This is a make-up assignment, in the style of the take-home midterm exam. You may refer to the textbook by Jackson, any material linked to the class website, and any second classical electromagnetism textbook of your choosing. Reference to integrals or other mathematical facts, and any personal handwritten notes are also OK. However, you should *not* collaborate with anyone else (this includes AI help such as ChatGPT).

There is no need to rederive results that have been previously obtained in the textbook, the class notes or the class handouts. But if you make use of any previously derived result, please cite the source of the result. Otherwise, you must show all work involved in obtaining the solution to obtain full credit for a problem solution.

The completed assignment must be returned either to my ISB mailbox or to me via email by 4 pm on Wednesday May 6.

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1. Consider an oversimplified model of an antenna consisting of a thin wire of length ℓ and negligible cross section, carrying a harmonically varying current density flowing in the z direction. The (complex) current in the wire is given by $Ie^{-i\omega t}$, where I is a constant (independent of position).

(a) Show that the (complex) current density takes the form:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \hat{\boldsymbol{z}} I e^{-i\omega t}\delta(x)\delta(y)[\Theta(z+\tfrac{1}{2}\ell) - \Theta(z-\tfrac{1}{2}\ell)]\,, \tag{1}$$

by verifying that eq. (1) implies that the corresponding current is given by $Ie^{-i\omega t}$, where the step function $\Theta(x) \equiv 1$ if $x > 0$ and $\Theta(x) \equiv 0$ if $x < 0$. Here, we have assumed that the point $z = 0$ corresponds to the midpoint of the antenna.

(b) Prove that there is an oscillating charge density at $z = \pm\frac{1}{2}\ell$ (i.e., at both ends of the antenna), but the charge density vanishes at any interior point on the antenna.

(c) Show that the antenna acts like an oscillating electric dipole moment, $\vec{\boldsymbol{p}}e^{-i\omega t}$. Evaluate $\vec{\boldsymbol{p}}$ in terms of the current I, the antenna length ℓ and the angular frequency ω.

(d) Calculate the angular distribution of the radiated power, $dP/d\Omega$, assuming that $\lambda \gg \ell$, where λ is the wavelength of the emitted radiation. Express your answer in terms of the current I, the antenna length ℓ and the wavelength λ. Integrate over angles to obtain the total radiated power.

2. Evaluate the following quantity:

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\left[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\right],$$

where $f(r)$ is an arbitrary function of the radial variable $r \equiv |\vec{\boldsymbol{x}}|$, and $\vec{\boldsymbol{X}}_{\ell m}((\theta,\varphi)$ is the vector spherical harmonic introduced by Jackson in Chapter 9.

3. A particle of mass m, charge q, moves in a plane perpendicular to a uniform, static, magnetic induction B.

(a) Calculate the total energy radiated per unit time, expressing it in terms of the constants already defined and the ratio γ of the particle's total energy to its rest energy.

(b) If at time $t = 0$ the particle has a total energy $E_0 = \gamma_0 mc^2$, show that it will have energy $E = \gamma mc^2 < E_0$, at a time t, where

$$t \simeq \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right),$$

provided that $\gamma \gg 1$.

(c) If the particle is initially nonrelativistic and has a *kinetic* energy T_0 at $t = 0$, what is its kinetic energy at time t?

4. A charged particle of mass m and charge e with relativistic velocity $\vec{\boldsymbol{v}}_0 = v_0\hat{\boldsymbol{z}}$ enters a medium where it is slowed down by a force that is proportional to its velocity. That is, $\vec{\boldsymbol{F}} = d\vec{\boldsymbol{p}}/dt = -\eta\vec{\boldsymbol{v}}$, where η is a positive dimensionful constant. The time t refers to the moving charge and $t = 0$ when the particle enters the medium.

(a) Using relativistic mechanics, determine the acceleration of the charged particle as a function of its velocity, mass and η.

HINT: This is a one dimensional problem, since the particle moves in a straight line.

(b) Determine the angular distribution of the instantaneous power radiated once the particle has entered the medium and slowed down to a velocity v. The polar and azimuthal angles of the emitted radiation are defined relative to the z-axis which lies along the direction of the particle velocity. In your calculation, you may neglect the effect of the medium on the emitted radiation (*i.e.*, you should treat the radiation as if it were emitted in the vacuum.)

(c) How much energy is emitted in the form of electromagnetic radiation from the time the particle enters the medium until it slows down and reaches zero velocity? Express your answer in terms of the parameters e, m, c, η and v_0.

HINT: First, compute the total power. The following integral may be useful,

$$\int_{-1}^{1} \frac{1 - x^2}{(1 - \beta x)^5}\, dx = \frac{4}{3}\gamma^6,$$

where $\beta \equiv v/c$ and $\gamma \equiv (1-\beta^2)^{-1/2}$. Then to determine the energy, change the integration variable from time t to velocity v and integrate from $v_0 \equiv v(t = 0)$ to zero velocity.

5. In a new theory, the Thomson scattering differential cross section for an incident wave with polarization $\hat{\boldsymbol{\epsilon}}_0$ and wave number $\vec{k}_0 = k\hat{\boldsymbol{n}}_0$ and an outgoing scattered wave with polarization $\hat{\boldsymbol{\epsilon}}$ and wave number $\vec{k} = k\hat{\boldsymbol{n}}$ is modified as follows:

$$\frac{d\sigma_{\lambda_0\lambda}}{d\Omega} = r_c^2 \left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*} + \eta\,(\hat{\boldsymbol{\epsilon}}_0^{\lambda_0} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{n}})\right|^2 , \tag{2}$$

where η is a real number and $r_c \equiv e^2/(mc^2)$ is the classical radius of the electron (in gaussian units). It is convenient to choose a coordinate system such that the incident wave approaches the origin of the coordinate system along the z-axis, and the scattered wave is detected at an angle θ with respect to the z axis in the x–z plane. We can then set $\hat{\boldsymbol{n}}_0 = \hat{\boldsymbol{z}}$.

(a) If the polarizations of the incident and the scattered waves are not measured, compute the angular distribution of the scattered wave. Evaluate the corresponding total cross section.

(b) An observer measures the polarization of the scattered wave at a fixed scattering angle of $\theta = 90°$. Show that if $\eta = 0$, then the scattered wave is 100% linearly polarized in the direction perpendicular to the scattering plane. Finally, if $\eta \neq 0$, show that the outgoing scattering wave detected at a fixed scattering angle of $\theta = 90°$ is only partially polarized, and compute the relative probability of detecting the two possible linear polarization states.