1. [Jackson, problem 14.4] Using the Liénard-Wiechert fields, discuss the time-averaged power radiated per unit solid angle in nonrelativistic motion of a point particle with charge $e$, moving:

(a) along the $z$ axis with instantaneous position $z(t) = a\cos\omega_0(t)$ ,

(b) in a circle of radius $R$ in the $x$–$y$ plane with constant angular frequency $\omega_0$.

Sketch the angular distribution of the radiation of the radiation and determine the total power radiated in each case.

(a) Case 1: Non-relativistic motion of a point particle with charge $e$ moving along the $z$-axis with instantaneous position $z(t) = a\cos\omega_0(t)$ .

We make use of eq. (14.20) of Jackson, which is relevant for non-relativistic motion,

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\left|\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\right|^2 , \tag{1}$$

where

$$\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{v}}}{c} = \frac{1}{c}\frac{d\vec{\boldsymbol{x}}}{dt} .$$

In this case, we have

$$\vec{\boldsymbol{x}}(t) = \hat{\boldsymbol{z}}\, a\cos\omega_0 t ,$$

which yields

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = -\hat{\boldsymbol{z}}\,\frac{a\omega_0^2}{c}\,\cos\omega_0 t .$$

Working out the absolute square of the triple product in eq. (1),

$$\left|\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\right|^2 = \left|\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right) - \frac{d\vec{\boldsymbol{\beta}}}{dt}\right|^2 = \left|\frac{d\vec{\boldsymbol{\beta}}}{dt}\right|^2 - \left(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2 \tag{2}$$

$$= \frac{a^2\omega_0^4}{c^2}\cos^2\omega_0 t\left[1-(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{z}})^2\right] = \frac{a^2\omega_0^4}{c^2}\cos^2\omega_0 t\,\sin^2\theta .$$

In obtaining the final result above, we chose to work in a coordinate system in which the origin corresponds to the instantaneous position of the charged particle, and the unit vector $\hat{\boldsymbol{n}}$ has polar angle $\theta$ and azimuthal angle $\phi$ with respect to the $z$-axis,

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\sin\theta\,\cos\phi + \hat{\boldsymbol{y}}\sin\theta\,\sin\phi + \hat{\boldsymbol{z}}\,\cos\theta . \tag{3}$$

The time-averaged power is easily obtained by noting that[1]

$$\langle\cos^2\omega_0 t\rangle = \tfrac{1}{2} .$$

[1] To compute the time-average of $\cos^2\omega_0 t$, note that the time averages satisfy $\langle\cos^2\omega_0 t\rangle = \langle\sin^2\omega_0 t\rangle$, and $\cos^2\omega_0 t + \sin^2\omega_0 t = 1$.

Hence, it follows that

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2 a^2 \omega_0^4}{8\pi c^3} \sin^2\theta \,. \tag{4}$$

In Figure 1, the angular distribution of the radiated power is exhibited as a polar plot.

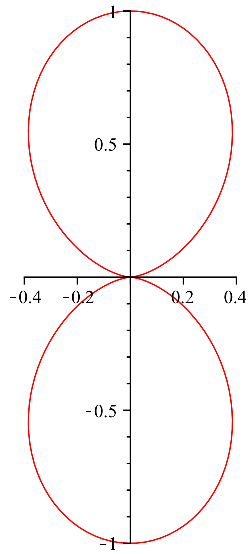



Figure 1: A polar plot of the angular distribution of the power radiated by a charged particle moving non-relativistically along the $z$ axis with instantaneous position $z(t) = a\cos\omega_0(t)$. The angular distribution is given by eq. (4) and is proportional to $\sin^2\theta$. This plot was created with Maple software.

Integrating over the solid angle yields the total radiated power,

$$\langle P \rangle = \frac{e^2 a^2 \omega_0^4}{3c^3} \,.$$

(b) Case 2: Non-relativistic motion of a point particle with charge $e$ moving in a circle of radius $R$ in the $x$–$y$ plane with constant angular frequency $\omega_0$.

For circular motion in the $x$–$y$ plane, the trajectory of the particle is given by

$$\vec{\boldsymbol{x}}(t) = R(\hat{\boldsymbol{x}} \cos\omega_0 t + \hat{\boldsymbol{y}} \sin\omega_0 t) \,.$$

Then, we easily compute

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{1}{c}\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = -\frac{\omega_0^2}{c}\,\vec{\boldsymbol{x}}(t) \,.$$

We again choose to work in a coordinate system in which the origin corresponds to the instantaneous position of the charged particle, and the unit vector $\hat{\boldsymbol{n}}$ given by eq. (3) has polar angle $\theta$ and azimuthal angle $\phi$ with respect to the $z$-axis. Consequently,

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt} = -\frac{\omega_0^2 R}{c}\left(\cos\omega_0 t\, \sin\theta\, \cos\phi + \sin\omega_0 t\, \sin\theta\, \sin\phi\right) \,.$$

Evaluating the absolute square of the triple cross product as in part (a) [cf. eq. (2)], we obtain:

$$\left|\hat{\boldsymbol{n}} \times \left(\hat{\boldsymbol{n}} \times \frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\right|^2 = \frac{\omega_0^4 R^2}{c^2}\left[1 - \sin^2\theta(\cos\phi\,\cos\omega_0 t + \sin\phi\,\sin\omega_0 t)^2\right]$$
$$= \frac{\omega_0^4 R^2}{c^2}\left[1 - \sin^2\theta\,\cos^2(\omega_0 t - \phi)\right]\,.$$

Using eq. (1), it follows that

$$\frac{dP}{d\Omega} = \frac{e^2\omega_0^4 R^2}{4\pi c^3}\left[1 - \sin^2\theta\,\cos^2(\omega_0 t - \phi)\right]\,.$$

The time-averaged power is easily obtained by noting that $\langle\cos^2(\omega_0 t - \phi)\rangle = \frac{1}{2}$. Employing the trigonometric identity, $1 - \frac{1}{2}\sin^2\theta = \frac{1}{2}(1 + \cos^2\theta)$, it follows that

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2\omega_0^4 R^2}{8\pi c^3}\left(1 + \cos^2\theta\right)\,. \qquad (5)$$

In Figure 2, the angular distribution of the radiated power is exhibited as a polar plot.

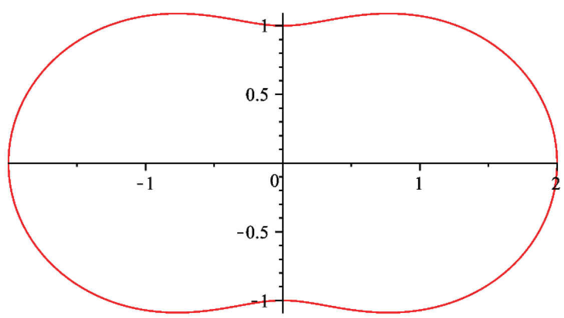



Figure 2: A polar plot of the angular distribution of the power radiated by a charged particle moving non-relativistically in a circle of radius $R$ in the $x$–$y$ plane with constant angular frequency $\omega_0$. The angular distribution is given by eq. (5) and is proportional to $1 + \cos^2\theta$. This plot was created with Maple software.

Integrating over solid angles yields the total radiated power,

$$\langle P\rangle = \frac{2e^2\omega_0^4 R^2}{3c^3}\,.$$

2. [Jackson, problem 14.8] A swiftly moving particle of charge $ze$ and mass $m$ passes a fixed point charge $Ze$ in an approximately straight-line path at impact parameter $b$ and nearly constant speed $v$. Show that the total energy radiated in the encounter is

$$\Delta W = \frac{\pi z^4 Z^2 e^6}{4m^2 c^4 \beta}\left(\gamma^2 + \frac{1}{3}\right)\frac{1}{b^3} \tag{6}$$

Using the Liénard result for the total power emitted by an accelerating charge given in eq. (14.26), suitably adjusted for a charge $ze$, we have

$$P = \frac{2z^2e^2}{3c}\,\gamma^6\left[\left(\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2 - \left(\vec{\boldsymbol{\beta}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2\right], \tag{7}$$

Note that

$$\left(\vec{\boldsymbol{\beta}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2 = \beta^2\left(\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2 - \left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2 .$$

Inserting this result back in eq. (7) and using $1-\beta^2=\gamma^{-2}$, it follows that

$$P = \frac{2z^2e^2}{3c}\,\gamma^4\left[\left(\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2 + \gamma^2\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2\right]. \tag{8}$$

We can solve for $d\vec{\boldsymbol{\beta}}/dt$ using the relativistic version of Newton's second law. Using eq. (23) of the class handout, *Examples of four-vectors*,

$$\vec{\boldsymbol{F}} = \gamma mc\left[\frac{d\vec{\boldsymbol{\beta}}}{dt} + \gamma^2\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\vec{\boldsymbol{\beta}}\right]. \tag{9}$$

Taking the dot product of this equation with $\vec{\boldsymbol{\beta}}$, and using $1+\gamma^2\beta^2=\gamma^2$,

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{F}}}{\gamma^3 mc}. \tag{10}$$

Inserting this back into eq. (9) yields,

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{1}{\gamma mc}\Big[\vec{\boldsymbol{F}} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{F}})\Big]. \tag{11}$$

We can now insert the results of eqs. (10) and (11) into eq. (8) to obtain:

$$\begin{aligned}
P &= \frac{2z^2e^2}{3m^2c^3}\Big[\gamma^2\big[\vec{\boldsymbol{F}} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{F}})\big]^2 - (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{F}})^2\Big] \\
&= \frac{2z^2e^2}{3m^2c^3}\Big[\gamma^2\vec{\boldsymbol{F}}^{\,2} + (1-2\gamma^2)(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{F}})^2 + (\gamma^2-1)(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{F}})^2\Big] \\
&= \frac{2z^2e^2\gamma^2}{3m^2c^3}\Big[\vec{\boldsymbol{F}}^{\,2} - (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{F}})^2\Big],
\end{aligned} \tag{12}$$

after using $\gamma^2\beta^2=\gamma^2-1$ in the penultimate step above.

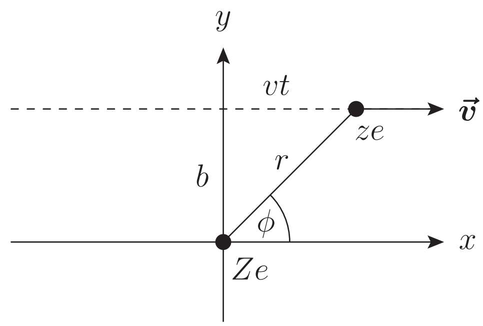
The two charges are subject to a Coulomb potential,

$$V(r) = \frac{zZe^2}{r}\,.$$

The corresponding force is

$$\vec{\boldsymbol{F}} = -\vec{\boldsymbol{\nabla}}V(r) = -\hat{\boldsymbol{r}}\frac{dV}{dr} = \frac{zZe^2}{r^2}\,\hat{\boldsymbol{r}}\,, \tag{13}$$

where $\hat{\boldsymbol{r}}$ is a unit vector that points in the radial direction. The geometry of the problem is exhibited below:




A particle with charge $ze$ is moving parallel to the $x$-axis along the dashed line[2] (which is a distance $b$ from the $x$-axis) with velocity $\vec{\boldsymbol{v}}$. A heavy particle with charge $Ze$ is located at the origin. The position of the charge $ze$ at time $t$ (which lies in the $x$–$y$ plane) has azimuthal angle $\phi$ and polar angle $\theta = \frac{1}{2}\pi$. Note that $\sin\phi = b/r$. Hence,

$$\vec{\boldsymbol{F}}^{\,2} - \bigl(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{F}}\bigr)^2 = \frac{z^2Z^2e^4}{r^4}\Bigl(1-\beta^2\cos^2\phi\Bigr)\,. \tag{14}$$

Noting that

$$\sin\phi = \frac{b}{r}\,, \qquad\qquad \cos\phi = \frac{\sqrt{r^2-b^2}}{r}\,, \tag{15}$$

it follows that

$$1-\beta^2\cos^2\phi = \frac{r^2(1-\beta^2)+\beta^2b^2}{r^2} = \frac{r^2+\gamma^2\beta^2b^2}{\gamma^2r^2} = \frac{r^2+(\gamma^2-1)b^2}{\gamma^2r^2}\,. \tag{16}$$

Hence, eq. (12) yields

$$P = \frac{2z^4Z^2e^6}{3m^2c^3}\left(\frac{r^2+(\gamma^2-1)b^2}{r^6}\right)\,. \tag{17}$$

The total radiated energy is obtained by integrating over the entire trajectory of the moving charge:

$$\Delta W = \int_{-\infty}^{\infty} P\,dt = 2\int_0^{\infty} P\,dt = 2\int_b^{\infty} P\,\frac{dt}{dr}\,dr\,. \tag{18}$$

[2] In this problem, under the assumption that the charge $ze$ is moving swiftly, we can approximate its velocity as being constant and its trajectory as being the straight dashed line shown in the above figure.

