r}\right)\rho(\vec{\boldsymbol{x}}) = \big[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\big]\frac{I}{ickrd}\left\{\frac{2\pi}{d}\sin\left(\frac{2\pi r}{d}\right)\Theta(\tfrac{1}{2}d - r) - \delta(r - \tfrac{1}{2}d)\right\}. \tag{87}$$

Using eq. (70), we end up with

$$\begin{aligned} -c\left(2 + r\frac{\partial}{\partial r}\right)\rho + ik\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{J}} = \frac{2\pi I}{ikrd^2}\left\{\left[1 - \left(\frac{kd}{2\pi}\right)^2\right]\sin\left(\frac{2\pi r}{d}\right)\Theta(\tfrac{1}{2}d - r) - \frac{d}{2\pi}\delta(r - \tfrac{1}{2}d)\right\} \\ \times\big[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\big]\,. \end{aligned} \tag{88}$$

We now insert eq. (88) into eq. (67). Using eq. (72), and performing some algebraic simplifications, it follows that

$$a_E(\ell,m)=\frac{Ik}{2}\sqrt{\frac{2\ell+1}{\pi\ell(\ell+1)}}\delta_{m0}\big[1+(-1)^\ell\big]\left\{\pi j_\ell(\tfrac{1}{2}kd)+\left[k^2-\left(\frac{2\pi}{d}\right)^2\right]\int_0^{d/2}\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)\,r\,dr\right\},$$

which reproduces eq. (73).

(b) Compare the shape of the angular distribution of the radiated power for the lowest nonvanishing multipole with the exact distribution obtained in Jackson, problem 9.16 (on Problem Set 3)

Using eq. (9.151) of Jackson, the angular distribution of power for a pure electric multipole of order (ℓ,m) is given by,

$$\frac{dP(\ell,m)}{d\Omega}=\frac{Z_0}{2k^2}|a_E(\ell,m)|^2|\vec{\boldsymbol{X}}_{\ell m}|^2\,.$$

We apply this result to the exact form of the pure electric multipole of order $(\ell,m)=(2,0)$ obtained in eq. (79), which we rewrite again here,

$$a_E(2,0)=Ik\sqrt{\frac{15}{2\pi^3}}\,.$$

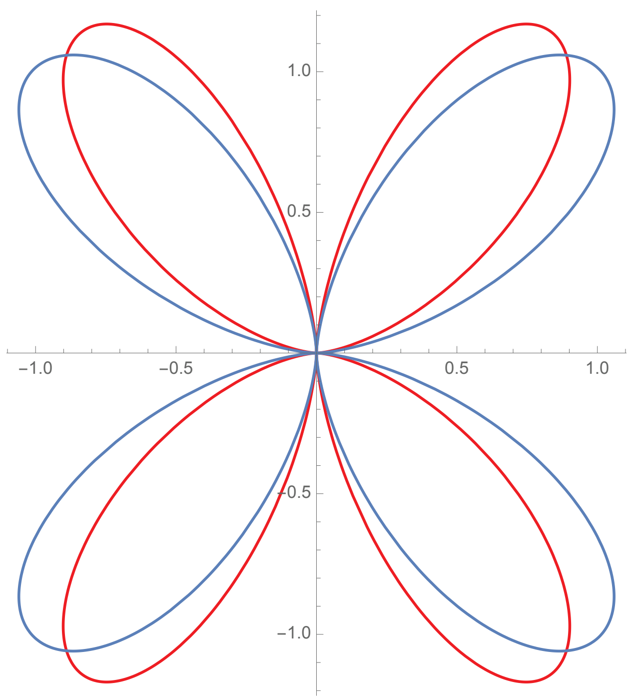

Figure 1: A polar plot of the antenna pattern of a thin linear antenna with a sinusoidal current that makes a full wavelength of oscillation. Normalization has been chosen such that $Z_0I^2=8\pi^2$. The angular distribution of the radiated power, shown in red, is given by eq. (90). This is compared with the corresponding angular distribution of the electric quadrupole component, shown in blue, which is given by eq. (89). This plot was created with Mathematica software.

Using Table 9.1 on p. 437 of Jackson,

$$|\vec{\boldsymbol{X}}_{\ell m}|^2 = \frac{15}{8\pi}\sin^2\theta\cos^2\theta\,,$$

Hence,

$$\frac{dP(2,0)}{d\Omega} = \frac{225 Z_0 I^2}{32\pi^4}\sin^2\theta\cos^2\theta\,. \tag{89}$$

This should be compared with the exact result,

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]^2\,. \tag{90}$$

obtained in Jackson, problem 9.16.

(c) Determine the total power radiated for the lowest multipole and the corresponding radiation resistance using both multipole moments from part (a). Compare with part (b) of Jackson, problem 9.16. Is there a paradox here?

