1. Consider an oversimplified model of an antenna consisting of a thin wire of length ℓ and negligible cross section, carrying a harmonically varying current density flowing in the z direction. The (complex) current in the wire is given by $Ie^{-i\omega t}$, where I is a constant (independent of position).

(a) Show that the (complex) current density takes the form:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)=\hat{\boldsymbol{z}}Ie^{-i\omega t}\delta(x)\delta(y)[\Theta(z+\tfrac{1}{2}\ell)-\Theta(z-\tfrac{1}{2}\ell)]\,, \tag{1}$$

by verifying that eq. (1) implies that the corresponding current is given by $Ie^{-i\omega t}$, where the step function $\Theta(x)\equiv 1$ if $x>0$ and $\Theta(x)\equiv 0$ if $x<0$. Here, we assume that the point $z=0$ corresponds to the midpoint of the antenna.

First we note that

$$\Theta(z+\tfrac{1}{2}\ell)-\Theta(z-\tfrac{1}{2}\ell)=\begin{cases}0\,, & z>\ell/2\,,\\ 1\,, & |z|<\ell/2\,,\\ 0\,, & z<-\ell/2\,.\end{cases}$$

Thus, $\vec{\boldsymbol{J}}=0$ if $z>\ell/2$ or $z<-\ell/2$. For $|z|<\ell/2$,

$$J_z=Ie^{-i\omega t}\delta(x)\delta(y)\,,\qquad\qquad J_x=J_y=0\,.$$

The current is obtained by computing

$$\int\vec{\boldsymbol{J}}\boldsymbol{\cdot}d\vec{\boldsymbol{a}}=\int\vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}\,dxdy=\int J_z\,dxdy=Ie^{-i\omega t}\,.$$

(b) Prove that there is an oscillating charge density at $z=\pm\frac{1}{2}\ell$ (*i.e.*, at both ends of the antenna), but the charge density vanishes at any interior point on the antenna.

The continuity equation is

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}+\frac{\partial\rho}{\partial t}=0\,.$$

For $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})e^{-i\omega t}$ and $\rho(\vec{\boldsymbol{x}},t)=\rho(\vec{\boldsymbol{x}})e^{-i\omega t}$, the continuity equation then reads:

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}=i\omega\rho(\vec{\boldsymbol{x}})\,.$$

Using $\vec{\boldsymbol{J}}$ given in part (a),

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}=\frac{\partial J_z}{\partial z}&=I\delta(x)\delta(y)\frac{\partial}{\partial z}\left[\Theta(z+\tfrac{1}{2}\ell)-\Theta(z-\tfrac{1}{2}\ell)\right]\\ &=I\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell)-\delta(z-\tfrac{1}{2}\ell)\right]\,.\end{aligned}$$

Setting this result to $i\omega\rho(\vec{\boldsymbol{x}})$, we conclude that:

$$\rho(\vec{\boldsymbol{x}}) = -\frac{iI}{\omega}\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right] ,$$

which corresponds to two point charges located at the two ends of the antenna. Moreover, $\rho(\vec{\boldsymbol{x}}, t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$ indicates that the point charges have magnitudes that oscillate in time. (As usual, we take the real part of $\rho(\vec{\boldsymbol{x}}, t)$ to find the corresponding physical quantity.)

(c) Show that the antenna acts like an oscillating electric dipole moment, $\vec{\boldsymbol{p}}\,e^{-i\omega t}$. Evaluate $\vec{\boldsymbol{p}}$ in terms of the current I, the antenna length ℓ and the angular frequency ω.

The electric dipole moment is given by

$$\vec{\boldsymbol{p}}(t) = \int \vec{\boldsymbol{x}}\rho(\vec{\boldsymbol{x}}, t)d^3x = e^{-i\omega t}\int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x = \vec{\boldsymbol{p}}\,e^{-i\omega t} , \tag{2}$$

after employing $\rho(\vec{\boldsymbol{x}}, t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$ and defining,

$$\vec{\boldsymbol{p}} \equiv \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x\,. \tag{3}$$

We therefore compute:

$$\begin{aligned}
\vec{\boldsymbol{p}} &= \int \vec{\boldsymbol{x}}\rho(\vec{\boldsymbol{x}})d^3x \\
&= \int (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}})\left(\frac{-iI}{\omega}\right)\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right] dxdydz \\
&= \frac{-iI}{\omega}\hat{\boldsymbol{z}}\int zdz\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right] = \frac{iI\ell}{\omega}\hat{\boldsymbol{z}}\,.
\end{aligned}$$

(d) Calculate the angular distribution of the radiated power, $dP/d\Omega$, assuming that $\lambda \gg \ell$, where λ is the wavelength of the emitted radiation. Express your answer in terms of the current I, the antenna length ℓ and the wavelength λ. Integrate over angles to obtain the total radiated power.