From the figure above, we see that $r = \sqrt{b^2 + v^2 t^2}$. Hence.

$$\frac{dt}{dr} = \left(\frac{dr}{dt}\right)^{-1} = \left(\frac{d}{dt}\sqrt{b^2 + v^2 t^2}\right)^{-1} = \left(\frac{v\sqrt{r^2 - b^2}}{r}\right)^{-1} = \frac{r}{v\sqrt{r^2 - b^2}}\,. \tag{19}$$

Thus, we end up with

$$\begin{aligned} \Delta W &= 2\int_b^\infty \frac{r\,dr}{v\sqrt{r^2 - b^2}}\,P \\ &= \frac{4z^4 Z^2 e^6}{3m^2 c^3}\int_b^\infty \frac{r\,dr}{v\sqrt{r^2 - b^2}}\left(\frac{r^2 + (\gamma^2 - 1)b^2}{r^6}\right)\,. \end{aligned} \tag{20}$$

Changing variables to $x = (r^2 - b^2)/b^2$ and writing $v = c\beta$,

$$\Delta W = \frac{2z^4 Z^2 e^6}{3m^2 c^4 \beta b^3}\int_0^\infty \frac{dx}{\sqrt{x}}\left(\frac{1}{(x+1)^2} + \frac{\gamma^2 - 1}{(x+1)^3}\right) \tag{21}$$

We now consult I.S. Gradshteyn and I.M. Rhyzik, *Table of Integrals, Series and Products*, 8th edition, edited by Daniel Zwillinger (Academic Press, Amsterdam, 2015). Eq. (8.380), no. 3 on p. 917 provides the following integral representation of the Beta function,

$$B(x, y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{t^{x-1}\,dx}{(1+t)^{x+y}}\,, \qquad \text{for } \operatorname{Re} x > 0, \text{ and } \operatorname{Re} y > 0. \tag{22}$$

Hence, it follows that

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)^2} = B\big(\tfrac{1}{2}, \tfrac{3}{2}\big) = \frac{\Gamma\big(\tfrac{1}{2}\big)\Gamma\big(\tfrac{3}{2}\big)}{\Gamma(2)} = \frac{\pi}{2}\,, \tag{23}$$

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)^3} = B\big(\tfrac{1}{2}, \tfrac{5}{2}\big) = \frac{\Gamma\big(\tfrac{1}{2}\big)\Gamma\big(\tfrac{5}{2}\big)}{\Gamma(3)} = \frac{3\pi}{8}\,, \tag{24}$$

after using $\Gamma\big(\tfrac{1}{2}\big) = \sqrt{\pi}$, $\Gamma\big(\tfrac{3}{2}\big) = \tfrac{1}{2}\Gamma\big(\tfrac{1}{2}\big) = \tfrac{1}{2}\sqrt{\pi}$, $\Gamma\big(\tfrac{5}{2}\big) = \tfrac{3}{2}\Gamma\big(\tfrac{3}{2}\big) = \tfrac{3}{4}\sqrt{\pi}$, and $\Gamma(n) = (n-1)!$ for positive integer $n$. Applying eqs. (23) and (24) to evaluate eq. (21), we end up with

$$\Delta W = \frac{\pi z^4 Z^2 e^6}{4m^2 c^4 \beta b^3}\left(\gamma^2 + \frac{1}{3}\right)\,, \tag{25}$$

in agreement with eq. (6).

3. In class, we showed that the angular distribution of the power radiated by a point particle of charge $e$ moving along a trajectory $\vec{\boldsymbol{r}}(t)$ at velocity $c\vec{\boldsymbol{\beta}}(t) \equiv d\vec{\boldsymbol{r}}(t)/dt$ is given by:

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} \frac{cr^2}{4\pi} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega''\, \vec{\boldsymbol{E}}^*_{\omega'}(\vec{\boldsymbol{x}}) \boldsymbol{\cdot} \vec{\boldsymbol{E}}_{\omega''}(\vec{\boldsymbol{x}})\, e^{i(\omega'-\omega'')t}\,,$$

where $r$ is the distance of the observer from the origin and the Fourier coefficient of the electric field vector is given by

$$\boldsymbol{E}_\omega(\vec{\boldsymbol{x}}) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) e^{i\omega t} \tag{26}$$

(a) Derive the following expression for the Fourier coefficient,

$$\boldsymbol{E}_\omega(\vec{\boldsymbol{x}}) = -\frac{ie\,\omega\, e^{i\omega r/c}}{2\pi rc} \int_{-\infty}^{\infty} dt\ \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\, e^{i\omega(t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c)}\,,$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing from the charge to the observer.[3]

If a point particle with charge $e$ moves along a trajectory $\vec{\boldsymbol{r}}(t)$ at velocity $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$ with acceleration $\vec{\boldsymbol{a}} = c\, d\vec{\boldsymbol{\beta}}/dt$, then the leading order behavior of the electric and magnetic fields at large distances (in gaussian units) is given by:

$$\vec{\boldsymbol{E}} = \frac{e}{cr}\left(\frac{\hat{\boldsymbol{n}}\times\left[(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\times\dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^3}\right)_{\vec{\boldsymbol{x}}\,'=\vec{\boldsymbol{r}}(t_{\rm ret})} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{27}$$

$$\vec{\boldsymbol{B}} = \hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{28}$$

where the retarded time is defined as $t_{\rm ret} \equiv t - \left|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t_{\rm ret})\right|/c$. Note that in eq. (27) the velocity $c\vec{\boldsymbol{\beta}}$ is equal to the derivative of $\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{r}}(t_{\rm ret})$ with respect to the retarded time,

$$\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{\beta}}(t_{\rm ret}) = \frac{1}{c}\frac{d\vec{\boldsymbol{r}}(t_{\rm ret})}{dt_{\rm ret}}\,. \tag{29}$$

Inserting eq. (27) into eq. (26) yields,

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{e}{2\pi cr}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\left(\frac{\hat{\boldsymbol{n}}\times\left[(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\times\dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^3}\right)_{\vec{\boldsymbol{x}}\,'=\vec{\boldsymbol{r}}(t_{\rm ret})}\,. \tag{30}$$

[3]As noted by Jackson below his eq. (14.62), assuming that the observation point $\vec{\boldsymbol{x}}$ is located very far away from the region of space where the acceleration occurs, the unit vector $\hat{\boldsymbol{n}}$ can be very well approximated as being constant in time.

Define $t' \equiv t_{\rm ret}$ and change the variable of integration in eq. (30),

$$t = t' + \frac{1}{c}|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')| \quad \Longrightarrow \quad dt = \frac{dt}{dt'}\,dt' = \left(1 - \frac{\bigl(\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')\bigr)\boldsymbol{\cdot} d\vec{\boldsymbol{r}}/dt'}{c|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')|}\right)dt' \,.$$

Noting that $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}} = d\vec{\boldsymbol{r}}(t')/dt'$ [cf. eq. (29)] and $\hat{\boldsymbol{n}} \equiv \bigl(\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')\bigr)/|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')|$, it follows that $dt = (1 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})dt'$. Hence,

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{e}{2\pi cr}\int_{-\infty}^{\infty} dt'\, e^{i\omega[t' + |\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')|/c]}\left(\frac{\hat{\boldsymbol{n}} \times \left[(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^2}\right). \tag{31}$$

For large values of $r \equiv |\vec{\boldsymbol{x}}|$, we can approximate

$$\hat{\boldsymbol{n}} = \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')|} = \frac{\vec{\boldsymbol{x}}}{r}\left[1 + \mathcal{O}\left(\frac{1}{r}\right)\right],$$

so that $\vec{\boldsymbol{x}} \simeq r\hat{\boldsymbol{n}}$ and

$$\begin{aligned} t' + \frac{1}{c}\,|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')| &= t' + \frac{1}{c}\sqrt{r^2 - 2\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t') + r'^2} = t' + \frac{r}{c}\left[1 - \frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] \\ &\simeq t' + \frac{1}{c}\Bigl(r - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')\Bigr), \end{aligned}$$

where $r' \equiv |\vec{\boldsymbol{r}}(t')|$. Inserting the above result into eq. (31) yields

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{e}{2\pi cr}\,e^{i\omega r/c}\int_{-\infty}^{\infty} dt'\, e^{i\omega[t' - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]}\left(\frac{\hat{\boldsymbol{n}} \times \left[(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^2}\right). \tag{32}$$

Employing the identity,

$$\frac{\hat{\boldsymbol{n}} \times \left[(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1 - \hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})^2} = \frac{d}{dt'}\left(\frac{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})}{1 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}}\right), \tag{33}$$

we can integrate by parts and drop the surface term.[4] Hence, eqs. (32) and (33) yield,

$$\begin{aligned} \vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) &= -\frac{e}{2\pi cr}\,e^{i\omega r/c}\int_{-\infty}^{\infty} dt'\left(\frac{\hat{\boldsymbol{n}} \times \left[(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{1 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}}\right)\frac{d}{dt'}\,e^{i\omega[t' - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]} \\ &= -\frac{ie\omega}{2\pi cr}\,e^{i\omega r/c}\int_{-\infty}^{\infty} dt\;\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})\,e^{i\omega[t - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]}\,, \end{aligned} \tag{34}$$

after dropping the primed superscripts by writing $t$ in place of $t'$ (which after all is just a dummy integration variable), and using $c\,\vec{\boldsymbol{\beta}}(t') = d\vec{\boldsymbol{r}}(t')/dt'$.

[4] The justification for dropping the surface term is discussed on pp. 675–676 of Jackson.

(b) [Jackson, problem 14.13] Using the results of part (a) and the Poisson sum formula, show explicitly that if the motion of a radiating particle repeats itself with periodicity $T$, then the continuous frequency spectrum becomes a discrete spectrum containing frequencies that are integral multiples of the fundamental. Show that a general expression for the time-averaged power radiated per unit solid angle in each multiple $m$ of the fundamental frequency $\omega_0 = 2\pi/T$ is given by

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{T}\int_0^T dt\, \frac{dP}{d\Omega} \equiv \sum_{m=1}^{\infty} \frac{dP_m}{d\Omega}\,,$$

where

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}\left| \int_0^{2\pi/\omega_0} \vec{\boldsymbol{v}}(t)\times\hat{\boldsymbol{n}}\, \exp\left[im\omega_0\left(t - \frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)}{c}\right)\right] dt\right|^2 . \tag{35}$$

It is convenient to rewrite the integral in eq. (34) as

$$\int_{-\infty}^{\infty} dt\; \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\, e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]} = \sum_{m=-\infty}^{\infty} \int_{mT}^{(m+1)T} dt\; \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\, e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]}\,. \tag{36}$$

Since the motion is periodic, we have

$$\vec{\boldsymbol{r}}(t+T) = \vec{\boldsymbol{r}}(t) \quad \text{and} \quad \vec{\boldsymbol{\beta}}(t+T) = \vec{\boldsymbol{\beta}}(t)\,,$$

where $T \equiv 2\pi/\omega_0$ defines the fundamental frequency $\omega_0$. Let us define a new variable, $t' \equiv t - mT$. Then, eq. (36) takes the following form,

$$\int_{-\infty}^{\infty} dt\; \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\, e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]} = \sum_{m=-\infty}^{\infty} e^{i\omega mT} \int_0^T dt'\; \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}}')\, e^{i\omega[t'-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]}\,, \tag{37}$$

where $\vec{\boldsymbol{\beta}}' \equiv \vec{\boldsymbol{\beta}}(t')$.

