1. Consider the case of *localized* sources consisting of a charge density $\rho(\vec{\boldsymbol{x}}, t)$ and a current density $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t)$.

(a) Suppose that the charge density is *independent* of time. Derive the following two identities:

$$\int J^i \, d^3x = \int \partial_k (J^k x^i) d^3x = 0\,, \tag{1}$$

$$\int (J^i x^j + J^j x^i)\, d^3x = \int \partial_k (J^k x^i x^j) d^3x = 0\,, \tag{2}$$

where there is an implicit sum over the repeated index k. Then, using eq. (2), show that

$$\int J^i x^j \, d^3x = -\tfrac{1}{2}\epsilon_{ijk} \int (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}})^k \, d^3x = -c\,\epsilon_{ijk} m^k\,, \tag{3}$$

where m^k is the kth component of the magnetic dipole moment vector (in gaussian units).

The continuity equation states that

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{J}} + \frac{\partial \rho}{\partial t} = 0\,. \tag{4}$$

Thus, if ρ is time-independent, then $\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{J}} = 0$. It follows that

$$\partial_k (J^k x^i) = x^i \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{J}} + J^k \partial_k x^i = J^i\,, \tag{5}$$

after using $\partial_k x^i = \delta^i_k$. Hence, integrating both sides of eq. (5) over all space, the left hand side is zero since localized sources by definition vanish at spatial infinity. It then follows that

$$\int J^i \, d^3x = 0\,, \tag{6}$$

which establishes eq. (1).

A similar analysis shows that

$$\partial_k (J^k x^i x^j) = x^i x^j \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{J}} + J^k \big(x^j \partial_k x^i + x^i \partial_k x^j\big) = J^i x^j + J^j x^i\,. \tag{7}$$

Integrating both sides of eq. (5) over all space, it follows that

$$\int (J^i x^j + J^j x^i)\, d^3x = 0\,, \tag{8}$$

establishes eq. (2).

Finally, to derive eq. (3), we first write

$$J^i x^j = \tfrac{1}{2}(J^i x^j + J^j x^i) + \tfrac{1}{2}(J^i x^j - J^j x^i)\,. \tag{9}$$

Next, we note that

$$-\epsilon_{ijk}(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}})^k = J^i x^j - J^j x^i\,. \tag{10}$$

The proof of eq. (10) follows from

$$-\epsilon_{ijk}(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}})^k = -\epsilon_{ijk}\epsilon_{k\ell m}x^\ell J^m = (\delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm})x^\ell J^m = J^i x^j - J^j x^i\,. \tag{11}$$

Eqs. (9) and (10) yield:

$$J^i x^j = \tfrac{1}{2}(J^i x^j + J^j x^i) - \tfrac{1}{2}\epsilon_{ijk}(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}})^k\,. \tag{12}$$

Integrating over all space and using eq. (2), we end up with

$$\int J^i x^j\, d^3x = -\tfrac{1}{2}\epsilon_{ijk}\int (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}})^k\, d^3x\,, \tag{13}$$

which establishes eq. (3). Recalling the definition of the magnetic moment (in gaussian units),

$$\vec{\boldsymbol{m}} = \frac{1}{2c}\int \vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}}\, d^3x\,, \tag{14}$$

we can rewrite eq. (13) as

$$\int J^i x^j\, d^3x = -c\,\epsilon_{ijk}m^k\,. \tag{15}$$

(b) Suppose that the charge and current densities are harmonic in time. That is, $\rho(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$ and $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})e^{-i\omega t}$. How are the identities obtained in eqs. (1) and (3) modified?