For $\lambda \gg \ell$, the electric dipole approximation is very accurate. Hence, we can neglect all other multipole contributions. Using eq. (9.23) of Jackson (in SI units),

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|\vec{\boldsymbol{p}}|^2\sin^2\theta = \frac{c^2 Z_0}{32\pi^2}\frac{I^2\ell^2 k^4}{\omega^2}\sin^2\theta\,, \tag{4}$$

where $Z_0 \equiv \sqrt{\mu_0/\varepsilon_0}$ is the impedance of free space. Recalling that $\omega = kc$ and $k = 2\pi/\lambda$, the above result can be written as:

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8}\left(\frac{\ell}{\lambda}\right)^2\sin^2\theta\,. \tag{5}$$

Integrating over angles by using

$$\int \sin^2\theta\, d\Omega = 2\pi \int_{-1}^{1} (1-\cos^2\theta)\, d\cos\theta = \frac{8\pi}{3}\,,$$

we end up with:

$$P = \frac{\pi Z_0 I^2}{3}\left(\frac{\ell}{\lambda}\right)^2\,. \tag{6}$$

Note that one can also derive eq. (6) by using eq. (9.24) of Jackson,

$$P = \frac{c^2 Z_0 k^4}{12\pi}\,|\vec{\boldsymbol{p}}|^2 = \frac{c^2 Z_0}{12\pi}\left(\frac{2\pi}{\lambda}\right)^4\left(\frac{I^2\ell^2}{c^2}\right)\left(\frac{\lambda}{2\pi}\right)^2 = \frac{\pi Z_0 I^2}{3}\left(\frac{\ell}{\lambda}\right)^2\,. \tag{7}$$

The results above have been given in SI units. To convert eqs. (4)–(7) to Gaussian units, one can simply replace $Z_0 \to 4\pi/c$. This can be understood by writing the impedance of free space [cf. eq. (9.5) of Jackson] as,

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\epsilon_0 c}\,. \tag{8}$$

Moreover, using Table 3 on p. 782 of Jackson, we must replace $I \to \sqrt{4\pi\epsilon_0}\, I$, when converting a formula expressed in SI units to gaussian units. Thus,

$$Z_0 I^2 \to \frac{1}{\epsilon_0 c}\, 4\pi\epsilon_0 I^2 = \frac{4\pi I^2}{c} \tag{9}$$

which is consistent with replacing Z_0 with $4\pi/c$ as asserted above.

2. Evaluate the following quantity:

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\left[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\right]\,, \tag{10}$$

where $f(r)$ is an arbitrary function of the radial variable $r \equiv |\vec{\boldsymbol{x}}|$, and $\vec{\boldsymbol{X}}_{\ell m}((\theta,\varphi)$ is the vector spherical harmonic introduced by Jackson in Chapter 9.

In light of eq. (9.119) of Jackson,

$$\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi) = \frac{1}{\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{L}}\, Y_{\ell m}(\theta,\varphi)\,, \quad \text{where } \vec{\boldsymbol{L}} = -i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}. \tag{11}$$

We first employ the vector identity,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\psi\vec{\boldsymbol{A}}) = \vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi + \psi\,\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}\,, \tag{12}$$

for any vector quantity $\vec{\boldsymbol{A}}$ and scalar quantity ψ. Thus,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\left[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\right] = \vec{\boldsymbol{X}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}f(r) + f(r)\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{X}} = \vec{\boldsymbol{X}}\boldsymbol{\cdot}\hat{\boldsymbol{r}}\,\frac{\partial f}{\partial r} + f(r)\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{X}}\,, \tag{13}$$

where $\hat{\boldsymbol{r}} \equiv |\vec{\boldsymbol{x}}|/r$ is the unit vector in the radial direction. Note that[1]

$$\vec{\boldsymbol{X}}\cdot\hat{\boldsymbol{r}} = \frac{-i}{r\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{x}}\cdot\big[\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\theta,\varphi)\big] = 0\,, \tag{14}$$

since $\vec{\boldsymbol{x}}$ is orthogonal to $\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}}$ for any vector quantity $\vec{\boldsymbol{A}}$. Moreover,