At this point, we can apply the Poisson sum formula,[5]

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{i\omega mT} = \sum_{m=-\infty}^{\infty} \delta(\omega T - 2\pi m)\,.$$

Hence, eq. (37) can be rewritten as

$$\int_{-\infty}^{\infty} dt\; \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\, e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]} = \sum_{m=-\infty}^{\infty} \delta(\omega T - 2\pi m) \int_0^T dt\; \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\, e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]}\,,$$

after again removing the superscript primes from $t'$ from the right hand side above as a notational convenience. Note that the $\delta$-function enforces the condition,

$$\omega = \frac{2\pi m}{T} = m\omega_0\,, \qquad \text{for } m = 0, \pm 1, \pm 2, \ldots,$$

which implies that the frequency spectrum is discrete.

[5] See Section 5 of the class handout entitled *Generalized Functions for Physics*.

Thus we can rewrite eq. (34), obtained in part (a), as

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = -\frac{ie\omega}{2\pi cr}\, e^{i\omega r/c} \sum_{m=-\infty}^{\infty} \delta(\omega T - 2\pi m) \int_0^T dt\; \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})\, e^{i\omega[t - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]}\,. \tag{38}$$

The power radiated per unit solid angle (in gaussian units) is given by

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} r^2\, \vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,, \qquad \text{where } \vec{\boldsymbol{S}} = \frac{c}{4\pi}(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}})\,,$$

and $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are the real physical fields. For large distances $r$, we have $\vec{\boldsymbol{B}} \simeq \hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}}$, as noted in eq. (28), in which case

$$\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{E}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}}) = \hat{\boldsymbol{n}}|\vec{\boldsymbol{E}}|^2\,,$$

after using $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 0$ (i.e., the electromagnetic radiation is transverse). Hence, it follows that

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} \lim_{r\to\infty} r^2 |\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)|^2\,. \tag{39}$$

Inverting the Fourier transform defined in eq. (26),

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \int_{-\infty}^{\infty} d\omega\, \vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}})\, e^{-i\omega t}\,,$$

and inserting the result into eq. (39) yields

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} \frac{cr^2}{4\pi} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega''\, \vec{\boldsymbol{E}}^*_{\omega'}\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{\omega''}\, e^{i(\omega'-\omega'')t}\,.$$

Using eq. (38) in the above expression, we obtain

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega''\, \delta(\omega' T - 2\pi m)\, \delta(\omega'' T - 2\pi n)\, e^{i(\omega'-\omega'')(t - r/c)}\, \omega'\omega''$$
$$\times \int_0^T dt'\; \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}')\, e^{i\omega[t' - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]} \int_0^T dt''\; \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}'')\, e^{i\omega''[t'' - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t'')/c]}\,.$$

We can now perform the integrals over $\omega'$ and $\omega''$ using the $\delta$-functions, which set $\omega' = m\omega_0$ and $\omega'' = n\omega_0$, respectively (where $\omega_0 \equiv 2\pi/T$). Thus,

$$\frac{dP}{d\Omega} = \frac{e^2\omega_0^2}{4\pi c T^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} mn\, e^{i\omega_0(m-n)(t-r/c)} \int_0^T dt' \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}')\, e^{i\omega[t' - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]}$$
$$\times \int_0^T dt''\; \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}'')\, e^{i\omega''[t'' - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t'')/c]}\,. \tag{40}$$

Since $dP/d\Omega$ depends on $t$, we shall integrate over one cycle,

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{T}\int_0^T \frac{dP}{d\Omega}\,dt\,.$$

Taking the time-average of eq. (40), the integration over $t$ is straightforward, as it depends only on the following integral,

$$\frac{1}{T}\int_0^T e^{i\omega_0 t(m-n)}\,dt = \delta_{mn}\,.$$

The sums over $m$ and $n$ in eq. (40) now collapse into a single sum over $m$. Noting that

$$\bigl[\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}}')\bigr]\boldsymbol{\cdot}\bigl[\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}}'')\bigr] = (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}}')\boldsymbol{\cdot}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}}'')\,, \tag{41}$$

the end result is,

$$\begin{aligned}\left\langle\frac{dP}{d\Omega}\right\rangle &= \frac{e^2\omega_0^2}{4\pi c T^2}\sum_{m=-\infty}^{\infty} m^2\left|\int_0^T(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{im\omega_0(t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c)}dt\right|^2 \\ &= \frac{e^2\omega_0^2}{2\pi c T^2}\sum_{m=1}^{\infty} m^2\left|\int_0^T(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{im\omega_0(t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c)}dt\right|^2\,, \end{aligned} \tag{42}$$

after noting that positive and negative $m$ contribute equally to the sum over $m$ (whereas the $m=0$ contribution to the sum vanishes). Thus, using $\vec{\boldsymbol{v}}(t) = c\vec{\boldsymbol{\beta}}$ and $T = 2\pi/\omega_0$ in eq. (42), we can write:

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \sum_{m=1}^{\infty}\frac{dP_m}{d\Omega}\,, \tag{43}$$

where

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}\left|\int_0^{2\pi/\omega_0}\vec{\boldsymbol{v}}(t)\times\hat{\boldsymbol{n}}\,\exp\left[im\omega_0\left(t-\frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)}{c}\right)\right]dt\right|^2\,. \tag{44}$$

4. [Jackson, problem 13.9] Assuming that Plexiglas or Lucite has an index of refraction of 1.50 in the visible region, compute the angle of emission of visible Cherenkov radiation for electrons and protons as a function of their kinetic energies in MeV. Determine how many quanta with wavelengths between 4000 and 6000 Å are emitted per centimeter of the path in Lucite by a 1 MeV electron, a 500 MeV proton, and a 5 GeV proton.

Using eq. (13.50) of Jackson, the angle of emission $\theta_c$ is obtained from:

$$\cos\theta_c = \frac{1}{\beta\sqrt{\epsilon}}\,, \tag{45}$$

and the index of refraction is $n_r = \sqrt{\epsilon}$. To compute $\beta$ given the kinetic energy $T$, we recall that

$$T = E - mc^2 = \gamma mc^2 - mc^2 = mc^2 \left[\sqrt{\frac{1}{1-\beta^2}} - 1\right] .$$

Solving for $\beta$, it follows that

$$\beta^2 = 1 - \frac{1}{\left(1 + \dfrac{T}{mc^2}\right)^2} ,$$

from which $\beta$ is easily obtained,

$$\beta = \frac{T\sqrt{1 + \dfrac{2mc^2}{T}}}{T + mc^2} .$$

Hence, eq. (45) yields

$$\cos\theta_c = \frac{1}{n_r}\left(1 + \frac{mc^2}{T}\right)\left(1 + \frac{2mc^2}{T}\right)^{-1/2} . \tag{46}$$

Note that $mc^2 = 0.511$ MeV for the electron and $mc^2 = 938$ MeV for the proton. Inserting these numbers along with $n_r = 1.5$ in eq. (46), one obtains the angle of emission of visible Cherenkov radiation for electrons and protons as a function of their kinetic energies in MeV.

To determine the number of quanta emitted per path length, we first use eq. (13.48) of Jackson:[6]

$$\left(\frac{dE}{dx}\right)_{\rm rad} = \frac{e^2}{c^2}\int_{n_r > 1/\beta} \omega\left(1 - \frac{1}{\beta^2 n_r^2}\right) d\omega .$$

Assuming that $n_r$ is independent of $\omega$ in the frequency range of interest, we integrate from $\omega = \omega_1$ to $\omega = \omega_2$ to obtain,

$$\left(\frac{dE}{dx}\right)_{\rm rad} = \frac{e^2}{2c^2}\left(1 - \frac{1}{\beta^2 n_r^2}\right)(\omega_2^2 - \omega_1^2) .$$

For the range 4000 Å$\leq \lambda \leq$ 6000Å, where 1 Å$= 10^{-8}$ cm, we have

$$\omega_1 = \frac{2\pi c}{\lambda_1} = \frac{2\pi(3\times 10^{10}\ {\rm cm\cdot s^{-1}})}{4\times 10^{-5}\ {\rm cm}} = 4.71\times 10^{15}\ {\rm s}^{-1} ,$$

$$\omega_2 = \frac{2\pi c}{\lambda_2} = \frac{2}{3}\omega_1 = 3.14\times 10^{15}\ {\rm s}^{-1} .$$

The energy of one quantum is $\hbar\omega$. Hence, it follows that

$$\frac{dN}{d\omega\, dx} = \frac{1}{\hbar\omega}\frac{dE}{d\omega\, dx} ,$$

[6] Note that the charge of the moving particle is denoted by $ze$ in Jackson. In this problem, $z = \pm 1$ for the proton and electron, respectively, so that $z^2 = 1$.

where $N$ is the number of quanta radiated. Thus,

$$\left(\frac{dN}{dx}\right)_{\rm rad} = \frac{e^2}{\hbar c^2}\int_{\omega_1}^{\omega_2}\left(1-\frac{1}{\beta^2 n_r^2}\right)\,d\omega = \frac{e^2}{\hbar c^2}\left(1-\frac{1}{\beta^2 n_r^2}\right)(\omega_2-\omega_1)\,.$$

We can rewrite the above equation by using $\omega = kc/n_r = 2\pi c/(n_r\lambda)$ [cf. eq. (7.5) of Jackson] and by introducing the fine structure constant,

$$\alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,.$$

It follows that

$$\left(\frac{dN}{dx}\right)_{\rm rad} = \frac{2\pi\alpha}{n_r}\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\beta^2 n_r^2}\right)\,. \tag{47}$$

We now plug in the relevant numbers into eq. (47).

Case 1: For a $T = 1$ MeV electron,

$$\frac{1}{\beta} = \left(1+\frac{mc^2}{T}\right)\left(1+\frac{2mc^2}{T}\right)^{-1/2} = 1.0626\,.$$

Hence,

$$\cos\theta_c = \frac{2}{3\beta} = 0.7084 \qquad\Longrightarrow\qquad \theta_c = 44.9^\circ\,,$$

and

$$1-\frac{1}{\beta^2 n_r^2} = 1-\frac{(1.0626)^2}{(1.5)^2} = 0.4982\,.$$

Eq. (47) then yields,

$$\left(\frac{dN}{dx}\right)_{\rm rad} = \frac{2\pi}{1.5}\left(\frac{1}{137}\right)\left(\frac{1}{4\times 10^{-5}~{\rm cm}}-\frac{1}{6\times 10^{-5}~{\rm cm}}\right)(0.4982) = 127~{\rm quanta/cm}\,.$$

Case 2: For a $T = 500$ MeV proton,

$$\frac{1}{\beta} = 1.3193\,.$$

Hence,

$$\cos\theta_c = 0.8795 \qquad\Longrightarrow\qquad \theta_c = 28.4^\circ\,.$$

Eq. (47) then yields,

$$\left(\frac{dN}{dx}\right)_{\rm rad} = 58~{\rm quanta/cm}\,.$$

Case 3: For a $T = 5$ GeV proton,

$$\frac{1}{\beta} = 1.0127\,.$$

Hence,

$$\cos\theta_c = 0.6751 \quad \Longrightarrow \quad \theta_c = 47.5^\circ\,.$$

Eq. (47) then yields,

$$\left(\frac{dN}{dx}\right)_{\rm rad} = 140\text{ quanta/cm}\,.$$

5. [Jackson, problem 10.1]

(a) Show that for arbitrary initial polarizations, the scattering cross section of a perfectly conducting sphere of radius $a$, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega}(\hat{\boldsymbol{\epsilon}}_0,\hat{\boldsymbol{n}}_0,\hat{\boldsymbol{n}}) = k^4a^6\left[\tfrac{5}{4} - |\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}|^2 - \tfrac{1}{4}|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)|^2 - \hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\right],$$

where $\hat{\boldsymbol{n}}_0$ and $\hat{\boldsymbol{n}}$ are the directions of the incident and scattered electromagnetic waves, respectively, while $\hat{\boldsymbol{\epsilon}}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\hat{\boldsymbol{\epsilon}}_0^*\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = 1\,;\ \hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = 0$.)