The total power radiated by a pure electric multipole of order (ℓ, m) is given by eq. (9.154) of Jackson,

$$P(\ell,m) = \frac{Z_0}{2k^2}|a(\ell,m)|^2\,.$$

In part (b) we obtained two expressions for $a_E(2,0)$. The first expression was exact for $kd = 2\pi$ [cf. eq. (79)],

$$a_E(2,0) = Ik\sqrt{\frac{15}{2\pi^3}}\,. \tag{91}$$

The second was computed in the long-wavelength limit, but with $kd = 2\pi$ [cf. eq. (80)],

$$a_E(2,0) \simeq Ik\sqrt{\frac{2\pi}{15}}\,. \tag{92}$$

If we use the exact electric quadrupole result [eq. (91)], then we obtain

$$P(2,0) = \frac{15 Z_0 I^2}{4\pi^3}\,.$$

The corresponding radiative resistance (in ohms) is equal to the coefficient of $\frac{1}{2}I^2$ [cf. the text below eq. (9.29) of Jackson]. Thus, using $Z_0 = 376.7$ ohms [given below eq. (7.11)$'$ of Jackson],

$$R_{\rm rad} = \frac{15 Z_0}{2\pi^3} = 91.1\text{ ohms}\,, \tag{93}$$

which is remarkably close to the exact result,

$$R_{\rm rad} = (3.114)\,\frac{Z_0}{4\pi} = 93.3\text{ ohms}\qquad[\text{exact result}]\,, \tag{94}$$

obtained in part (b) of Jackson, problem 9.16. In contrast, had we used eq. (92), we would have obtained $R_{\rm rad} = 2\pi Z_0/15 = 157.8$ ohms, which is a terrible approximation, as one might have expected.

There is no paradox here. The discussion in Jackson on pp. 446–448 makes clear that keeping the lowest nonvanishing multipole but computing it exactly (i.e., without assuming that $kd \ll 1$) yields an accurate result to the exact antenna problem even for values of kd as large as 2π. Presumably, if one computes the next non-trivial multipole (in this problem, that wold be $\ell = 4$) its numerical contribution, the result would be a rather small correction to the power even when $kd = 2\pi$.

Perhaps the paradox that Jackson is alluding to is based on the expectation that,

$$P(2,0) < P_{\rm exact}\,,$$

since according to eq. (9.155) of Jackson, the total power is equal to an incoherent sum of contributions from all the multipoles. Indeed in our computations above, we did confirm that $P(2,0) < P_{\rm exact}$, or equivalently the radiation resistance of the electric quadrupole contribution given in eq. (93) is less than the exact result obtained in eq. (94). In contrast, the opposite (incorrect) conclusion would have been drawn had we used the expression for $P(2,0)$ based on setting $kd = 2\pi$ in the long wavelength limit [e.g., eq. (92)]. Of course, this latter result is an artifact of a poor approximation.

5. [Jackson, problem 12.3] A particle with mass m and charge e moves in a uniform, static, electric field $\vec{\boldsymbol{E}}_0$.

(a) Solve for the velocity and position of the particle as explicit functions of time, assuming that the initial velocity $\vec{\boldsymbol{v}}_0$ was perpendicular to the electric field.

Using eqs. (12.1) and (12.2) of Jackson and setting $\vec{\boldsymbol{B}} = 0$, we have:

$$\frac{d\vec{\boldsymbol{p}}}{dt} = e\vec{\boldsymbol{E}}\,, \qquad\qquad \frac{dW}{dt} = e\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{E}}\,,$$

where W is the total mechanical energy (usually called E, but we have renamed this W in order to better distinguish it from the electric field) and $\vec{\boldsymbol{v}}$ is the particle velocity (which is denoted as $\vec{\boldsymbol{u}}$ by Jackson).

Clearly, the motion takes place in a plane containing the $\vec{\boldsymbol{E}}$-field. Without loss of generality, we assume that

$$\vec{\boldsymbol{E}} = E\hat{\boldsymbol{x}}\,,$$

and assume that the motion takes place in the x–y plane. By assumption, $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{E}} = 0$ at $t = 0$, in which case $p_x = 0$ at $t = 0$. Solving the equations,

$$\frac{dp_x}{dt} = eE\,, \qquad\qquad \frac{dp_y}{dt} = 0\,, \tag{95}$$

in follows that

$$p_x = eEt\,, \qquad\qquad p_y = p_0\,,$$

where p_0 is a constant.