$$\sqrt{\ell(\ell+1)}\,\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{X}} = -i\sqrt{\ell(\ell+1)}\,\vec{\boldsymbol{\nabla}}\cdot\big[\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\theta,\varphi)\big] = 0\,, \tag{15}$$

where we have used the following result,

$$\vec{\boldsymbol{\nabla}}\cdot\big(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\big) = \epsilon_{ijk}\partial_i(x_j\partial_k) = \epsilon_{ijk}\big[\delta_{ij}\partial_k + x_j\partial_i\partial_k\big] = 0\,, \tag{16}$$

after the implicit sum over the three pairs of repeated indices. Eq. (16) is a consequence of $\partial_i x_j = \delta_{ij}$ and the identities $\epsilon_{ijk}\delta_{ij} = 0$ and $\epsilon_{ijk}\partial_i\partial_k = 0$. These latter two identities are obtained after noting that ϵ_{ijk} is a completely antisymmetric tensor, whereas δ_{ij} and $\partial_i\partial_k$ are both symmetric under the interchange of their two indices.

We therefore conclude that

$$\vec{\boldsymbol{\nabla}}\cdot\big[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\big] = 0\,. \tag{17}$$

<u>*REMARK*</u>*:* Starting from eq. (13), we can make use of eqs. (18) and (19) of the class handout entitled, *Properties of the differential operator* $\vec{\boldsymbol{L}}$:

$$\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{L}} = 0\,, \tag{18}$$

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{L}} = \vec{\boldsymbol{L}}\cdot\vec{\boldsymbol{\nabla}} = 0\,. \tag{19}$$

It then follows that

$$\vec{\boldsymbol{X}}\cdot\hat{\boldsymbol{r}} = \frac{1}{r}\,\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{X}} = \frac{1}{r\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{L}}\,Y_{\ell m}(\theta,\varphi) = 0\,, \tag{20}$$

after using eq. (18). Furthermore,

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{X}} = \frac{1}{\sqrt{\ell(\ell+1)}}\vec{\boldsymbol{\nabla}}\cdot\big(\vec{\boldsymbol{L}}\,Y_{\ell m}(\theta,\varphi)\big) = \frac{1}{\sqrt{\ell(\ell+1)}}\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{L}}\,Y_{\ell m}(\theta,\varphi) + \vec{\boldsymbol{L}}\cdot\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\theta,\varphi) = 0\,, \tag{21}$$

after using eq. (19). Hence, we reproduce the result of eq. (17),

$$\vec{\boldsymbol{\nabla}}\cdot\big[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\big] = \vec{\boldsymbol{X}}\cdot\hat{\boldsymbol{r}}\,\frac{\partial f}{\partial r} + f(r)\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{X}} = 0\,. \tag{22}$$

By the way, in light of eqs. (9), (11), and (25) of the class handout entitled, *The tensor spherical harmonics*, we can identify $\vec{\boldsymbol{X}}(\theta,\varphi) = \vec{\boldsymbol{Y}}_{\ell\ell m}(\hat{\boldsymbol{n}})$. Then, eq. (103) of this same handout states that $\vec{\boldsymbol{\nabla}}\cdot\big[f(r)\vec{\boldsymbol{Y}}_{\ell\ell m}(\hat{\boldsymbol{n}})\big] = 0$, which is equivalent to eq. (17).

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[1]Since $\vec{\boldsymbol{L}}$ is a purely angular operator (with no component in the $\hat{\boldsymbol{r}}$ direction), it follows that $\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{L}} = 0$. Hence $\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{X}} = 0$, which is another way to establish eq. (14).

3. A particle of mass m, charge q, moves in a plane perpendicular to a uniform, static, magnetic induction B.

(a) Calculate the total energy radiated per unit time, expressing it in terms of the constants already defined and the ratio γ of the particle's total energy to its rest energy.

