This is an open book exam with a time limit of three hours. You may refer to Jackson's text, a second textbook of your choosing, your own personal notes, and any reference of integrals or other mathematical facts. Regarding laptop use, you are permitted to consult any documents located on the Physics 214 course webpage. However, you may not *otherwise use the Internet, nor collaborate with anyone else during the exam.*

Note that you do not have to derive all results from scratch. However, if you use a particular result, you should cite its source (e.g. an equation in Jackson, an equation derived in a class lecture, or an equation that appears in a solution set or in a class handout).

This exam consists of four problems with ten individual parts in total, each part worth ten points. Use this information to manage your time appropriately during the exam.

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1. Consider the case of *localized* sources consisting of a charge density $\rho(\vec{\boldsymbol{x}}, t)$ and a current density $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)$.

(a) Suppose that the charge density is *independent* of time. Derive the following two identities:

$$\int J^i\, d^3x = \int \partial_k(J^k x^i) d^3x = 0\,, \tag{1}$$

$$\int (J^i x^j + J^j x^i)\, d^3x = \int \partial_k(J^k x^i x^j) d^3x = 0\,, \tag{2}$$

where there is an implicit sum over the repeated index k. Then, using eq. (2), show that

$$\int J^i x^j\, d^3x = -\tfrac{1}{2}\epsilon_{ijk} \int (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}})^k\, d^3x = -c\, \epsilon_{ijk} m^k\,, \tag{3}$$

where m^k is the kth component of the magnetic dipole moment vector (in gaussian units).

(b) Suppose that the charge and current densities are harmonic in time. That is, $\rho(\vec{\boldsymbol{x}}, t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$ and $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})e^{-i\omega t}$. How are the identities obtained in eqs. (1) and (3) modified?

2. Consider a magnetic dipole moment $\vec{\boldsymbol{m}}_0$ derived from a *steady* localized current density $\vec{\boldsymbol{J}}_0$ in the rest frame of the magnetic dipole, denoted by K_0'. Moreover, in frame K_0', the charge density ρ_0 is equal to zero. The frame K_0' moves with velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ with respect to the laboratory frame K. The two reference frames coincide at time $t = t' = 0$. Do *not* assume that $\beta \equiv |\vec{\boldsymbol{\beta}}| \ll 1$.

(a) The magnetic moment (in gaussian units) in reference frame K_0' is defined by

$$\vec{\boldsymbol{m}}_0 = \frac{1}{2c}\int \vec{\boldsymbol{x}}\,' \times \vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\, d^3x'\,.$$

Show that

$$\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}_0 = \frac{1}{c}\int \vec{\boldsymbol{x}}\,'\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\big]\, d^3x\,.$$

(b) Find the charge and current densities $\rho(\vec{\boldsymbol{x}}, t)$ and $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)$ in the frame K. Verify your result by explicitly showing that the continuity equation is satisfied in the frame K.

HINT: To solve this problem, first write out the expressions for the spacetime coordinates $(x_0'\,;\,\vec{\boldsymbol{x}}\,')$ of frame K_0' in terms of the spacetime coordinates $(x_0\,;\,\vec{\boldsymbol{x}})$ in frame K. You will need these expressions for both parts (b) and (c) of this problem.

(c) Determine the electric dipole moment measured in the frame K using

$$\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}}, t)\, d^3x\,,$$

where t is held fixed. Express your result in terms of $\vec{\boldsymbol{m}}_0$.

HINT: The most straightforward way to proceed is to multiply both sides of the equation obtained for $\rho(\vec{\boldsymbol{x}}, t)$ in part (b) by $\vec{\boldsymbol{x}}\,'\, d^3x'$. Then, rewrite $\vec{\boldsymbol{x}}\,'\rho(\vec{\boldsymbol{x}})\, d^3x'$ in terms of $\vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}}, t)\, d^3x$, in light of the HINT for part (b).

3. An electron of charge e and mass m moves in a plane perpendicular to a uniform magnetic field B. If the energy loss by radiation is neglected, the orbit is a circle of some radius R. Let E be the total relativistic energy of the electron, and assume that $E \gg mc^2$ (corresponding to ultra-relativistic motion).

(a) Express B analytically in terms of the parameters given above. Compute numerically the required magnetic field B, in gauss, for the case of $R = 30$ meters and $E = 2.5$ GeV.

(b) In fact, the electron radiates electromagnetic energy. Suppose that the energy loss per revolution, ΔE, is small compared to E. Express the ratio $\Delta E/E$ analytically in terms of the parameters given above.

HINT: You can assume that the power is constant (independent of time) during the time Δt it takes for the electron to complete one orbit. In the ultra-relativistic limit, $v \simeq c$, so you can easily compute Δt.

(c) Evaluate the ratio obtained in part (b) numerically using the values of R and E given in part (a). Note that the rest mass of the electron is $mc^2 = 511$ keV.

4. The Thomson scattering cross section for an incoming wave with polarization $\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)}$ and an outgoing wave with polarization $\hat{\boldsymbol{\epsilon}}^{(\lambda)}$ was obtained in class,

$$\frac{d\sigma_{\lambda_0\lambda}}{d\Omega} = r_c^2 \left|\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)} \boldsymbol{\cdot} \hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 ,$$

where $r_c \equiv e^2/(mc^2)$ is the classical radius of the electron (in gaussian units). Imagine another theory, in which the corresponding cross section is given by

$$\frac{d\tilde{\sigma}_{\lambda_0\lambda}}{d\Omega} = r_c^2 \left|\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 ,$$

where the notation $\tilde{\sigma}$ distinguishes this case from Thomson scattering. In both cases, it is convenient to choose a coordinate system such that the incident wave approaches the origin of the coordinate system along the z-axis, and the scattered wave is detected at an angle θ with respect to the z axis in the x–z plane.

(a) If the polarizations of the incident and the scattered waves are not measured, compute the angular distribution of the scattered wave for both cases introduced above. Evaluate the corresponding total cross sections.

(b) An observer measures the polarization of the scattered wave at a scattering angle of $\theta = 90^\circ$. Compare the properties of the observed polarizations corresponding to the two cases introduced above.