Our starting point is eq. (10.14) of Jackson,

$$\frac{d\sigma}{d\Omega} = k^4a^6\left|\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^*)\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)\right|^2\,.$$

For arbitrary initial polarization $\hat{\boldsymbol{\epsilon}}_0$, the scattering cross section summed over the final state polarizations is

$$\frac{d\sigma}{d\Omega} = k^4a^6\sum_\lambda\left|\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)\right|^2\,. \tag{48}$$

We shall evaluate the polarization sum using the following identity derived in the class handout entitled *Polarization Vectors and Polarization Sums*,

$$\sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)} = \delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\,, \tag{49}$$

where the $\hat{\boldsymbol{n}}_i$ ($i\in\{1,2,3\}$) are the Cartesian components of the unit vector $\hat{\boldsymbol{n}}\equiv\vec{\boldsymbol{k}}/k$. Expanding out the terms in eq. (48), we first evaluate

$$\sum_\lambda|\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 = \sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)}(\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{\epsilon}}_0^*)_j = (\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{\epsilon}}_0^*)_j\left[\delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\right] = 1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\,, \tag{50}$$

after using $\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^* = 1$ in the final step.

Similarly,

$$\sum_\lambda|(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)|^2 = \sum_\lambda \hat{\boldsymbol{\epsilon}}_{ijk}\,\hat{\boldsymbol{n}}_j\hat{\boldsymbol{\epsilon}}_k^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_{\ell mn}\,\hat{\boldsymbol{n}}_m\hat{\boldsymbol{\epsilon}}_n^{(\lambda)}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0^*)_\ell\,,$$

where the summation over repeated index pairs is implied by the Einstein summation convention. Using the polarization sum identity given by eq. (49),

$$\sum_\lambda |(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0)|^2 = \epsilon_{ijk}\epsilon_{\ell mn}(\delta_{kn}-\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_n)(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0^*)_\ell\,.$$

Since $\epsilon_{ijk}$ is a totally antisymmetric tensor, it follows that $\epsilon_{ijk}\,\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k=0$. Employing the identity,

$$\epsilon_{ijk}\epsilon_{\ell mn}\delta_{kn}=\epsilon_{ijk}\epsilon_{\ell mk}=\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell}\,,$$

we end up with

$$\begin{aligned}\sum_\lambda |(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)|^2 &= (\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell})\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_m(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0^*)_\ell \\ &= |\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0|^2-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)|^2\,,\end{aligned}$$

after noting that $\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_m\delta_{jm}=\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}=1$. Finally, we can expand out the square of the cross product,

$$|\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0|^2 = 1-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2=1\,,$$

after using $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0=0$ (which follows from the fact that the polarization vector is transverse to the direction of propagation of the electromagnetic wave). Hence, we conclude that

$$\sum_\lambda |(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)|^2 = 1-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)|^2\,.$$

All that remains is to evaluate the cross-term in eq. (48).

$$\begin{aligned}\sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}(\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)})_j(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0^*)_j &= \sum_\lambda \epsilon_{jk\ell}\,\hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_\ell^{(\lambda)}(\hat{\boldsymbol{\epsilon}}_0)_i\hat{\boldsymbol{n}}_k(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0^*)_j \\ &= \epsilon_{jk\ell}(\delta_{i\ell}-\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_\ell)(\hat{\boldsymbol{\epsilon}}_0)_i\hat{\boldsymbol{n}}_k(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0^*)_j \\ &= \epsilon_{jk\ell}\,\hat{\boldsymbol{n}}_k(\hat{\boldsymbol{\epsilon}}_0)_\ell(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0^*)_j = (\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0)\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0^*) \\ &= (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*)-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 \\ &= \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\,.\end{aligned}$$

Collecting all the above results, it follows that

$$\frac{d\sigma}{d\Omega}=k^4a^6\left\{1-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2+\tfrac{1}{4}\left[1-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)|^2\right]-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\right\},$$

which simplifies to

$$\frac{d\sigma}{d\Omega}=k^4a^6\left[\tfrac{5}{4}-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2-\tfrac{1}{4}|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)|^2-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\right], \tag{51}$$

as required.

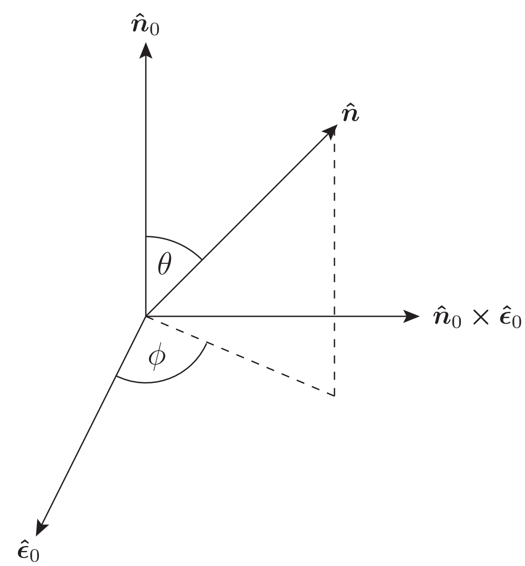
REMARK: If you express the square of the vector cross product in eq. (48) as the sum of products of dot products before carrying out the polarization sums, you will arrive at a different form for eq. (51). Nevertheless, it is possible to show that the two forms are equivalent. This alternative method is provided at the end of the solution to part (c) of this problem.

(b) If the incident radiation is linearly polarized, show that cross section is

$$\frac{d\sigma}{d\Omega}(\hat{\boldsymbol{\epsilon}}_0, \hat{\boldsymbol{n}}_0, \hat{\boldsymbol{n}}) = k^4 a^6 \left[\tfrac{5}{8}(1+\cos^2\theta) - \cos\theta - \tfrac{3}{8}\sin^2\theta\cos 2\phi\right],$$

where $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0 = \cos\theta$ and the azimuthal angle $\phi$ is measured from the direction of the linear polarization.

We set up our coordinate system as follows:




The components of the corresponding unit vectors are:

$$\hat{\boldsymbol{\epsilon}}_0 = (1\,,\,0\,,\,0), \qquad \hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0 = (0\,,\,1\,,\,0)\,, \qquad \hat{\boldsymbol{n}} = (\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)\,.$$

It follows that

$$\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,, \qquad \hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0) = \sin\theta\sin\phi\,, \qquad \hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \cos\theta\,.$$

Hence, eq. (51) yields

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\tfrac{5}{4} - \sin^2\theta\cos^2\phi - \tfrac{1}{4}\sin^2\theta\sin^2\phi - \cos\theta\right]. \tag{52}$$

Writing $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$ and $\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$, eq. (52) takes the following form,

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\tfrac{5}{8}(1+\cos^2\theta) - \cos\theta - \tfrac{3}{8}\sin^2\theta\cos 2\phi\right]. \tag{53}$$

(c) What is the ratio of the scattered intensities at $\theta = \frac{1}{2}\pi$, $\phi = 0$ and $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$? Explain physically in terms of the induced multipoles and their radiation patterns.

Using eq. (53), it follows that

$$\frac{\dfrac{d\sigma}{d\Omega}\left(\theta = \tfrac{1}{2}\pi\,,\,\phi = 0\right)}{\dfrac{d\sigma}{d\Omega}\left(\theta = \tfrac{1}{2}\pi\,,\,\phi = \tfrac{1}{2}\pi\right)} = \frac{1}{4}\,.$$

If we trace back the origin of the various contributions, we see that the electric dipole scattering originates from

$$1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}|^2 = 1 - \sin^2\theta\cos^2\phi \quad \underset{\theta=\frac{1}{2}\pi}{\longrightarrow} \quad \sin^2\phi\,,$$

whereas the magnetic dipole scattering originates from

$$\tfrac{1}{4}\left[1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2\right] = \tfrac{1}{4}\left(1 - \sin^2\theta\sin^2\phi\right) \quad \underset{\theta=\frac{1}{2}\pi}{\longrightarrow} \quad \tfrac{1}{4}\cos^2\phi\,.$$

Thus, at $\theta = \frac{1}{2}\pi$, $\phi = 0$, we have pure magnetic dipole scattering. In contrast, at $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$, we have pure electric dipole scattering, whose contribution is four times larger than the magnetic dipole scattering contribution at $\theta = \frac{1}{2}\pi$, $\phi = 0$. The factor of four originates from the relative factor of two between the electric dipole moment $\vec{\boldsymbol{p}}$ [cf. eq. (10.12) of Jackson] and the magnetic dipole moment $\vec{\boldsymbol{m}}$ [cf. eq. (10.13) of Jackson] that are induced by the electric and magnetic fields of the incoming plane wave.

At $\theta = \frac{1}{2}\pi$, $\phi = 0$, we see that $\hat{\boldsymbol{n}}$ points in the direction of $\hat{\boldsymbol{\epsilon}}_0$. But $\hat{\boldsymbol{n}}$ points in the direction of the outgoing wave, whereas $\hat{\boldsymbol{\epsilon}}_0$ is parallel to the direction of the electric field of the incoming plane wave. Since the latter is also parallel to the direction of $\vec{\boldsymbol{p}}$, we conclude that in this case $\hat{\boldsymbol{n}}$ is parallel to $\vec{\boldsymbol{p}}$. It follows that $\hat{\boldsymbol{\epsilon}}^*$ must be perpendicular to $\vec{\boldsymbol{p}}$ (since the former is necessarily perpendicular to $\hat{\boldsymbol{n}}$), in which case $\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\vec{\boldsymbol{p}} = 0$. Eq. (10.4) of Jackson then implies that the scattering in this case is entirely due to the magnetic dipole term.

Similarly, at $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$, we see that $\hat{\boldsymbol{n}}$ points in the direction of $\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0$, which is parallel to the direction of the magnetic field of the incoming plane wave. Since the latter is also parallel to $\vec{\boldsymbol{m}}$, we conclude that in this case $\hat{\boldsymbol{n}}$ is parallel to $\vec{\boldsymbol{m}}$. It follows that $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}_0^*$ must be perpendicular to $\vec{\boldsymbol{m}}$, in which case $(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}_0^*)\boldsymbol{\cdot}\vec{\boldsymbol{m}} = 0$. Eq. (10.4) of Jackson then implies that the scattering in this case is entirely due to the electric dipole term.