Using eq. (14.46) of Jackson, the total radiated power is given by:

$$P(t') = \frac{2q^2\gamma^4}{3c^3}\left|\frac{d\boldsymbol{\vec{v}}}{dt'}\right|^2 , \tag{23}$$

where $t' \equiv t_{\rm ret} = t - |\boldsymbol{\vec{x}} - \boldsymbol{\vec{r}}(t_{\rm ret})|/c$ and $\boldsymbol{\vec{r}}(t_{\rm ret})$ is the particle trajectory at the retarded time. Recall that the angular velocity is related to velocity by $\boldsymbol{\vec{v}} = \boldsymbol{\vec{\omega}} \times \boldsymbol{\vec{x}}$. It follows that

$$v^2 \equiv |\boldsymbol{\vec{v}}|^2 = |\boldsymbol{\vec{\omega}} \times \boldsymbol{\vec{x}}|^2 = \omega^2 r^2 - (\boldsymbol{\vec{\omega}}\boldsymbol{\cdot}\boldsymbol{\vec{x}})^2 ,$$

where $r \equiv |\boldsymbol{\vec{x}}|$ and $\omega \equiv |\omega|$. For planar circular motion of radius R, we have $r = R$ and $\boldsymbol{\vec{\omega}}\boldsymbol{\cdot}\boldsymbol{\vec{x}} = 0$, since the vector $\boldsymbol{\vec{\omega}}$ points in the direction perpendicular to the plane of motion. Thus,

$$|\boldsymbol{\vec{a}}| \equiv \left|\frac{d\boldsymbol{\vec{v}}}{dt'}\right| = \frac{v^2}{R} = \omega^2 R .$$

The angular velocity, $\omega = \omega_B$, is the gyration frequency given by eq. (12.39) of Jackson,

$$\omega_B = \frac{qB}{\gamma mc} . \tag{24}$$

Hence, eq. (23) yields

$$P(t') = \frac{2q^2\gamma^4 R^2\omega_B^4}{3c^3} = \frac{2q^6B^4R^2}{3m^4c^7} . \tag{25}$$

To express $P(t')$ in terms of γ and the constants m, q and B, we need to eliminate R. Using eq. (24), and noting that

$$\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}} \qquad \Longrightarrow \qquad \frac{v}{c} = \sqrt{1-\frac{1}{\gamma^2}} ,$$

it follows that

$$R = \frac{v}{\omega_B} = \frac{\gamma mcv}{qB} = \frac{mc^2}{qB}\left(\gamma^2-1\right)^{1/2} .$$

Inserting this result into eq. (25) then gives:

$$P(t') = \frac{2q^4B^2}{3m^2c^3}\left(\gamma^2-1\right) . \tag{26}$$

(b) If at time $t = 0$ the particle has a total energy $E_0 = \gamma_0 mc^2$, show that it will have energy $E = \gamma mc^2 < E_0$, at a time t, where

$$t \simeq \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right) ,$$

provided that $\gamma \gg 1$.

Since $P(t')$ is the energy loss of the particle per unit retarded time,

$$P(t') = -\frac{dE}{dt'} \,.$$

Hence, eq. (26) yields the differential equation,

$$\frac{dE}{\gamma^2 - 1} = -\frac{2q^4 B^2}{3m^2 c^3}\, dt' \,.$$

Using $E = \gamma mc^2$, it follows that $dE = mc^2 d\gamma$. It follows that

$$\int_{\gamma_0}^{\gamma} \frac{d\gamma}{\gamma^2 - 1} = -\frac{2q^4 B^2}{3m^3 c^5} \int_0^t dt' \,. \tag{27}$$

Assuming $\gamma \gg 1$, we can approximate $\gamma^2 - 1 \simeq \gamma^2$ in the denominator on the left-hand side of eq. (27). In this case, the integrals of eq. (27) are trivial. In particular,

$$\int_{\gamma_0}^{\gamma} \frac{d\gamma}{\gamma^2} = \frac{1}{\gamma_0} - \frac{1}{\gamma} \,.$$

Hence, eq. (27) yields:

$$t \simeq \frac{3m^3 c^5}{2q^4 B^2} \left(\frac{1}{\gamma} - \frac{1}{\gamma_0} \right) \,, \tag{28}$$

under the assumption that $\gamma\,,\ \gamma_0 \gg 1$.

(c) If the particle is initially nonrelativistic and has a *kinetic* energy T_0 at $t = 0$, what is its kinetic energy at time t?