In light of the continuity equation [eq. (4)],

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = i\omega\rho(\vec{\boldsymbol{x}})\,. \tag{16}$$

Thus, eq. (5) is modified to

$$\partial_k(J^k x^i) = x^i\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} + J^k\partial_k x^i = i\omega x^i\rho + J^i\,, \tag{17}$$

$$\partial_k(J^k x^i x^j) = x^i x^j\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} + J^k\left(x^j\partial_k x^i + x^i\partial_k x^j\right) = i\omega x^i x^j\rho + J^i x^j + J^j x^i\,. \tag{18}$$

Integrating over space, we obtain

$$\int \vec{\boldsymbol{J}}\, d^3x = -i\omega\vec{\boldsymbol{p}}\,, \tag{19}$$

$$\int (J^i x^j + J^j x^i) d^3x = -i\omega\int x^i x^j \rho\, d^3x\,, \tag{20}$$

where

$$\vec{\boldsymbol{p}} \equiv \int \vec{\boldsymbol{x}}\,\rho\,d^3x\,, \tag{21}$$

is the electric dipole moment vector. Using eq. (12), it follows that

$$\int J^i x^j\,d^3x = -\tfrac{1}{2}i\omega \int x^i x^j\,\rho\,d^3x - c\,\epsilon_{ijk}m^k\,, \tag{22}$$

where $\vec{\boldsymbol{m}}$ is the magnetic dipole moment [eq. (14)].

It is convenient to introduce the electric quadrupole moment,

$$Q^{ij} = \int (3x^i x^j - r^2\delta^{ij})\,\rho\,d^3x\,, \tag{23}$$

where $r \equiv |\vec{\boldsymbol{x}}|$. Then,

$$\int x^i x^j\,\rho\,d^3x = \frac{1}{3}\left(Q^{ij} - \delta^{ij}\int r^2\,\rho\,d^3x\right)\,. \tag{24}$$

Then, one can rewrite eq. (22) as

$$\int J^i x^j\,d^3x = -\frac{i\omega}{6}\left(Q^{ij} - \delta^{ij}\int r^2\,\rho\,d^3x\right) - c\,\epsilon_{ijk}m^k\,. \tag{25}$$

2. Consider a magnetic dipole moment $\vec{\boldsymbol{m}}_0$ derived from a *steady* localized current density $\vec{\boldsymbol{J}}_0$ in the rest frame of the magnetic dipole, denoted by K_0'. Moreover, in frame K_0', the charge density ρ_0 is equal to zero. The frame K_0' moves with velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ with respect to the laboratory frame K. The two reference frames coincide at time $t = t' = 0$. Do *not* assume that $\beta \equiv |\vec{\boldsymbol{\beta}}| \ll 1$.

(a) The magnetic moment (in gaussian units) in reference frame K_0' is defined by

$$\vec{\boldsymbol{m}}_0 = \frac{1}{2c}\int \vec{\boldsymbol{x}}\,' \times \vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\,d^3x'\,. \tag{26}$$

Show that

$$\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}_0 = \frac{1}{c}\int \vec{\boldsymbol{x}}\,'\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\big]\,d^3x\,. \tag{27}$$

Since the current $\vec{\boldsymbol{J}}_0$ is a steady current density in frame K_0', we have $\vec{\boldsymbol{\nabla}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0 = 0$. Thus, we may use eq. (15), which was derived under the stated assumptions. Hence,

$$\frac{1}{c}\int x'^{\,j}\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\big]\,d^3x' = \frac{\beta^i}{c}\int x'^{\,j} J_0^i(\vec{\boldsymbol{x}}\,')\,d^3x' = -\epsilon_{ijk}\beta^i m_0^k = \big(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}_0\big)^j\,. \tag{28}$$

(b) Find the charge and current densities $\rho(\vec{\boldsymbol{x}},t)$ and $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)$ in the frame K. Verify your result by explicitly showing that the continuity equation is satisfied in the frame K.