## Alternative evaluation of eq. (48)

In the evaluation of eq. (48), one might be tempted to employ the vector identity,

$$(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0) = (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0) - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\,. \tag{54}$$

Then, eq. (48) takes the following form:

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \sum_\lambda \left| \big(\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big)\big[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\big] + \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\big] \right|^2 . \tag{55}$$

Expanding out the squared quantity yields

$$\begin{aligned}\frac{d\sigma}{d\Omega} = k^4 a^6 \Bigg[ &\big[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\big]^2 \sum_\lambda \big|\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big|^2 + \tfrac{1}{4}\big|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big|^2 \sum_\lambda \big|\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\big|^2 \\ &+ \mathrm{Re}\bigg\{ \big[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\big](\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*) \sum_\lambda \big(\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big)\big(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}\big) \bigg\} \Bigg]. \end{aligned} \tag{56}$$

We have already used eqs. (49) and (50) to obtain,

$$\sum_\lambda |\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 = \sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)}(\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{\epsilon}}_0^*)_j = (\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{\epsilon}}_0^*)_j\,[\delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j] = 1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 . \tag{57}$$

Similarly,

$$\sum_\lambda \big|\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\big|^2 = \sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)}(\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}}_0)_j = (\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}}_0)_j\,[\delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j] = 1 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2 . \tag{58}$$

and

$$\sum_\lambda \big(\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big)\big(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}\big) = \sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)}(\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}}_0)_j = (\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}}_0)_j\,[\delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j] = -(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\,, \tag{59}$$

where we have used the fact that $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = 0$ (since the electromagnetic wave is transverse to the direction of propagation). Inserting the polarization sums obtained above into eq. (56) yields,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= k^4 a^6 \bigg\{ \big[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\big]^2\big[1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\big] + \tfrac{1}{4}\big|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big|^2\big[1 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2\big] \\ &\qquad - \big[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\big](\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\big|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big|^2 \bigg\} \\ &= k^4 a^6 \big[1 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0 + \tfrac{1}{4}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2 - \tfrac{3}{4}\big|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big|^2\big]. \end{aligned} \tag{60}$$

To see that eq. (60) is equivalent to eq. (51), we need to evaluate

$$|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 = \epsilon_{ijk}\,\hat{\boldsymbol{n}}_i(\hat{\boldsymbol{n}}_0)_j(\hat{\boldsymbol{\epsilon}}_0)_k\,\epsilon_{mpq}\hat{\boldsymbol{n}}_m(\hat{\boldsymbol{n}}_0)_p(\hat{\boldsymbol{\epsilon}}_0^*)_q\,, \tag{61}$$

where the Einstein summation convention is being used to sum over the repeated indices. Eq. (61) can be simplified by employing the following identity,

$$\begin{aligned}\epsilon_{ijk}\epsilon_{mpq} &= \det\begin{pmatrix} \delta_{im} & \delta_{ip} & \delta_{iq} \\ \delta_{jm} & \delta_{jp} & \delta_{jq} \\ \delta_{km} & \delta_{kp} & \delta_{kq} \end{pmatrix} \\ &= \delta_{im}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) - \delta_{ip}(\delta_{jm}\delta_{kq} - \delta_{jq}\delta_{km}) + \delta_{iq}(\delta_{jm}\delta_{kp} - \delta_{jp}\delta_{km})\,. \end{aligned} \tag{62}$$

Thus, it follows that

$$
\begin{aligned}
&|\hat{\boldsymbol{n}}\cdot(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 = \epsilon_{ijk}\hat{\boldsymbol{n}}_i(\hat{\boldsymbol{n}}_0)_j(\hat{\boldsymbol{\epsilon}}_0)_k\,\epsilon_{mpq}\hat{\boldsymbol{n}}_m(\hat{\boldsymbol{n}}_0)_p(\hat{\boldsymbol{\epsilon}}^*_0)_q \\
&= \big[\delta_{im}(\delta_{jp}\delta_{kq}-\delta_{jq}\delta_{kp}) - \delta_{ip}(\delta_{jm}\delta_{kq}-\delta_{jq}\delta_{km}) + \delta_{iq}(\delta_{jm}\delta_{kp}-\delta_{jp}\delta_{km})\big]\hat{\boldsymbol{n}}_i(\hat{\boldsymbol{n}}_0)_j(\hat{\boldsymbol{\epsilon}}_0)_k\hat{\boldsymbol{n}}_m(\hat{\boldsymbol{n}}_0)_p(\hat{\boldsymbol{\epsilon}}^*_0)_q \\
&= \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}\big[(\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{\epsilon}}_0\cdot\hat{\boldsymbol{\epsilon}}^*_0) - (\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{\epsilon}}_0)(\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{\epsilon}}^*_0)\big] - \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_0\big[(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{\epsilon}}_0\cdot\hat{\boldsymbol{\epsilon}}^*_0) - (\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{\epsilon}}^*_0)(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0)\big] \\
&\qquad\qquad + \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}^*_0\big[(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{\epsilon}}_0) - (\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0)\big]\,. \qquad (63)
\end{aligned}
$$

The above expression can be further simplified by using the fact that $\hat{\boldsymbol{\epsilon}}_0$ is a complex unit vector that satisfies $\hat{\boldsymbol{\epsilon}}_0\cdot\hat{\boldsymbol{\epsilon}}^*_0 = 1$ and the real unit vectors $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}_0$ satisfy $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}_0 = 1$. In addition, due to the transverse nature of the incoming electromagnetic wave, it follows that $\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{\epsilon}}_0 = \hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{\epsilon}}^*_0 = 0$, as previously noted. Thus, we end up with

$$
|\hat{\boldsymbol{n}}\cdot(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 = 1 - (\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_0)^2 - \big|\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0\big|^2\,. \qquad (64)
$$

Plugging the result of eq. (64) into eq. (51) yields,

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= k^4a^6\Big[\tfrac{5}{4} - |\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0|^2 - \tfrac{1}{4}\big[1-(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_0)^2 - \big|\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0\big|^2\big] - \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_0\Big] \\
&= k^4a^6\big[1 - \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_0 + \tfrac{1}{4}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_0)^2 - \tfrac{3}{4}\big|\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0\big|^2\big], \qquad (65)
\end{aligned}
$$

which coincides with eq. (60).

6. [Jackson, problem 10.4] An unpolarized wave of frequency $\omega = ck$ is scattered by a *slightly* lossy uniform isotropic dielectric sphere of radius $R$ much smaller than a wavelength. The sphere is characterized by an ordinary real dielectric constant $\epsilon_r$ and a real conductivity $\sigma$. The parameters are such that the skin depth $\delta$ is very *large* compared to the radius $R$.

(a) Calculate the differential and total *scattering* cross sections.

Consider the electric field of the incident unpolarized wave, $\vec{\boldsymbol{E}}_{\rm inc}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{E}}_{\rm inc}(\vec{\boldsymbol{x}})e^{-i\omega t}$, where

$$
\vec{\boldsymbol{E}}_{\rm inc}(\vec{\boldsymbol{x}}) = E_0\hat{\boldsymbol{\epsilon}}_0 e^{ik\hat{\boldsymbol{n}}_0\cdot\vec{\boldsymbol{x}}}\,, \qquad (66)
$$

with polarization vector $\hat{\boldsymbol{\epsilon}}_0$ and incident direction along the unit vector $\hat{\boldsymbol{n}}_0$. Since the radius of the sphere, $R$, is much smaller than the wavelength of the radiation (i.e., $kR \ll 1$), one can treat the electric field of the incident wave as uniform over the whole volume of the sphere by approximating $e^{ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}} \simeq 1$. The effect of this electric field is to induce an electric dipole moment. Thus, we can use eq. (4.56) of Jackson, which provides the electric dipole moment induced by an external uniform electric field for a small isotropic dielectric sphere of radius $R$,

$$
\vec{\boldsymbol{p}} = 4\pi\epsilon_0\left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right)R^3E_0\hat{\boldsymbol{\epsilon}}_0\,. \qquad (67)
$$

In light of eq. (7.57) of Jackson,

$$\frac{\epsilon}{\epsilon_0} = \epsilon_r + i\frac{\sigma}{\epsilon_0\omega}\,, \tag{68}$$

where $\epsilon_r$ is the ordinary real dielectric constant and $\sigma$ is the real conductivity. Using $\omega = kc$ and the impedance of free space, $Z_0 = \sqrt{\mu_0/\epsilon_0} = c\mu_0 = 1/(c\epsilon_0)$, we can rewrite eq. (68) as

$$\frac{\epsilon}{\epsilon_0} = \epsilon_r + i\frac{Z_0\sigma}{k}\,. \tag{69}$$

Along with the incident electric field given in eq. (66), there is an incident magnetic field, $\vec{\boldsymbol{H}}_{\rm inc}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{H}}_{\rm inc}(\vec{\boldsymbol{x}})e^{-i\omega t}$, where

$$\vec{\boldsymbol{H}}_{\rm inc} = \hat{\boldsymbol{n}}_0 \times \vec{\boldsymbol{E}}_{\rm inc}/Z_0 = (\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)c\epsilon_0 E_0 e^{ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}}\,. \tag{70}$$

Since $kR \ll 1$ by assumption, one can treat the magnetic field of the incident wave as uniform over the whole volume of the sphere by approximating $e^{ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}} \simeq 1$. Moreover as shown in class, due to the nonzero conductivity, the harmonic $\vec{\boldsymbol{H}}$ field of the incident wave induces a circulating $\vec{\boldsymbol{E}}$ field inside the dielectric sphere due to Faraday's law (in frequency space), $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = i\omega\vec{\boldsymbol{B}} = i\omega\mu_0\vec{\boldsymbol{H}}$, which yields

$$\vec{\boldsymbol{E}}_{\rm induced} = -\tfrac{1}{2}i\omega c\epsilon_0\mu_0 E_0\,\vec{\boldsymbol{x}} \times (\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)\,. \tag{71}$$

This induced electric field drives eddy currents $\vec{\boldsymbol{J}} = \sigma\vec{\boldsymbol{E}}_{\rm induced}$ via Ohm's law,[7] which generate an induced magnetic dipole moment $\vec{\boldsymbol{m}}$,

$$\vec{\boldsymbol{m}} = \frac{1}{2}\int_0^R r^2\,dr\int d\Omega\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}} = \frac{2\pi ic\epsilon_0\mu_0 E_0}{15}\sigma\omega R^5\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0\,. \tag{72}$$

Note that the skin depth, defined in eq. (5.15) of Jackson, is given by

$$\delta = \sqrt{\frac{2}{\mu_0\sigma\omega}} = \sqrt{\frac{2}{Z_0\sigma k}}\,, \tag{73}$$

where we have assumed that the magnetic permeability is given by its value in free space. Hence, eq. (72) yields

$$\frac{\vec{\boldsymbol{m}}}{c} = i\,\frac{4\pi\epsilon_0 R^3 E_0}{15}\,\frac{R^2}{\delta^2}\,\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0\,. \tag{74}$$

[7] In obtaining eq. (71), we note that Faraday's Law determines the electric field up to the gradient of a scalar, since $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}}\psi) = 0$. We can determine the scalar $\psi$ by imposing two conditions. First, the continuity equation implies that $\vec{\boldsymbol{J}}$ is a steady current (i.e., $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} = 0$), since the charge density $\rho$ inside the sphere must be time-independent (due to the conservation of charge). This condition (via Ohm's law) yields $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}}_{\rm induced} = 0$, and thus implies that $\vec{\boldsymbol{\nabla}}^2\psi = 0$. Second, as the current resides completely inside the dielectric sphere, it follows that $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}}(r = R) = 0$ (where $\hat{\boldsymbol{n}}$ is a radial unit vector), which yields $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}_{\rm induced}(r = R) = 0$, and thus implies that $(\partial\psi/\partial r)_{r=R} = 0$. These two conditions imply that $\vec{\boldsymbol{\nabla}}\psi = 0$, and hence $\vec{\boldsymbol{E}}_{\rm induced}$ is uniquely determined. Finally, using the vector identity $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{v}}) = -2\vec{\boldsymbol{v}}$ for any constant vector $\vec{\boldsymbol{v}}$, we confirm the result obtained in eq. (71).