Using $\vec{\boldsymbol{p}} = \gamma m\vec{\boldsymbol{v}}$ and $W = \gamma mc^2$, it follows that[5]

$$\vec{\boldsymbol{v}} = \frac{c^2\vec{\boldsymbol{p}}}{W} = \frac{c^2\vec{\boldsymbol{p}}}{\sqrt{|\vec{\boldsymbol{p}}|^2c^2 + m^2c^4}}\,. \tag{96}$$

Hence,

$$v_x = \frac{c^2eEt}{\sqrt{(p_0^2 + e^2E^2t^2)c^2 + m^2c^4}}\,, \qquad v_y = \frac{c^2p_0}{\sqrt{(p_0^2 + e^2E^2t^2)c^2 + m^2c^4}}\,. \tag{97}$$

Since $\vec{\boldsymbol{v}} = d\vec{\boldsymbol{x}}/dt$, it follows that

$$x = c^2eE\int\frac{tdt}{\sqrt{W_0^2 + (ceEt)^2}}\,, \qquad y = c^2p_0\int\frac{dt}{\sqrt{W_0^2 + (ceEt)^2}}\,, \tag{98}$$

where $W_0^2 = p_0^2c^2 + m^2c^4$.

We shall define the origin of the coordinate system to coincide with $t = 0$. Then computing the integrals in eq. (98) yields

$$x(t) = \frac{1}{eE}\left[\sqrt{W_0^2 + (ceEt)^2} - W_0\right]\,, \qquad y(t) = \frac{p_0c}{eE}\sinh^{-1}\left(\frac{ceEt}{W_0}\right)\,. \tag{99}$$

<u>*REMARKS*</u>:

There is some temptation to first derive a differential equation for $\vec{\boldsymbol{v}}$ before attempting a solution. For example, starting from $\vec{\boldsymbol{v}} = c^2\vec{\boldsymbol{p}}/W$ [cf. eq. (96)], it follows that

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{c^2}{W}\frac{d\vec{\boldsymbol{p}}}{dt} - \frac{c^2\vec{\boldsymbol{p}}}{W^2}\frac{dW}{dt} = \frac{ec^2}{W}\vec{\boldsymbol{E}} - \frac{ec^2\vec{\boldsymbol{p}}}{W^2}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,. \tag{100}$$

Using $\vec{\boldsymbol{p}} = \gamma m\vec{\boldsymbol{v}}$ and $E = \gamma mc^2$, we obtain

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{\gamma m}\left[\vec{\boldsymbol{E}} - \frac{\vec{\boldsymbol{v}}}{c}\left(\frac{\vec{\boldsymbol{v}}}{c}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\right)\right]\,. \tag{101}$$

In terms of the x and y components of the velocity, eq. (101) is equivalent to:

$$\frac{dv_x}{dt} = \frac{eE}{\gamma m}\left(1 - \frac{v_x^2}{c^2}\right)\,, \tag{102}$$

$$\frac{dv_y}{dt} = -\frac{eEv_xv_y}{\gamma mc^2}\,, \tag{103}$$

where

$$\gamma \equiv \left(1 - \frac{v_x^2 + v_y^2}{c^2}\right)^{-1/2}\,, \tag{104}$$

subject to the boundary condition $v_x(t=0) = 0$ and $v_y(t=0) \equiv v_0$.

[5] Normally, we write the relativistic energy is given by $E = \gamma mc^2$. However, to avoid confusion with the electric field, I have denoted the relativistic energy by W.

If we were tasked to solve eqs. (102) and (103), it might not be obvious how to proceed. However, in light of the solution to eq. (95), the method is clear. Namely, we can multiply eqs. (102) and (103) by γ and use

$$\gamma\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d}{dt}(\gamma\vec{\boldsymbol{v}}) - \vec{\boldsymbol{v}}\,\frac{d\gamma}{dt}\,. \tag{105}$$

Next we would make use of

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1-\frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}}{c^2}\right)^{-1/2} = \frac{\gamma^3}{c^2}\vec{\boldsymbol{v}}\cdot\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{mc^2}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{E}} = \frac{eEv_x}{mc^2}\,, \tag{106}$$

after employing eq. (101) in the penultimate step above. Thus, eqs. (102) and (103) yield

$$\frac{d}{dt}(\gamma v_x) = v_x\frac{d\gamma}{dt} + \frac{eE}{m}\left(1-\frac{v_x^2}{c^2}\right) = \frac{eE}{m}\,, \tag{107}$$

$$\frac{d}{dt}(\gamma v_y) = v_y\frac{d\gamma}{dt} - \frac{eEv_xv_y}{mc^2} = 0\,, \tag{108}$$

subject to the boundary condition $v_x(t=0)=0$ and $v_y(t=0)\equiv v_0$. Of course, we have simply reproduced eq. (95). For completeness, we can now trivially solve eqs. (107) and (108):

$$\gamma v_x = \frac{eEt}{m}\,, \qquad\qquad \gamma v_y = \gamma_0 v_0 = \frac{p_0}{m}\,, \tag{109}$$