Frame K_0' moves with velocity $\vec{\boldsymbol{v}}=\vec{\boldsymbol{\beta}}c$ with respect to frame K. Since $x^\mu=(ct\,;\,\vec{\boldsymbol{x}})$ is a four-vector, the spacetime coordinates of an observer in frame K_0' are related to those in frame K by,

$$x_0'=\gamma(x_0-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\,,\qquad\qquad \vec{\boldsymbol{x}}\,'=\vec{\boldsymbol{x}}+\frac{\gamma-1}{\beta^2}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}}-\gamma\vec{\boldsymbol{\beta}}x_0\,, \tag{29}$$

where $x_0=x^0\equiv ct$ and $x_0'=x'^0\equiv ct'$. Since $J^\mu=(c\rho\,;\,\vec{\boldsymbol{J}})$ is a four vector, it follows that under a Lorentz boost from frame K' back to frame K, the transformed charge and current densities are given by

$$c\rho(\vec{\boldsymbol{x}},t)=\gamma\big[c\rho_0(\vec{\boldsymbol{x}}\,')+\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\big]\,, \tag{30}$$

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')+\frac{\gamma-1}{\beta^2}\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\big]\vec{\boldsymbol{\beta}}+\gamma\vec{\boldsymbol{\beta}}c\rho_0(\vec{\boldsymbol{x}}\,')\,. \tag{31}$$

Since $\rho_0=0$ by assumption, it follows that in frame K,

$$c\rho(\vec{\boldsymbol{x}},t)=\gamma\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\,, \tag{32}$$

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')+\frac{\gamma-1}{\beta^2}\big[\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\big]\vec{\boldsymbol{\beta}}\,. \tag{33}$$

Note that because $\vec{\boldsymbol{J}}_0$ is a steady current, it satisfies $\vec{\boldsymbol{\nabla}}\,'\boldsymbol{\cdot}\,\vec{\boldsymbol{J}}_0=0$.

To check the continuity equation, we must verify that eq. (4) is satisfied by ρ and $\vec{\boldsymbol{J}}$ as given by eq. (33). The relevant derivatives can be evaluated by using the chain rule. Note that the right hand sides of eqs. (32) and (33) depend only on $\vec{\boldsymbol{x}}\,'$ (and not on t'). Hence,

$$\frac{\partial}{\partial t}=\frac{\partial\vec{\boldsymbol{x}}\,'}{\partial t}\boldsymbol{\cdot}\frac{\partial}{\partial\vec{\boldsymbol{x}}\,'}=-\gamma c\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,, \tag{34}$$

$$\frac{\partial}{\partial x^i}=\frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial x'^j}=\left(\delta^{ij}+\frac{\gamma-1}{\beta^2}\beta^i\beta^j\right)\frac{\partial}{\partial x'^j}=\Big[\vec{\boldsymbol{\nabla}}\,'+\frac{\gamma-1}{\beta^2}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,')\Big]_i\,, \tag{35}$$

after making use of eq. (29) in evaluating $\partial\vec{\boldsymbol{x}}\,'/\partial t$ and $\partial x'^j/\partial x^i$. Applying eqs. (34) and (35) to eqs. (32) and (33), we obtain

$$\frac{\partial\rho}{\partial t}=-\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,')\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\big)\,, \tag{36}$$

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}&=\left(\vec{\boldsymbol{\nabla}}\,'+\frac{\gamma-1}{\beta^2}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,')\right)\boldsymbol{\cdot}\left[\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')+\frac{\gamma-1}{\beta^2}\vec{\boldsymbol{\beta}}\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\big)\right]\\&=\frac{2(\gamma-1)+(\gamma-1)^2}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,')\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\big)=\gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,')\big(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}'(\vec{\boldsymbol{x}}\,')\big)\,,\end{aligned} \tag{37}$$

after employing $\vec{\boldsymbol{\nabla}}\,'\boldsymbol{\cdot}\,\vec{\boldsymbol{J}}_0=0$ and using the identity $\beta^2\gamma^2=\gamma^2-1$. Hence, we have verified that $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}+\partial\rho/\partial t=0$ is satisfied.

(c) Determine the electric dipole moment measured in the frame K using

$$\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{x}}\, \rho(\vec{\boldsymbol{x}}, t)\, d^3x \,, \tag{38}$$

where t is held fixed. Express your result in terms of $\vec{\boldsymbol{m}}_0$.