Using eq. (10.4) of Jackson,

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\vec{\boldsymbol{p}} + (\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^*)\boldsymbol{\cdot}\vec{\boldsymbol{m}}/c\right|^2, \tag{75}$$

where $\hat{\boldsymbol{\epsilon}}$ is the polarization vector of the scattered wave. In light of eqs. (67) and (74), we see that

$$\frac{|\vec{\boldsymbol{m}}|/c}{|\vec{\boldsymbol{p}}|} \sim \mathcal{O}\left(\frac{\delta^2}{R^2}\right). \tag{76}$$

Given that the problem states that the parameters are such that the skin depth $\delta$ is very *large* compared to the radius $R$, it follows that we may neglect the contribution of the induced magnetic dipole moment.[8] Henceforth, we shall set $\vec{\boldsymbol{m}} = 0$.

Inserting the result of eq. (69) back into eq. (67),

$$\vec{\boldsymbol{p}} = 4\pi\epsilon_0\left(\frac{\epsilon_r - 1 + iZ_0\sigma/k}{\epsilon_r + 2 + iZ_0\sigma/k}\right)R^3 E_0\hat{\boldsymbol{\epsilon}}_0\,. \tag{77}$$

To simplify the notation, we introduce the following notation:

$$\zeta \equiv \frac{\epsilon_r - 1 + iZ_0\sigma/k}{\epsilon_r + 2 + iZ_0\sigma/k}. \tag{78}$$

The differential scattering cross section is given by eq. (10.3) of Jackson,

$$\frac{d\sigma}{d\Omega} = \frac{r^2\left|\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{\rm sc}(\vec{\boldsymbol{x}})\right|^2}{\left|\hat{\boldsymbol{\epsilon}}_0^*\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{\rm inc}(\vec{\boldsymbol{x}})\right|^2}, \tag{79}$$

where $r \equiv |\vec{\boldsymbol{x}}|$ and $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$, and the scattered wave in the electric dipole approximation, in the far (radiation) zone, is given by eq. (10.2) of Jackson,

$$\vec{\boldsymbol{E}}_{\rm sc}(\vec{\boldsymbol{x}}) = \frac{1}{4\pi\epsilon_0}k^2\,\frac{e^{ikr}}{r}\big(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}}\big)\times\hat{\boldsymbol{n}}\,. \tag{80}$$

Hence, using eqs. (77)–(80),

$$\frac{d\sigma}{d\Omega} = k^4 R^6\,|\zeta|^2\left|\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\big[\big(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0\big)\times\hat{\boldsymbol{n}}\big]\right|^2. \tag{81}$$

Using $\big(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0\big)\times\hat{\boldsymbol{n}} = \hat{\boldsymbol{\epsilon}}_0 - \hat{\boldsymbol{n}}\big(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big)$, and noting that $\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\hat{\boldsymbol{n}} = 0$ (since the scattered electromagnetic wave is transverse), it follows that

$$\left|\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\big[\big(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0\big)\times\hat{\boldsymbol{n}}\big]\right|^2 = \left|\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right|^2. \tag{82}$$

Inserting this result back into eq. (81), we end up with

$$\frac{d\sigma}{d\Omega} = k^4 R^6\,|\zeta|^2\left|\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right|^2. \tag{83}$$

[8]Note that in light of eqs. (69) and (73), the assumption that the sphere is only slightly lossy implies that $Z_0\sigma/k \ll \epsilon_r$, which is a stronger constraint than $\delta \gg R$ (assuming $kR \ll 1$), since typical values of $\epsilon_r \sim \mathcal{O}(1)$.

Since the incoming wave is unpolarized and the polarization of thee outgoing wave is not measured, we must average over initial state polarizations and sum over final state polarizations. That is, the unpolarized differential scattering cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol} = k^4 R^6\,|\zeta|^2\,\frac{1}{2}\sum_{\lambda_0,\lambda}\left|\hat{\boldsymbol{\epsilon}}^{\lambda\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\right|^2 . \tag{84}$$

The sum over polarizations can be computed using the results of the class handout entitled *Polarization Vectors and Polarization Sums*,

$$\begin{aligned}\sum_{\lambda_0,\lambda}\left|\hat{\boldsymbol{\epsilon}}^{\lambda\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\right|^2 &= \left(\sum_{\lambda_0}\bigl(\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\bigr)_i\bigl(\hat{\boldsymbol{\epsilon}}_0^{\lambda_0\,*}\bigr)_j\right)\left(\sum_{\lambda}\bigl(\hat{\boldsymbol{\epsilon}}^{\lambda\,*}\bigr)_i\bigl(\hat{\boldsymbol{\epsilon}}^{\lambda}\bigr)_j\right)\\ &= \bigl[\delta_{ij}-(\hat{n}_0)_i(\hat{n}_0)_j\bigr]\bigl[\delta_{ij}-(\hat{n})_i(\hat{n})_j\bigr] = 1+\bigl(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\bigr)^2 .\end{aligned} \tag{85}$$

If $\theta$ is the scattering angle, then it follows that $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0=\cos\theta$. Hence, we end up with

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol} = \tfrac{1}{2}k^4 R^6\,|\zeta|^2\bigl(1+\cos^2\theta\bigr)\,, \tag{86}$$

after making use of eq. (78). Integrating over solid angles,

$$\int d\Omega(1+\cos^2\theta) = 2\pi\int_{-1}^{1}(1+\cos^2\theta)\,d\cos\theta = \frac{16\pi}{3}\,. \tag{87}$$

Hence, the total scattering cross section is given by

$$\sigma_{\rm sc} = \frac{8\pi k^4 R^6}{3}\left(\frac{(\epsilon_r-1)^2+Z_0^2\sigma^2/k^2}{(\epsilon_r+2)^2+Z_0^2\sigma^2/k^2}\right)\,. \tag{88}$$

(b) Show that the absorption cross section is

$$\sigma_{\rm abs} = 12\pi R^2\frac{RZ_0\sigma}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2} \tag{89}$$

The absorption cross section in SI units is

$$\frac{d\sigma_{\rm abs}}{d\Omega} = \frac{P_{\rm abs}}{\text{incident flux}} = \frac{2Z_0P_{\rm abs}}{\left|\hat{\boldsymbol{\epsilon}}_0^*\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{\rm inc}\right|^2} = \frac{2Z_0P_{\rm abs}}{|E_0|^2}\,, \tag{90}$$

where the power absorbed at the surface of the sphere of radius $R$ is given by eq. (10.134) of Jackson:

$$P_{\rm abs} = -\tfrac{1}{2}R^2\,{\rm Re}\oint(\vec{\boldsymbol{E}}_{\rm inc}+\vec{\boldsymbol{E}}_{\rm sc})\times(\vec{\boldsymbol{H}}^*_{\rm inc}+\vec{\boldsymbol{H}}^*_{\rm sc})\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega\,, \tag{91}$$

where

$$\vec{\boldsymbol{E}}_{\rm inc} = \hat{\boldsymbol{\epsilon}}_0 E_0 e^{ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}}\,, \qquad\qquad \vec{\boldsymbol{H}}_{\rm inc} = \hat{\boldsymbol{n}}_0\times\vec{\boldsymbol{E}}_{\rm inc}/Z_0 = (\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)Z_0^{-1}E_0 e^{ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}}\,, \tag{92}$$

However, one cannot use eq. (80) for $\vec{\boldsymbol{E}}_{\rm sc}$ and a corresponding expression, $\vec{\boldsymbol{B}}_{\rm sc} = \hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}_{\rm sc}/Z_0$, with $r = R$ since these two latter expressions are valid only in the asymptotic limit of large $r$. Since $kR \ll 1$ by assumption, it is appropriate to employ the near zone fields given in Jackson eq. (9.20), namely,

$$\vec{\boldsymbol{E}}_{\rm sc} = \frac{3\hat{\boldsymbol{n}}\big(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}\big)-\vec{\boldsymbol{p}}}{4\pi\epsilon_0 R^3}\big[1+\mathcal{O}(kR)\big]\,, \qquad\qquad \vec{\boldsymbol{H}}_{\rm sc} = \frac{i\omega(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})}{4\pi R^2}\big[1+\mathcal{O}(kR)\big]\,, \tag{93}$$

where $\vec{\boldsymbol{p}}$ is given by [cf. eqs. (77) and (78)]:

$$\vec{\boldsymbol{p}} = 4\pi\epsilon_0 E_0 R^3\zeta\hat{\boldsymbol{\epsilon}}_0\,. \tag{94}$$

Note that the expressions exhibited in eq. (93) differ from the exact expressions for the electric dipole fields given by eq. (9.18) of Jackson by terms of $\mathcal{O}(kR)$ or higher. Inserting eq. (94) for $\vec{\boldsymbol{p}}$ into eq. (93) and putting $Z_0^{-1} = \epsilon_0 c$ and $\omega = kc$, we end up with

$$\vec{\boldsymbol{E}}_{\rm sc} = E_0\zeta\big(3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0)-\hat{\boldsymbol{\epsilon}}_0\big)\,, \qquad\qquad \vec{\boldsymbol{H}}_{\rm sc} = iZ_0^{-1}E_0 kR\zeta\big(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0\big)\,. \tag{95}$$

Using eqs. (92) and (95) in evaluating eq. (91), we obtain:

$$\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm inc} = Z_0^{-1}|E_0|^2\,\hat{\boldsymbol{\epsilon}}_0\times(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}^*_0)\,, \tag{96}$$

$$\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc} = -iZ_0^{-1}|E_0|^2 kR\zeta^* e^{ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}}\,\hat{\boldsymbol{\epsilon}}_0\times(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^*_0)\,, \tag{97}$$

$$\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc} = Z_0^{-1}|E_0|^2\zeta e^{-ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}}\,\big[3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0)-\hat{\boldsymbol{\epsilon}}_0\big]\times(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}^*_0)\,, \tag{98}$$

$$\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc} = -iZ_0^{-1}|E_0|^2 kR|\zeta|^2\,\big[3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0)-\hat{\boldsymbol{\epsilon}}_0\big]\times(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^*_0)\,. \tag{99}$$

Using $\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{\epsilon}}_0 = 0$ (due to the transverse nature of the incident waves) and expanding out the triple cross products, the above results simplify:

$$(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm inc})\cdot\hat{\boldsymbol{n}} = Z_0^{-1}|E_0|^2\,\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}\,, \tag{100}$$

$$(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc})\cdot\hat{\boldsymbol{n}} = -iZ_0^{-1}|E_0|^2 kR\zeta^* e^{ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}}\,\big(1-|\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0|^2\big)\,, \tag{101}$$

$$(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc})\cdot\hat{\boldsymbol{n}} = -Z_0^{-1}|E_0|^2\zeta e^{-ikR\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}}\hat{\boldsymbol{n}}_0\cdot\hat{\boldsymbol{n}}\,, \tag{102}$$

$$(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc})\cdot\hat{\boldsymbol{n}} = iZ_0^{-1}|E_0|^2 kR|\zeta|^2\big(1-|\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0|^2\big)\,. \tag{103}$$

We immediately note that [cf. eq. (111)]:

$$\oint\big(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm inc}\big)\cdot\hat{\boldsymbol{n}}\,d\Omega = Z_0^{-1}|E_0|^2\,\hat{\boldsymbol{n}}_0\cdot\oint\hat{\boldsymbol{n}}\,d\Omega = 0\,. \tag{104}$$

Thus, we only need to consider the contributions of eqs. (101)–(103) to the integral given in eq. (91).