where $\gamma_0 \equiv (1-v_0^2/c^2)^{-1/2}$ and we have taken the constant of integration to be $\gamma_0 v_0 \equiv p_0/m$, which defines p_0. Squaring these two equations and adding, we obtain

$$\gamma^2 v^2 = \frac{p_0^2 + e^2E^2t^2}{m^2}\,, \tag{110}$$

where $v^2 = v_x^2 + v_y^2$. Inserting $\gamma^2v^2 = c^2(\gamma^2-1)$ above, it follows that

$$\gamma = \frac{\sqrt{m^2c^2 + p_0^2 + e^2E^2t^2}}{mc}\,. \tag{111}$$

Plugging eq. (111) into eq. (109), we recover the expressions for v_x and v_y previously obtained in eq. (97).

(b) Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define "short" and "long" times).

We can eliminate t from eq. (99),

$$t = \frac{W_0}{ceE}\sinh\left(\frac{eEy}{p_0c}\right)\,.$$

Inserting this into the equation for $x(t)$ and using the identity $\cosh^2 z - \sinh^2 z = 1$, it follows that

$$x = \frac{W_0}{eE}\left[\cosh\left(\frac{eEy}{p_0c}\right) - 1\right]\,,$$

which is the equation for a catenary curve.

To describe the shape of the path for short and long times, we note that $W_0/(ceE)$ has units of time. This we can define short and long times relative to this quantity. For $t \ll W_0/(ceE)$, we have

$$\sqrt{W_0^2 + (ceEt)^2} \simeq W_0 + \frac{(ceEt)^2}{2W_0}\,, \qquad\qquad \sinh^{-1}\left(\frac{ceEt}{W_0}\right) \simeq \frac{ceEt}{W_0}\,.$$

Hence the approximate form of eq. (99) is

$$x(t) \simeq \frac{c^2eEt^2}{2W_0}\,, \qquad\qquad y(t) \simeq \frac{p_0c^2t}{W_0}\,.$$

Solving for t and inserting the result back into the above equations yields

$$x \simeq \frac{eEW_0y^2}{2p_0^2c^2}\,.$$

Since $v_0 = c^2p_0/W_0$, we can eliminate W_0 from the above expression to obtain,

$$x \simeq \frac{eEy^2}{2p_0v_0}\,. \tag{112}$$

That is, as short times, the motion is parabolic.[6]

For $t \gg W_0/(ceE)$, eq. (99) yields:

$$x(t) \simeq ct\,, \qquad\qquad y(t) \simeq \frac{p_0c}{eE}\ln\left(\frac{2ceEt}{W_0}\right)\,.$$

In the latter case, we used:

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right) \simeq \ln 2z\,, \quad \text{for } z \gg 1\,.$$

Hence, to a good approximation,

$$y \simeq \frac{p_0c}{eE}\ln\left(\frac{2eEx}{W_0}\right)\,,$$

or equivalently,

$$x \simeq \frac{W_0}{2eE}\exp\left(\frac{eEy}{p_0c}\right)\,.$$

That is, at long times the motion is exponential.

[6]The result of eq. (112) also coincides with the non-relativistic limit (in which case $p_0 = mv_0$). To verify this assertion, we can perform a formal expansion in powers of $1/c$. In this limit, $W_0 \simeq mc^2$ and

$$t \ll \frac{W_0}{ceE} \simeq \frac{mc}{eE}\,,$$

which is always true in the limit of $c \to \infty$ (which is equivalent to taking the non-relativistic limit).

6. [Jackson, problem 12.11] Consider the precession of the spin of a muon, initially longitudinally polarized, as the muon moves in a circular orbit in a plane perpendicular to a uniform magnetic field $\vec{\boldsymbol{B}}$.

(a) Show that the difference Ω of the spin precession frequency and the orbital gyration frequency is

$$\Omega = \frac{eBa}{m_\mu c}\,,$$

independent of the muon's energy, where $a = \frac{1}{2}(g-2)$ is the magnetic moment anomaly. Find the equations of motion for the components of the spin along the mutually perpendicular directions defined by the particle's velocity, the radius vector from the center of the circle to the particle, and the magnetic field.