Our strategy is to multiply both sides of eq. (32) by $\vec{\boldsymbol{x}}\,'\, d^3x'$. We will then evaluate the left had side of the resulting equation by expressing $\vec{\boldsymbol{x}}\,'$ in terms of $\vec{\boldsymbol{x}}$ using eq. (29), and changing the integration variable from x' to x (holding $x_0 = ct$ fixed). In particular,

$$d^3x' = \left|\frac{\partial(x'^1, x'^2, x'^3)}{\partial(x^1, x^2, x^3)}\right| d^3x = \left|\det\left(\delta^{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j\right)\right| d^3x = \big[1 + (\gamma - 1)\big] = \gamma\, d^3x \,, \tag{39}$$

where the determinant of the Jacobian matrix has been evaluated using the formula derived in the class handout entitled *A determinantal identity*, $\det(\delta_{ij} + a_i a_j) = 1 + |\vec{\boldsymbol{a}}|^2$.

We therefore obtain:

$$\int c\rho(\vec{\boldsymbol{x}}, t)\Big[\vec{\boldsymbol{x}} + \frac{\gamma-1}{\beta^2}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}x_0\Big]\, d^3x = \int \vec{\boldsymbol{x}}\,'\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}_0(\vec{\boldsymbol{x}}\,')\, d^3x' \,, \tag{40}$$

after canceling an overall factor of γ from both sides of the above equation. Note that $\gamma\vec{\boldsymbol{\beta}}x_0$ is a constant (at fixed x_0). Thus,

$$\int \rho(\vec{\boldsymbol{x}}, t) d^3x = 0 \,, \tag{41}$$

since by assumption $\rho_0 = 0$ which implies that the total charge of the source (which is a Lorentz invariant) is zero. Using eqs. (28) and (38), it then follows from eq. (40) that

$$\vec{\boldsymbol{p}} + \frac{\gamma-1}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})\vec{\boldsymbol{\beta}} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}_0 \,. \tag{42}$$

Taking the dot product of this equation with $\vec{\boldsymbol{\beta}}$, and noting that $\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\big(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}_0\big) = 0$, it follows that $\gamma\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} = 0$. Inserting this result back into eq. (42), we conclude that

$$\vec{\boldsymbol{p}} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{m}}_0 \,. \tag{43}$$

3. An electron of charge e and mass m moves in a plane perpendicular to a uniform magnetic field B. If the energy loss by radiation is neglected, the orbit is a circle of some radius R. Let E be the total relativistic energy of the electron, and assume that $E \gg mc^2$ (corresponding to ultra-relativistic motion).

(a) Express B analytically in terms of the parameters given above. Compute numerically the required magnetic field B, in gauss, for the case of $R = 30$ meters and $E = 2.5$ GeV.

For circular motion,

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = -\frac{v^2}{R}\,\hat{\boldsymbol{r}}\,. \tag{44}$$

Since the circular motion is in a plane that is perpendicular to the magnetic field $\vec{\boldsymbol{B}}$, it follows that $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{v}}$ and $\hat{\boldsymbol{r}}$ are mutually orthogonal vectors. Moreover, eqs. (12.38) and (12.39) on p. 585 of Jackson yield (in gaussian units),

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{\gamma mc}\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\,. \tag{45}$$

Thus, if $\vec{\boldsymbol{B}}$ points in the z-direction, then $\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{\theta}}$ and the circular motion is clockwise in the x–y plane. Combining eqs. (44) and (45), it follows that

$$B = \frac{\gamma mvc}{eR}\,.$$

In the ultra-relativistic limit, we have $v \simeq c$ so that

$$B \simeq \frac{\gamma mc^2}{eR} = \frac{E}{eR}\,, \tag{46}$$

where $E = \gamma mc^2$ is the total relativistic energy of the electron. Note that eq. (46) has been derived using gaussian units. Plugging in numbers, and recalling that 1 eV $= 1.6 \times 10^{-19}$ J and 1 J $= 10^7$ ergs, it follows that

$$B = \frac{(2.5 \times 10^9 \text{ eV})(1.6 \times 10^{-12} \text{ ergs/eV})}{(4.8 \times 10^{-10} \text{ esu})(3 \times 10^3 \text{ cm})} = 2.78 \times 10^3 \text{ gauss}\,. \tag{47}$$

