DUE: TUESDAY, MARCH 3, 2026

1. In SI units, the leading $\mathcal{O}(r^{-1})$ contributions to the multipole expansions of the magnetic field, $\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})e^{-i\omega t}$, and the electric field, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t}$, in the far (radiation) zone are given by:

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(-i)^{\ell+1}}{\sqrt{\ell(\ell+1)}}\left\{a_E(\ell,m)\frac{e^{ikr}}{kr}\vec{\boldsymbol{L}}\big[Y_{\ell m}(\theta,\phi)\big]-\frac{i}{k}a_M(\ell,m)\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\left[\frac{e^{ikr}}{kr}Y_{\ell m}(\theta,\phi)\right]\right\},$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})=Z_0\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(-i)^{\ell+1}}{\sqrt{\ell(\ell+1)}}\left\{\frac{i}{k}a_E(\ell,m)\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\left[\frac{e^{ikr}}{kr}Y_{\ell m}(\theta,\phi)\right]+a_M(\ell,m)\frac{e^{ikr}}{kr}\vec{\boldsymbol{L}}\big[Y_{\ell m}(\theta,\phi)\big]\right\},$$

where $\vec{\boldsymbol{L}}\equiv -i\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}$ is a differential operator that acts on functions of $\vec{\boldsymbol{x}}=(r,\theta,\phi)$, $k=\omega/c$, $r\equiv|\vec{\boldsymbol{x}}|$, Z_0 is the impedance of free space, and $a_E(\ell,m)$ and $a_M(\ell,m)$ are related to the electric and magnetic multipole moment spherical tensors, $Q_{\ell m}$ and $M_{\ell m}$, respectively:

$$a_E(\ell,m)=-\frac{ick^{\ell+2}}{(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}Q_{\ell m}\,,\qquad a_M(\ell,m)=\frac{ik^{\ell+2}}{(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}M_{\ell m}\,,$$

after setting the intrinsic magnetization to zero for simplicity.

(a) Using the multipole expansions above, derive the corresponding multipole expansions for $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})$ and $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$, where $\hat{\boldsymbol{n}}\equiv\vec{\boldsymbol{x}}/r$.

HINT: In the large r approximation, the leading contributions to $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}})$ and $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$ are of $\mathcal{O}(1/r^2)$. All terms of this order should be retained, and terms of $\mathcal{O}(1/r^3)$ may be neglected. Express your final results in terms of $Q_{\ell m}$ and $M_{\ell m}$.

(b) Using the results of part (a), express the first two terms of the multipole expansion of $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$ in terms of the electric dipole vector $\vec{\boldsymbol{p}}$ defined in eq. (9.17) of Jackson, and the electric quadrupole vector $\vec{\boldsymbol{Q}}$ defined in eq. (9.43) of Jackson.

2. Jackson, problem 9.12

HINT: The charge density can be expressed as $\rho(\vec{\boldsymbol{x}},t)=\rho_0\Theta(R(\theta)-r)$, where the step function $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$. The constant ρ_0 can be determined in terms of the total charge Q which is conserved and hence time-independent. Find the relation between ρ_0 and Q assuming that $\beta\ll 1$. Then writing $\beta=\beta_0 e^{-i\omega t}$, expand the expression for $\rho(\vec{\boldsymbol{x}},t)$ to linear order in β_0. You can now use this expression to evaluate the electric multipole moments. You can also evaluate the current density $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)$ by making use of the continuity equation. This will be needed to evaluate the magnetic multipole moments.

3. Jackson, problem 9.15

The first sentence of the problem posed by Jackson is likely to be misinterpreted. Thus, I have rewritten the first sentence below to make the statement of the problem clearer:

> Two fixed electric dipoles of dipole moment p, with their axes parallel but their moments directed oppositely, are located in the x–y plane a distance $2a$ apart. Both dipole moment axes are perpendicular to the plane (i.e., they point parallel and antiparallel to the z-axis, respectively).

4. Jackson, problem 9.17

5. Jackson, problem 12.3

6. Jackson, problem 12.11

HINT 1: In part (a), instead of making use of the parenthetical hint provided by Jackson, you should proceed as follows. Begin your analysis with the Thomas equation, which is given in eq. (11.170) of Jackson. Note that the Thomas equation is an equation of the form

$$\frac{d\vec{\boldsymbol{s}}}{dt} = \vec{\boldsymbol{s}} \times \vec{\boldsymbol{\omega}}\,,$$

where $\vec{\boldsymbol{s}}$ is the spin vector and $\vec{\boldsymbol{\omega}}$ is a vector whose magnitude is the spin precession frequency.

HINT 2: Note that the energy values given in part (c) correspond to the total relativistic energies.