In the nonrelativistic limit, $\gamma_0 \simeq 1$. Since the accelerating particle loses energy by emitting radiation it follows that the particle remains nonrelativistic for all $t > 0$. That is, $\gamma(t) \simeq 1$. Hence, we can approximate

$$\gamma^2 - 1 = (\gamma + 1)(\gamma - 1) \simeq 2(\gamma - 1) \,.$$

In this case,

$$\int_{\gamma_0}^{\gamma} \frac{d\gamma}{\gamma^2 - 1} \simeq \tfrac{1}{2} \int_{\gamma_0}^{\gamma} \frac{d\gamma}{\gamma - 1} = \tfrac{1}{2} \ln\left(\frac{\gamma - 1}{\gamma_0 - 1} \right) \,.$$

Using eq. (27), we obtain

$$\ln\left(\frac{\gamma - 1}{\gamma_0 - 1} \right) = -\frac{4q^4 B^2 t}{3m^3 c^5} \,. \tag{29}$$

The kinetic energy is defined as

$$T = E - mc^2 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2 \,. \tag{30}$$

Thus, exponentiating eq. (29) yields

$$T(t) = T_0 \exp\left(-\frac{4q^4 B^2 t}{3m^3 c^5} \right) \,. \tag{31}$$

REMARK:

It is not difficult to repeat parts (b) and (c) for a particle whose initial velocity is arbitrary (without making any assumptions of relativistic or nonrelativistic motion). Without any approximations, eq. (27) yields

$$\ln\left(\frac{\gamma+1}{\gamma-1}\right)-\ln\left(\frac{\gamma_0+1}{\gamma_0-1}\right)=\frac{4q^4B^2t}{3m^3c^5}\,. \tag{32}$$

Thus,

$$t=\frac{3m^3c^5}{4q^4B^2}\left[\ln\left(\frac{\gamma+1}{\gamma-1}\right)-\ln\left(\frac{\gamma_0+1}{\gamma_0-1}\right)\right]\,. \tag{33}$$

As a check, if γ, $\gamma_0\gg 1$ then

$$\ln\left(\frac{\gamma+1}{\gamma-1}\right)=\ln\left(\frac{1+\gamma^{-1}}{1-\gamma^{-1}}\right)\simeq\frac{2}{\gamma}\,. \tag{34}$$

and we end up with

$$t\simeq\frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma}-\frac{1}{\gamma_0}\right)\,, \tag{35}$$

thereby reproducing eq. (28).

We can also manipulate eq. (32) to obtain

$$\frac{\gamma+1}{\gamma-1}=\left(\frac{\gamma_0+1}{\gamma_0-1}\right)\exp\left(\frac{4q^4B^2t}{3m^3c^5}\right)\,. \tag{36}$$

Writing

$$\frac{\gamma+1}{\gamma-1}=1+\frac{2}{\gamma-1}\,, \tag{37}$$

it follows that

$$\gamma-1=\frac{\gamma_0-1}{\tfrac{1}{2}(\gamma_0+1)\exp\left(\dfrac{4q^4B^2t}{3m^3c^5}\right)-\tfrac{1}{2}(\gamma_0-1)}\,. \tag{38}$$

Multiplying both sides of the above equation by mc^2 and making use of eq. (30), we end up with the following expression for the kinetic energy at time t,

$$T(t)=T_0\left[\left(1+\frac{T_0}{mc^2}\right)\exp\left(\frac{4q^4B^2t}{3m^3c^5}\right)-\frac{T_0}{2mc^2}\right]^{-1}\,. \tag{39}$$

This is an exact expression. If the particle is initially nonrelativistic, it follows that $T_0\ll mc^2$ and eq. (39) reduces to

$$T(t)=T_0\,\exp\left(-\frac{4q^4B^2t}{3m^3c^5}\right)\,. \tag{40}$$

thereby reproducing eq. (31).

4. A charged particle of mass m and charge e with relativistic velocity $\vec{\boldsymbol{v}}_0 = v_0\hat{\boldsymbol{z}}$ enters a medium where it is slowed down by a force that is proportional to its velocity. That is, $\vec{\boldsymbol{F}} = d\vec{\boldsymbol{p}}/dt = -\eta\vec{\boldsymbol{v}}$, where η is a positive dimensionful constant. The time t refers to the moving charge and $t = 0$ when the particle enters the medium.

(a) Using relativistic mechanics, determine the acceleration of the charged particle as a function of its velocity, mass and η.