One can convert eq. (46) into SI units by replacing $eB \to ecB$.[1] In this case, eq. (47) would be replaced by

$$B = \frac{(2.5 \times 10^9 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})}{(1.6 \times 10^{-19} \text{ C})(30 \text{ m})(3 \times 10^8 \text{ m/s})} = 0.278 \text{ T}\,.$$

Converting tesla to gauss using 1 T $= 10^4$ gauss using Table 4 on p. 783 of Jackson, we recover the result of eq. (47).

(b) In fact, the electron radiates electromagnetic energy. Suppose that the energy loss per revolution, ΔE, is small compared to E. Express the ratio $\Delta E/E$ analytically in terms of the parameters given above.

Using eq. (14.46) on p. 671 of Jackson,

$$P = \frac{2e^2a^2}{3c^3}\gamma^4\,,$$

[1]The easiest way to see this is to note that the magnetic force on a charge e in SI units is $e\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$, whereas in gaussian units, the magnetic force is given by $e\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}/c$.

where $a \equiv |d\vec{\boldsymbol{v}}/dt|$ and t is the charge's own time (which is denoted by t' in section 3 of Chapter 14 of Jackson). For circular motion, the magnitude of the acceleration is $a = v^2/R$. One orbit covers a distance of $2\pi R$ in a time $\Delta t = 2\pi R/v$. Thus, the energy lost per orbit is

$$\Delta E = P\,\Delta t = \frac{2e^2}{3c^3}\left(\frac{v^2}{R}\right)^2\left(\frac{2\pi R}{v}\right)\gamma^4 = \frac{4\pi e^2}{3R}\left(\frac{v}{c}\right)^3\gamma^4\,.$$

In the ultra-relativistic limit, $v \simeq c$, so we end up with

$$\Delta E \simeq \frac{4\pi e^2}{3R}\gamma^4\,.$$

Dividing by the total relativistic energy $E = \gamma mc^2$, and substituting $\gamma \equiv E/(mc^2)$, it follows that

$$\frac{\Delta E}{E} \simeq \frac{4\pi e^2}{3mc^2 R}\left(\frac{E}{mc^2}\right)^3\,. \tag{48}$$

(c) Evaluate the ratio obtained in part (b) numerically using the values of R and E given in part (a). Note that the rest mass of the electron is $mc^2 = 511$ keV.

Plugging in the numbers into eq. (48),

$$\frac{\Delta E}{E} \simeq \frac{4\pi(4.8\times 10^{-10}\text{ esu})^2}{3(0.511\times 10^6\text{ eV})(1.6\times 10^{-12}\text{ ergs/eV})(3\times 10^3\text{ cm})}\left(\frac{2.5\times 10^9\text{ eV}}{0.511\times 10^6\text{ eV}}\right)^3 = 4.6\times 10^{-5}\,. \tag{49}$$

To check the dimensions (since $\Delta E/E$ must be dimensionless), note that in gaussian units,

$$1\text{ esu} = 1\text{ statcoulomb} = 1\text{ dyne}^{1/2}\!\cdot\!\text{cm} = 1\ (\text{erg}\!\cdot\!\text{cm})^{1/2}. \tag{50}$$