Since we eventually need to average over initial polarizations, let us take care of this now.

$$\tfrac{1}{2}\sum_{\lambda_0}|\boldsymbol{\hat{n}}\boldsymbol{\cdot}\boldsymbol{\hat{\epsilon}}_0|^2=\tfrac{1}{2}\bigl[1-(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})^2\bigr]\,. \tag{105}$$

Hence, it follows that

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\bigl(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\bigr)\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega=-\tfrac{1}{2}iZ_0^{-1}|E_0|^2kR\zeta^*\oint e^{ikR(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})}\left[1+(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})^2\right]d\Omega\,, \tag{106}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\bigl(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc}\bigr)\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega=-Z_0^{-1}|E_0|^2\zeta\oint e^{-ikR(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})}\,\boldsymbol{\hat{n}_0}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega\,. \tag{107}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\bigl(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\bigr)\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega=\tfrac{1}{2}iZ_0^{-1}|E_0|^2kR|\zeta|^2\oint\bigl[1+(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})^2\bigr]\,d\Omega\,. \tag{108}$$

In eq. (106), we can take $e^{ikR(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})}\simeq 1$, to leading order in $kR$. However, in eq. (107), the same approximation yields a vanishing integral [cf. eq. (111)]. Hence, in evaluating eq. (107), we must approximate $e^{ikR(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})}\simeq 1-ikR(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})$. Thus, to leading order in $kR$,

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\bigl(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\bigr)\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega=-\tfrac{1}{2}iZ_0^{-1}|E_0|^2kR\zeta^*\oint\bigl[1+(\boldsymbol{\hat{n}}_0\boldsymbol{\cdot}\boldsymbol{\hat{n}})^2\bigr]\,d\Omega\,, \tag{109}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\bigl(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc}\bigr)\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega=iZ_0^{-1}|E_0|^2kR\zeta\oint\bigl(\boldsymbol{\hat{n}_0}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\bigr)^2\,d\Omega\,. \tag{110}$$

The remaining integrals above are all very simple (see the class handout entitled, *Evaluation of some integrals over solid angles*). Using

$$\oint\hat{n}_i\,d\Omega=0\,,\qquad\qquad\oint\hat{n}_i\hat{n}_j\,d\Omega=\frac{4\pi}{3}\delta_{ij}\,, \tag{111}$$

it follows that to leading order in $kR$,

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\bigl(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\bigr)\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega=-\frac{8\pi i}{3}Z_0^{-1}|E_0|^2kR\zeta^*\,, \tag{112}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\bigl(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc}\bigr)\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega=\frac{4\pi i}{3}Z_0^{-1}|E_0|^2kR\zeta\,, \tag{113}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\bigl(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\bigr)\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,d\Omega=\frac{8\pi i}{3}Z_0^{-1}|E_0|^2kR|\zeta|^2\,. \tag{114}$$

Finally, in light pf eqs. (90) and (91), we obtain $\sigma_{\rm abs}$ by adding the results for three integrals above, taking the real part of the result and multiplying by $-R^2Z_0/|E_0|^2$. Noting that $\mathrm{Re}(-i\zeta^*)=\mathrm{Re}(i\zeta)=-\mathrm{Im}\,\zeta$, we end up with

$$\sigma_{\rm abs}=4\pi kR^3\,\mathrm{Im}\,\zeta\,. \tag{115}$$

If we now make use of eq. (78), it follows that $\operatorname{Im}\zeta = (3Z_0\sigma/k)/[(\epsilon_r+2)^2+(Z_0\sigma/k)^2]$, and we end up with

$$\sigma_{\rm abs} = 12\pi R^2 \frac{RZ_0\sigma}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}\,, \tag{116}$$

in agreement with eq. (89).

Admittedly, there is a much simpler method for deriving eq. (116). Details can be found at the very end of this problem set.

## A more precise calculation that does not rely on $kR \ll 1$

In obtaining eq. (116), we employed the electric dipole fields in the near field approximation. However, a more precise calculation that does not assume that $kR \ll 1$ can be given by making use of the exact electric dipole fields in the source-free region. Using eq. (9.18) of Jackson,

$$\vec{\boldsymbol{H}} = \frac{ck^2}{4\pi}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\,, \tag{117}$$

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\times\hat{\boldsymbol{n}}\,\frac{e^{ikr}}{r} + \big[3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})-\vec{\boldsymbol{p}}\big]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}. \tag{118}$$

Inserting the result of eq. (94), we replace the scattering fields given by eq. (95) with:

$$\vec{\boldsymbol{E}}_{\rm sc} = E_0\zeta e^{ikR}\left\{k^2R^2\big[\hat{\boldsymbol{\epsilon}}_0-\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)\big]+(1-ikR)\big[3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)-\hat{\boldsymbol{\epsilon}}_0\big]\right\}, \tag{119}$$

$$\vec{\boldsymbol{H}}_{\rm sc} = iZ_0^{-1}E_0kR\zeta e^{ikR}(1-ikR)\big(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0\big)\,. \tag{120}$$

Eqs. (101)–(103) are then replaced by:

$$(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc})\boldsymbol{\cdot}\hat{\boldsymbol{n}} = -iZ_0^{-1}|E_0|^2kR(1+ikR)\zeta^*e^{ikR(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}-1)}\big(1-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\big)\,, \tag{121}$$

$$(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc})\boldsymbol{\cdot}\hat{\boldsymbol{n}} = -Z_0^{-1}|E_0|^2(1-ikR-k^2R^2)\zeta e^{-ikR(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}-1)}\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,, \tag{122}$$

$$(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc})\boldsymbol{\cdot}\hat{\boldsymbol{n}} = iZ_0^{-1}|E_0|^2kR(1+ikR)(1-ikR-k^2R^2)|\zeta|^2\big(1-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\big)\,. \tag{123}$$

Averaging over initial polarizations yields,

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\big(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = -\tfrac{1}{2}iZ_0^{-1}|E_0|^2kR(1+ikR)\zeta^*\oint e^{ikR(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}-1)}\big[1+(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\big]\,d\Omega\,, \tag{124}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\big(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = -Z_0^{-1}|E_0|^2(1-ikR-k^2R^2)\zeta\oint e^{-ikR(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}-1)}\,\hat{\boldsymbol{n}}_{\boldsymbol{0}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega\,, \tag{125}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\big(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = \tfrac{1}{2}iZ_0^{-1}|E_0|^2kR(1-ik^3R^3)|\zeta|^2\oint\big[1+(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\big]\,d\Omega\,. \tag{126}$$

To perform the integrals above, we choose the $z$-axis to point in the $\hat{\boldsymbol{n}}_0$ direction; that is, $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \cos\theta$. Setting $w = \cos\theta$, we obtain

$$\oint e^{ikR(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}-1)}\left[1 + (\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\right] d\Omega = 2\pi\int_{-1}^{1} e^{ikR(w-1)}(1+w^2)\,dw$$

$$= \frac{8\pi e^{-ikR}\left[kR\cos(kR) - (1-k^2R^2)\sin(kR)\right]}{k^3R^3}\,,$$

$$\oint e^{-ikR(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}-1)}\,(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})\,d\Omega = 2\pi\int_{-1}^{1} e^{-ikR(w-1)}\,w\,dw = -\,\frac{4\pi i e^{ikR}\left[\sin(kR) - kR\cos(kR)\right]}{k^2R^2}\,.$$

Hence, we end up with

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\big(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = -\,\frac{4\pi i Z_0^{-1}|E_0|^2(1+ikR)\zeta^* e^{-ikR}\left[kR\cos(kR) - (1-k^2R^2)\sin(kR)\right]}{k^2R^2}\,,$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\big(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = \frac{4\pi i Z_0^{-1}|E_0|^2(1-ikR-k^2R^2)\zeta e^{ikR}\left[\sin(kR) - kR\cos(kR)\right]}{k^2R^2}\,,$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint\big(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = \frac{8\pi i}{3}Z_0^{-1}|E_0|^2 kR(1-ik^3R^3)|\zeta|^2\,.$$

Taking the real part of each of the expressions above,

$$\tfrac{1}{2}\sum_{\lambda_0}\oint {\rm Re}\big(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = -\,\frac{4\pi Z_0^{-1}|E_0|^2\left[kR\cos(kR) - (1-k^2R^2)\sin(kR)\right]}{k^2R^2}$$
$$\times\Big\{\left[\sin(kR) - kR\cos(kR)\right]{\rm Re}\,\zeta + \left[kR\sin(kR) + \cos(kR)\right]{\rm Im}\,\zeta\Big\}\,, \tag{127}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint {\rm Re}\big(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = \frac{4\pi Z_0^{-1}|E_0|^2\left[\sin(kR) - kR\cos(kR)\right]}{k^2R^2}$$
$$\times\Big\{\left[(1-k^2R^2)\sin(kR) - kR\cos(kR)\right]{\rm Re}\,\zeta$$
$$+\left[kR\sin(kR) + (1-k^2R^2)\cos(kR)\right]{\rm Im}\,\zeta\Big\}, \tag{128}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint {\rm Re}\big(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\big)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = \frac{8\pi}{3}Z_0^{-1}|E_0|^2k^4R^4|\zeta|^2\,. \tag{129}$$

Adding the results of eqs. (127) and (128) yields a remarkably simple expression,

$$\tfrac{1}{2}\sum_{\lambda_0}\oint {\rm Re}\big[\big(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc}\big) + \big(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc}\big)\big]\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = -4\pi Z_0^{-1}|E_0|^2 kR\,{\rm Im}\,\zeta\,. \tag{130}$$

Finally, after making use of eqs. (90), (91), (104), (129), and (130), we end up with the exact result:

$$\sigma_{\rm abs} = 4\pi R^2\bigl[kR\,{\rm Im}\,\zeta - \tfrac{2}{3}(kR)^4|\zeta|^2\bigr]\,, \tag{131}$$

which differs from eq. (115) by a term of $\mathcal{O}\bigl((kR)^4\bigr)$. An alternative method of computing $\sigma_{\rm abs}$ that reproduces the result of eq. (131) is given at the end of the solution to part (c).

Using the result of part (a) for the scattering cross section $\sigma_{\rm sc}$, we obtain the total cross section.

$$\sigma_t = \sigma_{\rm sc} + \sigma_{\rm abs} = 4\pi k R^3\,{\rm Im}\,\zeta\,. \tag{132}$$

(c) Using the result of part (a), write down the forward scattering amplitude and use the optical theorem to evaluate the total cross section. Compare your answer with the sum of the scattering and absorption cross sections obtained in parts (a) and (b). Comment.

The scattering amplitude $\vec{\boldsymbol{F}}(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}_0)$ is defined on p. 485 of Jackson is equal to the coefficient of $e^{ikr}/r$ of $\vec{\boldsymbol{E}}_{\rm sc}$ in the far (radiation) zone, where $\vec{\boldsymbol{k}}_0\equiv k\hat{\boldsymbol{n}}_0$ and $\vec{\boldsymbol{k}}\equiv k\hat{\boldsymbol{n}}$. In light of eqs. (80) and (94), we can identify

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}_0) = E_0R^3\zeta\bigl(\vec{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_0\bigr)\times\vec{\boldsymbol{k}}\,. \tag{133}$$

Using eq. (10.138) of Jackson, we define the normalized scattering amplitude $\vec{\boldsymbol{f}}(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}_0)$ as follows,

$$\vec{\boldsymbol{f}}(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}_0) = \frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}_0)}{E_0} = R^3\zeta\bigl[k^2\hat{\boldsymbol{\epsilon}}_0 - \vec{\boldsymbol{k}}(\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\vec{\boldsymbol{k}})\bigr]\,, \tag{134}$$

after simplifying the triple product given in eq. (133).