Our starting point is the Thomas equation, which Jackson writes in the following form [cf. eq. (11.170) of Jackson]:

$$\frac{d\vec{\boldsymbol{s}}}{dt} = \frac{e}{mc}\,\vec{\boldsymbol{s}}\times\left\{\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\vec{\boldsymbol{B}} - \left(\frac{g}{2}-1\right)\frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}} - \left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\right\}\,, \tag{113}$$

where the time derivative of the velocity vector is given by [cf. eq. (11.168) of Jackson]:

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{e}{\gamma mc}\left[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\right]\,. \tag{114}$$

For a particle moving in a circular orbit in a plane perpendicular to a uniform magnetic field $\vec{\boldsymbol{B}}$, we have $\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = 0$, where $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$ is the particle velocity. Hence, eqs. (113) and (114) reduce to

$$\frac{d\vec{\boldsymbol{s}}}{dt} = \frac{e}{mc}\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\vec{\boldsymbol{s}}\times\vec{\boldsymbol{B}}\,, \qquad \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{\gamma mc}\,\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}\,, \tag{115}$$

since by assumption there is no electric field present ($\vec{\boldsymbol{E}} = 0$). That is, eq. (115) can be written in the form of precession equations,

$$\frac{d\vec{\boldsymbol{s}}}{dt} = \vec{\boldsymbol{s}}\times\vec{\boldsymbol{\omega}}\,, \qquad \frac{d\vec{\boldsymbol{v}}}{dt} = \vec{\boldsymbol{v}}\times\vec{\boldsymbol{\omega}}_B\,,$$

where the spin precession frequency $\vec{\boldsymbol{\omega}}$ and the orbital gyration frequency $\vec{\boldsymbol{\omega}}_B$ are given by:

$$\vec{\boldsymbol{\omega}} \equiv \frac{e}{\gamma mc}\left[1+\left(\frac{g-2}{2}\right)\gamma\right]\vec{\boldsymbol{B}}\,, \qquad \vec{\boldsymbol{\omega}}_B \equiv \frac{e}{\gamma mc}\vec{\boldsymbol{B}}\,.$$

The difference of these two frequencies is

$$\vec{\boldsymbol{\Omega}} \equiv \vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\omega}}_B = \frac{e}{mc}\left(\frac{g-2}{2}\right)\vec{\boldsymbol{B}}\,,$$

and the magnitude of this frequency difference is given by

$$\Omega = \frac{eBa}{mc}\,, \qquad \text{where } a \equiv \tfrac{1}{2}(g-2)\,.$$

To find the equations of motion for the components of the spin vector, we first decompose this vector into longitudinal and transverse components with respect to the direction of the velocity, $\hat{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{\beta}}/\beta$. That is, $\vec{\boldsymbol{s}} = \vec{\boldsymbol{s}}_\parallel + \vec{\boldsymbol{s}}_\perp$, where

$$\vec{\boldsymbol{s}}_\parallel = (\hat{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{s}})\hat{\boldsymbol{\beta}}\,, \qquad\qquad \vec{\boldsymbol{s}}_\perp = \vec{\boldsymbol{s}} - \vec{\boldsymbol{s}}_\parallel\,.$$

By construction,

$$\vec{\boldsymbol{s}}_\perp \boldsymbol{\cdot} \hat{\boldsymbol{\beta}} = 0\,. \tag{116}$$

We first work out $d\vec{\boldsymbol{s}}_\parallel/dt$.

$$\frac{d\vec{\boldsymbol{s}}_\parallel}{dt} = \frac{d}{dt}\Big((\hat{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{s}})\hat{\boldsymbol{\beta}}\Big) = \hat{\boldsymbol{\beta}}\,\frac{d}{dt}\Big(\hat{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{s}}\Big) + \vec{\boldsymbol{s}} \boldsymbol{\cdot} \hat{\boldsymbol{\beta}}\,\frac{d\hat{\boldsymbol{\beta}}}{dt}\,. \tag{117}$$

Jackson gives the following result in his eq. (11.171),

$$\frac{d}{dt}\Big(\hat{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{s}}\Big) = -\frac{e}{mc}\vec{\boldsymbol{s}}_\perp \boldsymbol{\cdot} \left[\left(\frac{g}{2} - 1\right)\hat{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}} + \left(\frac{g\beta}{2} - \frac{1}{\beta}\right)\vec{\boldsymbol{E}}\right]\,.$$

Setting $\vec{\boldsymbol{E}} = 0$ and defining $\hat{\boldsymbol{B}} \equiv \vec{\boldsymbol{B}}/B$ (where $B \equiv |\vec{\boldsymbol{B}}|$), we obtain

$$\frac{d}{dt}\Big(\hat{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{s}}\Big) = -\frac{eB}{mc}\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_\perp \boldsymbol{\cdot} (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})\,. \tag{118}$$

We also need to work out $d\hat{\boldsymbol{\beta}}/dt$.

$$\frac{d\hat{\boldsymbol{\beta}}}{dt} = \frac{d}{dt}\left(\frac{\vec{\boldsymbol{\beta}}}{\beta}\right) = \frac{1}{\beta}\,\frac{d\vec{\boldsymbol{\beta}}}{dt} - \frac{\vec{\boldsymbol{\beta}}}{\beta^2}\,\frac{d\beta}{dt}\,. \tag{119}$$