Eq. (48) has been derived using gaussian units. In SI units, e^2 is replaced by $e^2/(4\pi\epsilon_0)$. In this case, using the numerical value of ϵ_0 given at the bottom of Table 3 on p. 782 of Jackson, the modified eq. (48) would yield

$$\frac{\Delta E}{E} \simeq \frac{(1.6\times 10^{-19}\text{ C})^2}{3(0.511\times 10^6\text{ eV})(1.6\times 10^{-19}\text{ J/eV})(30\text{ m})(8.854\times 10^{-12}\text{ F/m})}\left(\frac{2.5\times 10^9\text{ eV}}{0.511\times 10^6\text{ eV}}\right)^3$$
$$= 4.6\times 10^{-5}\,, \tag{51}$$

which again produces the same result obtained in eq. (49). Note that the unit of capacitance (farad) is given by 1 F = 1 C/volt = 1 C^2/J, so that $\Delta E/E$ is dimensionless, as expected.

4. The Thomson scattering cross section for an incoming wave with polarization $\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)}$ and an outgoing wave with polarization $\hat{\boldsymbol{\epsilon}}^{(\lambda)}$ was obtained in class,

$$\frac{d\sigma_{\lambda_0\lambda}}{d\Omega} = r_c^2 \left|\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 , \tag{52}$$

where $r_c \equiv e^2/(mc^2)$ is the classical radius of the electron (in gaussian units). Imagine another theory, in which the corresponding cross section is given by

$$\frac{d\tilde{\sigma}_{\lambda_0\lambda}}{d\Omega} = r_c^2 \left|\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 , \tag{53}$$

where the notation $\tilde{\sigma}$ distinguishes this case from Thomson scattering. In both cases, it is convenient to choose a coordinate system such that the incident wave approaches the origin of the coordinate system along the z-axis, and the scattered wave is detected at an angle θ with respect to the z axis in the x–z plane.

(a) If the polarizations of the incident and the scattered waves are not measured, compute the angular distribution of the scattered wave for both cases introduced above. Evaluate the corresponding total cross sections.

To obtain the unpolarized cross sections, one must average over initial state polarizations and sum over final state polarizations. In the case of eq. (52), we repeat the computation provided in eq. (48) of the class handout entitled *Polarization Vectors and Polarization Sums*,

$$\tfrac{1}{2}\sum_{\lambda_0}\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 = \tfrac{1}{2}\hat{\boldsymbol{\epsilon}}_i^{(\lambda)}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)*}\sum_{\lambda_0}\big(\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)}\big)_j\big(\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)*}\big)_i = \tfrac{1}{2}\hat{\boldsymbol{\epsilon}}_i^{(\lambda)}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)*}(\delta_{ij}-\hat{z}_i\hat{z}_j) = \tfrac{1}{2}\big(1-|\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}|^2\big)\,. \tag{54}$$

It follows that

$$\begin{aligned}\tfrac{1}{2}\sum_{\lambda}\sum_{\lambda_0}\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 &= \tfrac{1}{2}\sum_{\lambda}\big(1-|\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}|^2\big) = 1-\tfrac{1}{2}\hat{z}_i\hat{z}_j(\delta_{ij}-\hat{n}_i\hat{n}_j)\\ &= 1-\tfrac{1}{2}\big[1-(\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\big] = \tfrac{1}{2}(1+\cos^2\theta)\,,\end{aligned} \tag{55}$$

where $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \cos\theta$. Hence, the angular distribution of the scattered electromagnetic wave is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol} = \tfrac{1}{2}\sum_{\lambda}\sum_{\lambda_0}\frac{d\sigma_{\lambda_0\lambda}}{d\Omega} = \tfrac{1}{2}r_c^2\sum_{\lambda}\sum_{\lambda_0}\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 = \tfrac{1}{2}r_c^2(1+\cos^2\theta)\,, \tag{56}$$

and the corresponding total cross section for unpolarized scattering is

$$\sigma_{\rm unpol} = \tfrac{1}{2}r_c^2\int_0^{2\pi}d\phi\int_{-1}^{1}(1+\cos^2\theta)\,d\cos\theta = \frac{8\pi r_c^2}{3}\,. \tag{57}$$