This is a one dimensional problem, since the particle moves in a straight line. The relativistic equation of motion for the particle is given by:

$$\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt} = \frac{d}{dt}(\gamma m\vec{\boldsymbol{v}})\,,$$

where $\vec{\boldsymbol{p}} = \gamma m\vec{\boldsymbol{v}}$ is the relativistic momentum. Writing $\gamma \equiv (1 - v^2/c^2)^{-1/2}$, where $v \equiv |\vec{\boldsymbol{v}}|$, and remembering that the velocity $\vec{\boldsymbol{v}}$ and γ depend on time,

$$\frac{d}{dt}(\gamma m\vec{\boldsymbol{v}}) = m\vec{\boldsymbol{v}}\frac{d\gamma}{dt} + \gamma m\frac{d\vec{\boldsymbol{v}}}{dt} = \gamma m\left[\frac{d\vec{\boldsymbol{v}}}{dt} + \frac{\gamma^2}{c^2}\vec{\boldsymbol{v}}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right)\right]\,, \tag{41}$$

where we have used

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{\gamma^3}{c^2}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right)\,.$$

For linear motion, $\vec{\boldsymbol{v}}$ and $d\vec{\boldsymbol{v}}/dt$ are parallel vectors. Thus,

$$\vec{\boldsymbol{v}}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right) = v^2\frac{d\vec{\boldsymbol{v}}}{dt}\,.$$

Inserting this result back into eq. (41) yields,

$$\vec{\boldsymbol{F}} = \frac{d}{dt}(\gamma m\vec{\boldsymbol{v}}) = \gamma m\frac{d\vec{\boldsymbol{v}}}{dt}\left(1 + \frac{v^2/c^2}{1 - v^2/c^2}\right) = \gamma^3 m\frac{d\vec{\boldsymbol{v}}}{dt}\,. \tag{42}$$

Here, we have rederived a result previously obtained in eq. (34) of the class handout entitled *Examples of four-vectors*.

Using $\vec{\boldsymbol{F}} = -\eta\vec{\boldsymbol{v}}$, we can solve eq. (42) for the acceleration $\vec{\boldsymbol{a}} \equiv d\vec{\boldsymbol{v}}/dt$,

$$\vec{\boldsymbol{a}} = -\frac{\eta}{\gamma^3 m}\vec{\boldsymbol{v}}\,.$$

If we denote $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{v}}$ and $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{v}}$, where $\hat{\boldsymbol{v}}$ is a unit vector in the direction of the motion, then

$$a = -\frac{\eta v}{\gamma^3 m}\,, \tag{43}$$

which indicates that the particle is *decelerating*.

(b) Determine the angular distribution of the instantaneous power radiated once the particle has entered the medium and slowed down to a velocity v. The polar and azimuthal angles of the emitted radiation are defined relative to the z-axis which lies along the direction of the particle velocity. In your calculation, you may neglect the effect of the medium on the emitted radiation (*i.e.*, you should treat the radiation as if it were emitted in the vacuum.)

Using the relativistic Larmor formula for an accelerating charge in linear motion given in eq. (14.39) on p. 669 of Jackson,[2]

$$\frac{dP}{d\Omega} = \frac{e^2 a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\,, \tag{44}$$

where θ is the direction of emitted radiation and $\beta \equiv v/c$. Inserting the result for the acceleration a obtained in eq. (43) into eq. (44), one obtains

$$\frac{dP}{d\Omega} = \frac{e^2\eta^2 v^2}{4\pi\gamma^6 m^2 c^3}\frac{1-\cos^2\theta}{(1-\beta\cos\theta)^5}\,. \tag{45}$$

(c) How much energy is emitted in the form of electromagnetic radiation from the time the particle enters the medium until it slows down and reaches zero velocity? Express your answer in terms of the parameters e, m, c, η and v_0.

To obtain the total instantaneous power emitted by the particle after slowing down to a velocity v, we simply integrate eq. (45) over the polar and azimuthal angles,

$$P = \frac{e^2\eta^2 v^2}{4\pi\gamma^6 m^2 c^3}\,2\pi\int_{-1}^{1}\frac{1-\cos^2\theta}{(1-\beta\cos\theta)^5}\,d\cos\theta\,.$$

Setting $x = \cos\theta$ and using

$$\int_{-1}^{1}\frac{1-x^2}{(1-\beta x)^5}\,dx = \frac{4}{3}\gamma^6\,,$$

we end up with

$$P = \frac{2e^2\eta^2 v^2}{3m^2c^3}\,. \tag{46}$$

Note that the instantaneous radiated power integrated over all solid angles can be directly obtained from eq. (14.26) on p. 666 of Jackson. Since the velocity and acceleration are parallel, it follows that

$$P = \frac{2e^2a^2}{3c^3}\gamma^6\,.$$

Using eq. (43) for the acceleration a in the above formula, we immediately recover eq. (46).