In terms of the total cross section $\sigma_t$ and $\vec{\boldsymbol{f}}$, the optical theorem reads [cf. eq.(10.139) of Jackson]:

$$\sigma_t = \frac{4\pi}{k}\,{\rm Im}\bigl[\hat{\boldsymbol{\epsilon}}_0^*\boldsymbol{\cdot}\vec{\boldsymbol{f}}(\vec{\boldsymbol{k}}=\vec{\boldsymbol{k}}_0)\bigr]\,. \tag{135}$$

Using eq. (134) and noting that $\hat{\boldsymbol{\epsilon}}_0^*\boldsymbol{\cdot}\vec{\boldsymbol{k}}_0 = k\hat{\boldsymbol{\epsilon}}_0^*\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0 = 0$ due to the transverse nature of the incoming plane wave, the optical theorem yields,

$$\sigma_t = 4\pi k R^3\,{\rm Im}\,\zeta\,. \tag{136}$$

It is instructive to compare eq. (136) with the sum of the scattering and absorption cross sections obtained in eqs. (88) and (115):

$$\sigma_t = 4\pi R^2\bigl[kR\,{\rm Im}\,\zeta + \tfrac{2}{3}(kR)^4|\zeta|^2\bigr]\,. \tag{137}$$

The resolution of this mismatch is discussed below eq. (10.151) of Jackson. Here, I quote the relevant comments from Jackson below:

> These seeming contradictions are reflections of the necessity of different orders of approximation required to obtain consistency between the two sides of the optical theorem. In the long-wavelength limit it is necessary to evaluate the forward scattering amplitude to higher order in powers of $\omega$ to find the scattering cross section contribution in the total cross sections by means of the optical theorem.

Indeed, at the end of the solution to part (b), we provided a more precise computation of $\sigma_{\rm abs}$ that captures higher order terms in $kR$. The result of this calculation given in eq. (132) was:

$$\sigma_{\rm abs} = 4\pi R^2\bigl[kR\,{\rm Im}\,\zeta - \tfrac{2}{3}(kR)^4|\zeta|^2\bigr]\,. \tag{138}$$

Adding this to the scattering cross section obtained in eq. (88) yields [cf. eq. (132)]:

$$\sigma_t = \sigma_{\rm sc} + \sigma_{\rm abs} = 4\pi kR^3\,{\rm Im}\,\zeta\,, \tag{139}$$

in agreement with the result of the optical theorem obtained in eq. (136).

## An alternative method for part (b)

An alternative method of obtaining the absorptive cross section is to compute the power absorbed at the surface of a very large sphere of radius $r \gg R$. In this case, $P_{\rm abs}$ is given by

$$P_{\rm abs} = -\tfrac{1}{2}r^2\,{\rm Re}\oint(\vec{\boldsymbol{E}}_{\rm inc}+\vec{\boldsymbol{E}}_{\rm sc})\times(\vec{\boldsymbol{H}}^*_{\rm inc}+\vec{\boldsymbol{H}}^*_{\rm sc})\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega\,. \tag{140}$$

Because $r \gg R$, we are now justified in employing the asymptotic (large $r$) form for the electric field of the electric dipole given in eq. (80). The incoming and scattered electric fields are given by eqs. (66) and (80), respectively, using eq. (94) for $\vec{\boldsymbol{p}}$ [cf. eqs. (77) and (78)], which are evaluated on the surface of a very large sphere of radius $r$,

$$\vec{\boldsymbol{E}}_{\rm inc} = \hat{\boldsymbol{\epsilon}}_0 E_0 e^{ikr\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}}\,,\qquad\qquad \vec{\boldsymbol{E}}_{\rm sc} = E_0k^2R^3\zeta\frac{e^{ikr}}{r}\bigl[\hat{\boldsymbol{\epsilon}}_0 - \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)\bigr]\,. \tag{141}$$

The corresponding magnetic fields in the far (radiation) zone are given in eqs. (10.1) and (10.2) of Jackson,

$$\vec{\boldsymbol{H}}_{\rm inc} = \hat{\boldsymbol{n}}_0\times\vec{\boldsymbol{E}}_{\rm inc}/Z_0 = (\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}_0)Z_0^{-1}E_0e^{ikr\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}}\,, \tag{142}$$

$$\vec{\boldsymbol{H}}_{\rm sc} = \hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}_{\rm sc}/Z_0 = (\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0)Z_0^{-1}E_0k^2R^3\zeta\frac{e^{ikr}}{r}\,. \tag{143}$$

At the end of the computation, we will take the limit of $r\to\infty$ to obtain the final result.

Plugging eqs. (141)–(143) eq. (140), the integrand is equal to the sum of the following four terms:

$$\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm inc} = Z_0^{-1}|E_0|^2\,\hat{\boldsymbol{\epsilon}}_0\times(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}^*_0)\,, \tag{144}$$

$$\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc} = Z_0^{-1}|E_0|^2k^2R^3r^{-1}\zeta^*e^{ikr[(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})-1]}\,\hat{\boldsymbol{\epsilon}}_0\times(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^*_0)\,, \tag{145}$$

$$\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc} = Z_0^{-1}|E_0|^2k^2R^3r^{-1}\zeta e^{-ikr[(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})-1]}\,\bigl[\hat{\boldsymbol{\epsilon}}_0-\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)\bigr]\times(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{\epsilon}}^*_0)\,, \tag{146}$$

$$\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc} = Z_0^{-1}|E_0|^2k^4|\zeta|^2R^4\,\bigl[\hat{\boldsymbol{\epsilon}}_0-\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)\bigr]\times(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}^*_0)\,. \tag{147}$$

Using $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = 0$ (due to the transverse nature of the incident waves) and expanding out the triple cross products, the above results simplify:

$$(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm inc})\boldsymbol{\cdot}\hat{\boldsymbol{n}} = Z_0^{-1}|E_0|^2\,\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,, \tag{148}$$

$$(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc})\boldsymbol{\cdot}\hat{\boldsymbol{n}} = Z_0^{-1}|E_0|^2k^2R^3r^{-1}\zeta^* e^{ikr[(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})-1]}\left(1-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\right), \tag{149}$$

$$(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc})\boldsymbol{\cdot}\hat{\boldsymbol{n}} = Z_0^{-1}|E_0|^2k^2R^3r^{-1}\zeta e^{-ikr[(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})-1]}\,\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,, \tag{150}$$

$$(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc})\boldsymbol{\cdot}\hat{\boldsymbol{n}} = Z_0^{-1}|E_0|^2k^4R^4|\zeta|^2\left(1-|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\right). \tag{151}$$

As previously noted,

$$\oint(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm inc})\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = Z_0^{-1}|E_0|^2\,\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\oint\hat{\boldsymbol{n}}\,d\Omega = 0\,. \tag{152}$$

Thus, we only need to consider the contributions of eqs. (149)–(151) to the integral given in eq. (140).

Next, we average over initial polarizations using eq. (105) to obtain:

$$\tfrac{1}{2}\sum_{\lambda_0}\oint(\vec{\boldsymbol{E}}_{\rm inc}\times\vec{\boldsymbol{H}}^*_{\rm sc})\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = \tfrac{1}{2}Z_0^{-1}|E_0|^2k^2R^3r^{-1}\zeta^*\,e^{-ikr}\oint e^{ikr(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})}\left[1+(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\right]d\Omega\,, \tag{153}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm inc})\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = Z_0^{-1}|E_0|^2k^2R^3r^{-1}\zeta\,e^{ikr}\oint e^{-ikr(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})}\,\hat{\boldsymbol{n}}_{\boldsymbol{0}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega\,. \tag{154}$$

$$\tfrac{1}{2}\sum_{\lambda_0}\oint(\vec{\boldsymbol{E}}_{\rm sc}\times\vec{\boldsymbol{H}}^*_{\rm sc})\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega = \tfrac{1}{2}Z_0^{-1}|E_0|^2k^4R^4|\zeta|^2\oint\left[1+(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\right]d\Omega\,. \tag{155}$$

To perform the integrals above, we choose the $z$-axis to point in the $\hat{\boldsymbol{n}}_0$ direction; that is, $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \cos\theta$. Using eqs. (90) and (140), we obtain:

$$\begin{aligned}\sigma_{\rm abs} &= -\pi k^2R^3r\biggl\{{\rm Re}\,\zeta\int_{-1}^1(1+\cos\theta)^2\cos\bigl[kr(\cos\theta-1)\bigr]d\cos\theta \\ &\qquad\qquad +{\rm Im}\,\zeta\int_{-1}^1(1+\cos\theta)^2\sin\bigl[kr(\cos\theta-1)\bigr]d\cos\theta\biggr\} - \tfrac{8}{3}\pi k^4R^6|\zeta|^2 \\ &= \frac{2\pi R^3}{k}\left\{\left[\frac{\sin(2kr)-2kr}{r^2}\right]{\rm Re}\,\zeta + \left[\frac{\cos(2kr)-1+2k^2r^2}{r^2}\right]{\rm Im}\,\zeta\right\} - \tfrac{8}{3}\pi k^4R^6|\zeta|^2\,.\end{aligned} \tag{156}$$

We can now take the $r\to\infty$ limit. We are then left with

$$\sigma_{\rm abs} = 4\pi R^2\bigl[kR\,{\rm Im}\,\zeta - \tfrac{2}{3}(kR)^4|\zeta|^2\bigr]\,, \tag{157}$$

in agreement with the more precise computation provided at the end of the solution to part (b) [cf. eq. (131)].

## Yet another alternative method for part (b)[9]

Start with

$$P_{\rm abs} = -\tfrac{1}{2}R^2\,{\rm Re}\oint (\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega\,, \tag{158}$$

where $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ are the total electric and magnetic fields on the surface of the sphere of radius $R$. Using the divergence theorem, we can rewrite eq. (158) as

$$P_{\rm abs} = -\tfrac{1}{2}\ {\rm Re}\int_V \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*)\,d^3x \tag{159}$$

where the volume integral is evaluated over the solid sphere of radius $R$. Note the well-known vector identity,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*) = \vec{\boldsymbol{H}}^*\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}) - \vec{\boldsymbol{E}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{H}}^*)\,. \tag{160}$$

Next, we make use of Maxwell's equations for harmonic fields. In particular,

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = i\omega\vec{\boldsymbol{B}} = ikZ_0\vec{\boldsymbol{H}}\,,\qquad\qquad \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} - i\omega\vec{\boldsymbol{D}} = \left(\sigma - \frac{ik}{Z_0}\right)\vec{\boldsymbol{E}}\,, \tag{161}$$

where we have used Ohm's Law ($\vec{\boldsymbol{J}} = \sigma\vec{\boldsymbol{E}}$) in the final step above. Hence, eq. (160) yields

$${\rm Re}\,\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*) = -\sigma|\vec{\boldsymbol{E}}|^2\,. \tag{162}$$

Inside the sphere of radius $R$, the electric field is constant, with a magnitude given by eq. (4.55) of Jackson,

$$|\vec{\boldsymbol{E}}| = \left|\frac{3E_0}{\epsilon/\epsilon_0 + 2}\right|\,, \tag{163}$$

where $\epsilon/\epsilon_0$ is given by eq. (69), and we have approximated the incident electric field to be constant inside the sphere by setting $e^{ikr\hat{\boldsymbol{n}}_0\cdot\hat{n}} \simeq 1$ for $r \leq R$ (since by assumption, $kR \ll 1$). That is,

$$|\vec{\boldsymbol{E}}| = \left|\frac{3E_0}{\epsilon_r + i\sigma Z_0/k + 2}\right|\,,\qquad \text{for } r \leq R. \tag{164}$$

In particular,

$$|\vec{\boldsymbol{E}}|^2 = \frac{9|E_0|^2}{(\epsilon_r+2)^2 + \sigma^2 Z_0^2/k^2}\,,\qquad \text{for } r \leq R. \tag{165}$$

Hence, eqs. (159), (162), and (165) yield

$$P_{\rm abs} = \frac{6\pi R^3\sigma|E_0|^2}{(\epsilon_r+2)^2 + \sigma^2 Z_0^2/k^2}\,. \tag{166}$$

Finally, in light of eq. (90), we obtain

$$\frac{d\sigma_{\rm abs}}{d\Omega} = \frac{12\pi R^3 Z_0\sigma}{(\epsilon_r+2)^2 + \sigma^2 Z_0^2/k^2}\,, \tag{167}$$

in agreement with eq. (89). However, due to its reliance on $kR \ll 1$, this method does not provide a way to derive the more precise result of eq. (131).

[9] Inspired by George Demetriou.