In the case of eq. (53), we note the vector identity,

$$\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 = 1-\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 . \tag{58}$$

It then follows that

$$\tfrac{1}{2}\sum_{\lambda}\sum_{\lambda_0}\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 = 2-\tfrac{1}{2}\sum_{\lambda}\sum_{\lambda_0}\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 = 2-\tfrac{1}{2}(1+\cos^2\theta) = \tfrac{1}{2}(3-\cos^2\theta)\,. \quad (59)$$

Hence, the angular distribution of the scattered electromagnetic wave is given by

$$\left(\frac{d\tilde{\sigma}}{d\Omega}\right)_{\rm unpol} = \tfrac{1}{2}\sum_{\lambda}\sum_{\lambda_0}\frac{d\tilde{\sigma}_{\lambda_0\lambda}}{d\Omega} = \tfrac{1}{2}r_c^2\sum_{\lambda}\sum_{\lambda_0}\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 = \tfrac{1}{2}r_c^2(3-\cos^2\theta)\,, \quad (60)$$

and the corresponding total cross section for unpolarized scattering is

$$\tilde{\sigma}_{\rm unpol} = \tfrac{1}{2}r_c^2\int_0^{2\pi}d\phi\int_{-1}^{1}(3-\cos^2\theta)\,d\cos\theta = \frac{16\pi r_c^2}{3}\,. \quad (61)$$

(b) An observer measures the polarization of the scattered wave at a scattering angle of $\theta = 90^\circ$. Compare the properties of the observed polarizations corresponding to the two cases introduced above.

Without loss of generality, we shall orient our coordinate system such that the scattering plane coincides with the x–z plane. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}$ for a scattering angle of 90°, and we may choose the basis for the polarization vectors of the scattered wave (which are orthogonal to $\hat{\boldsymbol{n}}$) to be $\{\hat{\boldsymbol{y}}\,,\,\hat{\boldsymbol{z}}\}$. In this case, we average over the initial state polarizations to obtain

$$\frac{d\sigma_\lambda}{d\Omega} = \tfrac{1}{2}r_c^2\sum_{\lambda_0}\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 = \tfrac{1}{2}r_c^2\big(1-|\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}|^2\big)\,. \quad (62)$$

Consequently,

$$\frac{d\sigma_{\hat{\boldsymbol{\epsilon}}=\hat{\boldsymbol{y}}}}{d\Omega}(\theta=\tfrac{1}{2}\pi) = \tfrac{1}{2}r_c^2\,, \qquad \frac{d\sigma_{\hat{\boldsymbol{\epsilon}}=\hat{\boldsymbol{z}}}}{d\Omega}(\theta=\tfrac{1}{2}\pi) = 0\,. \quad (63)$$

That is, the scattered wave emitted at $\theta = 90^\circ$ is 100% linearly polarized in the direction perpendicular to the scattering plane.

Repeating this exercise for the case of eq. (53),

$$\frac{d\tilde{\sigma}_\lambda}{d\Omega} = \tfrac{1}{2}r_c^2\sum_{\lambda_0}\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2 = \tfrac{1}{2}r_c^2\sum_{\lambda_0}\Big[1-\left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right|^2\Big] = \tfrac{1}{2}r_c^2\big(1+|\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}|^2\big)\,. \quad (64)$$

Consequently,

$$\frac{d\tilde{\sigma}_{\hat{\boldsymbol{\epsilon}}=\hat{\boldsymbol{y}}}}{d\Omega}(\theta=\tfrac{1}{2}\pi) = \tfrac{1}{2}r_c^2\,, \qquad \frac{d\tilde{\sigma}_{\hat{\boldsymbol{\epsilon}}=\hat{\boldsymbol{z}}}}{d\Omega}(\theta=\tfrac{1}{2}\pi) = r_c^2\,. \quad (65)$$

In this case, the scattered wave is partially polarized, with twice the probability to be detected as linear polarization in the scattering plane as compared to linear polarization perpendicular to the scattering plane.