Using

$$\frac{d\beta}{dt} = \frac{d}{dt}\Big(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{\beta}}\Big)^{1/2} = \frac{1}{2}\Big(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{\beta}}\Big)^{-1/2}\frac{d}{dt}\Big(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{\beta}}\Big) = \frac{1}{2\beta}\,2\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \frac{d\vec{\boldsymbol{\beta}}}{dt} = \hat{\boldsymbol{\beta}} \boldsymbol{\cdot} \frac{d\vec{\boldsymbol{\beta}}}{dt}\,,$$

in eq. (119), we conclude that

$$\frac{d\hat{\boldsymbol{\beta}}}{dt} = \frac{1}{\beta}\left[\frac{d\vec{\boldsymbol{\beta}}}{dt} - \hat{\boldsymbol{\beta}}\left(\hat{\boldsymbol{\beta}} \boldsymbol{\cdot} \frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\right]\,.$$

From eq. (115), we obtain

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{e}{\gamma mc}\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}\,.$$

Hence $\hat{\boldsymbol{\beta}} \boldsymbol{\cdot} d\vec{\boldsymbol{\beta}}/dt = 0$, and we end up with

$$\frac{d\hat{\boldsymbol{\beta}}}{dt} = \frac{eB}{\gamma mc}\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}}\,. \tag{120}$$

Inserting eqs. (118) and (120) into eq. (117), we obtain

$$\frac{ds_\parallel}{dt} = -\frac{eB}{mc}\left(\frac{g-2}{2}\right)[\vec{\boldsymbol{s}}_\perp \cdot (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})]\hat{\boldsymbol{\beta}} + \frac{eB}{\gamma mc}\,\vec{\boldsymbol{s}} \cdot \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})\,.$$

Since $\vec{\boldsymbol{s}}_\parallel \equiv (\vec{\boldsymbol{s}} \cdot \hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}}$, it immediately follows that

$$\vec{\boldsymbol{s}} \cdot \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}}) = \vec{\boldsymbol{s}}_\parallel \times \vec{\boldsymbol{B}}\,.$$

We can further simplify the quantity $[\vec{\boldsymbol{s}}_\perp \cdot (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})]\hat{\boldsymbol{\beta}}$ by using $\vec{\boldsymbol{s}}_\perp \cdot \hat{\boldsymbol{\beta}} = 0$ [cf. eq. (116)] and $\hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{B}} = 0$. First, consider the triple cross product

$$\vec{\boldsymbol{s}}_\perp \times \left[\hat{\boldsymbol{\beta}} \times (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})\right] = [\vec{\boldsymbol{s}}_\perp \cdot (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})]\hat{\boldsymbol{\beta}} - (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})\vec{\boldsymbol{s}}_\perp \cdot \hat{\boldsymbol{\beta}} = [\vec{\boldsymbol{s}}_\perp \cdot (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})]\hat{\boldsymbol{\beta}}\,.$$

However, $\hat{\boldsymbol{\beta}} \times (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}}) = \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{B}}) - \hat{\boldsymbol{B}} = -\hat{\boldsymbol{B}}$. Hence,

$$[\vec{\boldsymbol{s}}_\perp \cdot (\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}})]\hat{\boldsymbol{\beta}} = -\vec{\boldsymbol{s}}_\perp \times \hat{\boldsymbol{B}}\,.$$

Inserting eqs. (122) and (123) into eq. (121) then yields

$$\boxed{\frac{d\vec{\boldsymbol{s}}_\parallel}{dt} = \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_\perp + \frac{1}{\gamma}\,\vec{\boldsymbol{s}}_\parallel\right] \times \hat{\boldsymbol{B}}}$$

Using this result, we can evaluate $d\vec{\boldsymbol{s}}_\perp/dt$.

$$\frac{d\vec{\boldsymbol{s}}_\perp}{dt} = \frac{d}{dt}\left(\vec{\boldsymbol{s}} - \vec{\boldsymbol{s}}_\parallel\right) = \frac{eB}{mc}\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)(\vec{\boldsymbol{s}}_\parallel + \vec{\boldsymbol{s}}_\perp) \times \vec{\boldsymbol{B}} = \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_\perp + \frac{1}{\gamma}\,\vec{\boldsymbol{s}}_\parallel\right] \times \hat{\boldsymbol{B}}\,,$$

which simplifies to

$$\boxed{\frac{d\vec{\boldsymbol{s}}_\perp}{dt} = \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_\parallel + \frac{1}{\gamma}\,\vec{\boldsymbol{s}}_\perp\right] \times \hat{\boldsymbol{B}}}$$