The power is defined by $P = dE/dt$. It is convenient to rewrite dt in terms of dv^2. Note that

$$dv^2 = d(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{v}}) = 2\,\vec{\boldsymbol{v}}\boldsymbol{\cdot}d\vec{\boldsymbol{v}} = 2\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\,dt = 2\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\,dt\,.$$

[2] In this problem, the time t refers to the charge's own time. This is what Jackson denotes by $t' = t_{\rm ret}$ in section 3 of Chapter 14.

Using eq. (43), it follows that

$$dv^2 = -\frac{2\eta v^2}{m\gamma^3}\,dt\,.$$

or equivalently,

$$v^2 dt = -\frac{m\gamma^3}{2\eta} dv^2\,.$$

Writing $\gamma^3 = (1 - v^2/c^2)^{-3/2}$, it follows that

$$E = \int P\,dt = \frac{2e^2\eta^2}{3m^2c^3}\int v^2 dt = -\frac{e^2\eta}{3mc^3}\int_{v_0^2}^{0} \frac{dv^2}{(1 - v^2/c^2)^{3/2}}\,,$$

where $v_0 \equiv v(t = 0)$. The last integral is elementary, and the final result is,

$$E = \frac{2e^2\eta}{3mc}(\gamma_0 - 1)\,,$$

where $\gamma_0 \equiv (1 - v_0^2/c^2)^{-1/2}$.

<u>*NOTE*</u>: All the formulae given in Chapter 14 of Jackson are given in gaussian units. To convert the results of this problem to SI units, simply replace $e^2 \to e^2/(4\pi\epsilon_0)$.

5. In a new theory, the Thomson scattering differential cross section for an incident wave with polarization $\hat{\boldsymbol{\epsilon}}_0$ and wave number $\vec{k}_0 = k\hat{\boldsymbol{n}}_0$ and an outgoing scattered wave with polarization $\hat{\boldsymbol{\epsilon}}$ and wave number $\vec{k} = k\hat{\boldsymbol{n}}$ is modified as follows:

$$\frac{d\sigma_{\lambda_0\lambda}}{d\Omega} = r_c^2 \left|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*} + \eta\,(\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{n}})\right|^2, \tag{47}$$

where η is a real number and $r_c \equiv e^2/(mc^2)$ is the classical radius of the electron (in gaussian units). It is convenient to choose a coordinate system such that the incident wave approaches the origin of the coordinate system along the z-axis, and the scattered wave is detected at an angle θ with respect to the z axis in the x–z plane. We can then set $\hat{\boldsymbol{n}}_0 = \hat{\boldsymbol{z}}$.

(a) If the polarizations of the incident and the scattered waves are not measured, compute the angular distribution of the scattered wave. Evaluate the corresponding total cross section.

First, we note that

$$(\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{n}}) = -\left(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\right)\left(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right), \tag{48}$$

after using the transverse nature of the electromagnetic waves to set $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{\lambda_0} = \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*} = 0$.

Thus,

$$\tfrac{1}{2}\sum_{\lambda_0}\Big|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}-\eta\left(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\right)\left(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right)\Big|^2=\tfrac{1}{2}\hat{\boldsymbol{\epsilon}}_i^{(\lambda)}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)*}\sum_{\lambda_0}\bigg\{\left(\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)*}\right)_i\left(\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)}\right)_j$$
$$-\eta(\hat{\boldsymbol{n}}_0)_i\left(\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)}\right)_j\left(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)*}\right)-\eta(\hat{\boldsymbol{n}}_0)_j\left(\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)*}\right)_i\left(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)}\right)+\eta^2(\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}}_0)_j\big|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{(\lambda_0)}\big|^2\bigg\}.$$
$$=\tfrac{1}{2}\hat{\boldsymbol{\epsilon}}_i^{(\lambda)}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)*}\bigg\{\delta_{ij}-(\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}}_0)_j-\eta(\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}})_k\left[\delta_{jk}-(\hat{\boldsymbol{n}}_0)_j(\hat{\boldsymbol{n}}_0)_k\right]$$
$$-\eta(\hat{\boldsymbol{n}}_0)_j(\hat{\boldsymbol{n}})_k\left[\delta_{ik}-(\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}}_0)_k\right]+\eta^2(\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}}_0)_j\left[1-\left(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\right)^2\right]\bigg\}.$$
$$=\tfrac{1}{2}\bigg\{1+\big|\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}\big|^2\left[\eta^2-\left(1-\eta\,\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\right)^2\right]\bigg\}. \tag{49}$$

Finally, performing the polarization sum over the polarizations λ. The end result is:

$$\tfrac{1}{2}\sum_{\lambda}\sum_{\lambda_0}\Big|\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}-\eta\left(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\right)\left(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)*}\right)\Big|^2=\tfrac{1}{2}\sum_{\lambda}\bigg\{1+\big|\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}\big|^2\left[\eta^2-\left(1-\eta\,\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\right)^2\right]\bigg\}$$
$$=1+\tfrac{1}{2}\left[1-\left(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\right)^2\right]\left[\eta^2-\left(1-\eta\,\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\right)^2\right]. \tag{50}$$

We can identify $\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}=\cos\theta$. Therefore, we have obtained

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol}\equiv\tfrac{1}{2}\sum_{\lambda}\sum_{\lambda_0}\frac{d\sigma_{\lambda_0\lambda}}{d\Omega}=\tfrac{1}{2}r_c^2\left[1+\cos^2\theta+2\eta\cos\theta\sin^2\theta+\eta^2\sin^4\theta\right]. \tag{51}$$

Integrating over the solid angles to get the total cross section,

$$\sigma_{\rm unpol}=\frac{8\pi r_c^2}{3}\left(1+\frac{2\eta^2}{5}\right). \tag{52}$$

(b) An observer measures the polarization of the scattered wave at a fixed scattering angle of $\theta=90^\circ$. Show that if $\eta=0$, then the scattered wave is 100% linearly polarized in the direction perpendicular to the scattering plane. Finally, if $\eta\neq 0$, show that the outgoing scattering wave detected at a fixed scattering angle of $\theta=90^\circ$ is only partially polarized, and compute the relative probability of detecting the two possible linear polarization states.

Let us orient our coordinate system such that the scattering plane coincides with the x–z plane. In this case, $\hat{\boldsymbol{n}}=\hat{\boldsymbol{x}}$ for a scattering angle of 90°, and we may choose the basis for the polarization vectors of the scattered wave (which are orthogonal to $\hat{\boldsymbol{n}}$) to be $\{\hat{\boldsymbol{y}}\,,\,\hat{\boldsymbol{z}}\}$. In this case, we average over the initial state polarizations to obtain [see eq. (49)]:

$$\frac{d\sigma_\lambda}{d\Omega}=\tfrac{1}{2}r_c^2\left(1+\big|\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}\big|^2\left[\eta^2-\left(1-\eta\,\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\right)^2\right]\right). \tag{53}$$

Consequently,

$$\frac{d\sigma_{\hat{\boldsymbol{\epsilon}}=\hat{\boldsymbol{y}}}}{d\Omega}(\theta=\tfrac{1}{2}\pi)=\tfrac{1}{2}r_c^2\,,\qquad\qquad \frac{d\sigma_{\hat{\boldsymbol{\epsilon}}=\hat{\boldsymbol{z}}}}{d\Omega}(\theta=\tfrac{1}{2}\pi)=\tfrac{1}{2}r_c^2\eta^2\,. \tag{54}$$

That is, if $\eta=0$, then the scattered wave emitted at $\theta=90^\circ$ is 100% linearly polarized in the direction perpendicular to the scattering plane. On the other hand, if $\eta\neq 0$, then the scattered wave is only partially polarized, with corresponding probabilities

$$P_{\hat{\boldsymbol{\epsilon}}=\hat{\boldsymbol{y}}}=\frac{1}{1+\eta^2}\,,\qquad\qquad P_{\hat{\boldsymbol{\epsilon}}=\hat{\boldsymbol{z}}}=\frac{\eta^2}{1+\eta^2}\,. \tag{55}$$

REMARK:

This problem was motivated by axion electrodynamics. If one adds an axion to electrodynamics, then the Thomson cross section will be modified due to $\gamma e^-\to\gamma e^-$ scattering due to axion exchange. Thus, an interesting exercise is to compute the differential cross section for $\gamma e^-\to\gamma e^-$ in the long-wavelength approximation that includes the axion exchange contribution. The axion coupling to two photons leads to a Feynman rule of the form $\epsilon_{\alpha\beta\mu\nu}k_1^\alpha k_2^\beta\epsilon^\mu(k_1)\epsilon^\nu(k_2)^*$, where k_1 is the incident photon four-momentum and k_2 is the scattered photon four-momentum. An overall constant has been omitted. This expression reduces to $(\hat{\boldsymbol{\epsilon}}_0^{\lambda_0}\times\hat{\boldsymbol{\epsilon}}^{(\lambda)*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0\times\hat{\boldsymbol{n}})$, up to an overall factor, which motivates eq. (47).