Finally, we need to further decompose $\vec{\boldsymbol{s}}_\perp$ into components along the direction of the magnetic field and along the direction of the unit radius vector $\hat{\boldsymbol{r}}$ that points to the center of the circular path of the moving spin. In light of eq. (114) [with $\vec{\boldsymbol{E}} = 0$], $d\vec{\boldsymbol{v}}/dt \propto \hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{B}}$, where $\hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{B}} = 0$. But for circular motion, $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\beta}} = 0$ and the acceleration $d\vec{\boldsymbol{v}}/dt$ points radially into the origin, i.e. $d\vec{\boldsymbol{v}}/dt \propto -\hat{\boldsymbol{r}}$. It follows that $\hat{\boldsymbol{r}} = \hat{\boldsymbol{B}} \times \hat{\boldsymbol{\beta}}$, and we conclude that the unit vectors $\{\hat{\boldsymbol{B}}\,,\,\hat{\boldsymbol{\beta}}\,,\,\hat{\boldsymbol{r}}\}$ form a mutually orthonormal right-handed triad of vectors. Thus, we can write:

$$\vec{\boldsymbol{s}}_\perp \equiv \vec{\boldsymbol{s}}_B + \vec{\boldsymbol{s}}_r\,, \qquad \text{where} \;\; \vec{\boldsymbol{s}}_B \equiv (\vec{\boldsymbol{s}} \cdot \hat{\boldsymbol{B}})\hat{\boldsymbol{B}} \text{ and } \vec{\boldsymbol{s}}_r \equiv (\vec{\boldsymbol{s}} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,. \tag{121}$$

Note that

$$\frac{d\vec{\boldsymbol{s}}_B}{dt} = \left(\hat{\boldsymbol{B}} \cdot \frac{d\vec{\boldsymbol{s}}}{dt}\right)\hat{\boldsymbol{B}} = 0\,, \tag{122}$$

since $\vec{\boldsymbol{B}}$ is time-independent by assumption and

$$\vec{\boldsymbol{B}} \cdot \frac{d\vec{\boldsymbol{s}}}{dt} \propto \vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{s}} \times \vec{\boldsymbol{B}}) = 0\,,$$

in light of eq. (115). Thus, $\vec{\boldsymbol{s}}_B$ is a constant in time, from which it follows that

$$\frac{d\vec{\boldsymbol{s}}_r}{dt} = \frac{d}{dt}\left(\vec{\boldsymbol{s}}_\perp + \vec{\boldsymbol{s}}_B\right) = \frac{d\vec{\boldsymbol{s}}_\perp}{dt}\,. \tag{123}$$

Hence, the equations of motion for the components of the spin vector are:

$$\begin{aligned}
\frac{d\vec{\boldsymbol{s}}_B}{dt} &= 0\,,\\
\frac{d\vec{\boldsymbol{s}}_r}{dt} &= \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_\parallel + \frac{1}{\gamma}\vec{\boldsymbol{s}}_r\right]\times\hat{\boldsymbol{B}}\,,\\
\frac{d\vec{\boldsymbol{s}}_\parallel}{dt} &= \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_r + \frac{1}{\gamma}\vec{\boldsymbol{s}}_\parallel\right]\times\hat{\boldsymbol{B}}\,,
\end{aligned}$$

after using $\vec{\boldsymbol{s}}_B \times \hat{\boldsymbol{B}} = (\vec{\boldsymbol{s}}\boldsymbol{\cdot}\hat{\boldsymbol{B}})\hat{\boldsymbol{B}}\times\hat{\boldsymbol{B}} = 0$.

(b) For the CERN Muon Storage Ring, the orbit radius is $R = 2.5$ meters and $B = 17\times 10^3$ gauss. What is the momentum of the muon? What is the time dilation factor γ? How many periods of precession $T = 2\pi/\Omega$ occur per observed laboratory mean lifetime of the muons? [Relevant data: $m_\mu = 105.66$ MeV, $\tau_0 = 2.2\times 10^{-6}$ s, $a \simeq \alpha/(2\pi)$ where $\alpha \simeq 1/137$.]

For circular motion,

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = -\frac{v^2}{R}\,\hat{\boldsymbol{r}}\,. \tag{124}$$

Since the circular motion is in a plane that is perpendicular to the magnetic field $\vec{\boldsymbol{B}}$, it follows that $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{v}}$ and $\hat{\boldsymbol{r}}$ are mutually orthogonal vectors. Moreover, eqs. (12.38) and (12.39) of Jackson yield

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{\gamma mc}\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}\,. \tag{125}$$

Thus, if $\vec{\boldsymbol{B}}$ points in the z-direction, then $\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{\theta}}$ and the circular motion is clockwise in the x–y plane. Combining eqs. (124) and (125), it follows that

$$\gamma mv = \frac{eBR}{c}\,, \tag{126}$$

which we recognize as the relativistic momentum of the muon, p_μ. Using eq. (12.42) of Jackson, we can rewrite eq. (126) as

$$p_\mu\ (\text{MeV/c}) = 3\times 10^{-4}\ BR\ (\text{gauss-cm})\,. \tag{127}$$

The factor of 3×10^{-4} in eq. (127) arises as follows. In gaussian units, $e = 4.8\times 10^{-10}$ esu and 1 MeV$= 1.6\times 10^{-6}$ ergs. Hence, the conversion factor between ergs and MeV is

$$4.8\times 10^{-10}/1.6\times 10^{-6} = 3\times 10^{-4}.$$

Thus we end up with

$$p_\mu = (3 \times 10^{-4})(1.7 \times 10^4)(250)\ \text{MeV/c} = 1.275 \times 10^3\ \text{MeV/c}\,.$$

The γ-factor is

$$\gamma = \frac{E}{mc^2} = \frac{(p^2c^2 + m^2c^4)^{1/2}}{mc^2} = \left(\frac{p^2}{m^2c^2} + 1\right)^{1/2}\,.$$

The muon rest energy is $mc^2 = 105.66$ MeV. Hence,

$$\gamma = \left[1 + \frac{(1.275 \times 10^3)^2}{(105.66)^2}\right]^{1/2} = 12.11\,.$$

The number of periods of precession, $T = 2\pi/\Omega$, occurring per observed mean muon lifetime, $\gamma\tau_0 = \gamma(2.2 \times 10^{-6}\ \text{s})$, is given by[7]

$$\frac{\gamma\tau_0}{T} = \frac{\gamma\tau_0\Omega}{2\pi} = \frac{\gamma\tau_0 eBa}{2\pi mc} = \frac{\gamma^2\tau_0 va}{2\pi R}\,,$$

where eq. (126) was used to arrive at the final result above. Since $\gamma \gg 1$, we can approximate $v \simeq c$. In addition, we take

$$a = \tfrac{1}{2}(g-2) \simeq \frac{\alpha}{2\pi}\,, \qquad \text{where } \alpha \simeq \frac{1}{137}\,,$$

as predicted at lowest non-trivial order in quantum electrodynamics. Hence,

$$\frac{\gamma\tau_0}{T} \simeq \frac{\gamma^2\tau_0 c\alpha}{4\pi^2 R} = \frac{(12.11)^2(2.2 \times 10^{-6}\ \text{s})(3 \times 10^{10}\ \text{cm s}^{-1})}{4\pi^2(250\ \text{cm})(137)} = 7.156\,.$$

(c) Express the difference frequency Ω in units of orbital rotation frequency and compute how many precessional periods (at the difference frequency) occur per rotation for a 300 MeV muon, a 300 MeV electron, a 5 GeV electron (this last typical of the e^+e^- storage ring at Cornell).

NOTE: The energy values above correspond to the total relativistic energies.

For a 300 MeV muon,

$$\gamma = \frac{E}{mc^2} = \frac{300}{105.66} = 2.839\,,$$

and

$$\Omega = \frac{eBa}{mc} = \gamma\omega_B a \simeq \frac{\gamma\omega_B\alpha}{2\pi} = 3.3 \times 10^{-3}\omega_B\,.$$

[7]Note that in the laboratory frame, the observed muon lifetime is given by $\gamma\tau_0$, where τ_0 is the muon lifetime in the muon rest frame.

One revolution occurs in time $t = 2\pi R/v$. In this time, the number of periods of precession, $T = 2\pi/\Omega$, is given by

$$\frac{t}{T} = \left(\frac{2\pi R}{v}\right)\left(\frac{\Omega}{2\pi}\right) = \frac{\Omega R}{v}\,.$$

We can rewrite the above result using eq. (126), which yields

$$\frac{R}{v} = \frac{\gamma mc}{eB} = \frac{1}{\omega_B}\,.$$

Hence, for a 300 MeV muon, we have

$$\frac{t}{T} = \frac{\Omega}{\omega_B} \simeq \frac{\gamma\alpha}{2\pi} = 3.3 \times 10^{-3}\,.$$

For a 300 MeV electron, we use $m_e c^2 = 511$ keV to obtain $\gamma = 300/0.511 = 587$. Hence,

$$\frac{t}{T} = \frac{\Omega}{\omega_B} \simeq \frac{\gamma\alpha}{2\pi} = 0.682\,.$$

Finally, for a 5 GeV electron, we have $\gamma = 500/0.511 = 9.785 \times 10^3$. It follows that

$$\frac{t}{T} = \frac{\Omega}{\omega_B} \simeq \frac{\gamma\alpha}{2\pi} = 11.37\,.$$