## 1. The velocity four-vector

The velocity four-vector of a massive particle is defined by:[1]

$$u^\mu = \frac{dx^\mu}{d\tau} = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}}\,)\,, \tag{1}$$

where $d\tau = \gamma^{-1}dt$ is the differential proper time (which is a scalar quantity). Note that $\gamma \equiv (1-v^2/c^2)^{-1/2}$, where $v\equiv|\vec{\boldsymbol{v}}|$ is the magnitude of the velocity vector $\vec{\boldsymbol{v}}\equiv d\vec{\boldsymbol{x}}/dt$ that appears in eq. (1). The squared magnitude of the velocity four-vector,

$$u^2 \equiv g_{\mu\nu}u^\mu u^\nu = c^2 \tag{2}$$

is a Lorentz invariant. This quantity is most easily evaluated in the rest frame of the particle where $\vec{\boldsymbol{v}}=0$, in which case $u^\mu = c\,(1\,;\,\vec{\boldsymbol{0}})$. One then immediately obtains $u^2=c^2$. Note that $u$ is a timelike vector.

Let us now consider the following question. Suppose that the velocity vector is $u^\mu = (u^0\,;\,\vec{\boldsymbol{u}})$ in an inertial frame $K$. That is $u^0\equiv\gamma c$ and $\vec{\boldsymbol{u}}\equiv\gamma\vec{\boldsymbol{v}}$, where $\vec{\boldsymbol{v}}$ is the velocity as measured in $K$. Note that $\gamma$ depends implicitly on $\vec{\boldsymbol{v}}$ and is also frame dependent. A second inertial frame $K'$ is defined to be moving with relative velocity $\vec{\boldsymbol{w}}$ with respect to $K$. Note that I have chosen a different symbol for the relative velocity to avoid confusion with $\vec{\boldsymbol{v}}$ which is the velocity of the particle in the reference frame $K$.

We wish to relate the four-vector $u^\mu$ which describes the velocity of the particle in $K$ and the corresponding four-vector $u'^{\,\mu}$ which describes the velocity of the particle in $K'$. This is accomplished by a Lorentz boost:

$$u'^{\,\mu} = \Lambda^\mu{}_\nu u^\nu \tag{3}$$

with the boost matrix $\Lambda^\mu{}_\nu$ given by[2]

$$\Lambda = \begin{pmatrix} \gamma_w & -\gamma_w\,\vec{\boldsymbol{\beta}}_w \\ -\gamma_w\,\vec{\boldsymbol{\beta}}_w & \delta^{ij}+(\gamma_w-1)\dfrac{\beta^i_w\beta^j_w}{|\vec{\boldsymbol{\beta}}_w|^2}\end{pmatrix}\,, \tag{4}$$

where $\vec{\boldsymbol{\beta}}_w\equiv\vec{\boldsymbol{w}}/c$ and $\gamma_w\equiv(1-|\vec{\boldsymbol{\beta}}_w|^2)^{-1/2}$. Then, eqs. (3) and (4) imply that:

$$u'^{\,0} = \gamma_w(u^0 - \vec{\boldsymbol{\beta}}_w\boldsymbol{\cdot}\vec{\boldsymbol{u}}\,)\,, \tag{5}$$

$$\vec{\boldsymbol{u}}\boldsymbol{'} = \vec{\boldsymbol{u}} + \frac{(\gamma_w-1)}{|\vec{\boldsymbol{\beta}}_{\boldsymbol{w}}|^2}(\vec{\boldsymbol{\beta}}_w\boldsymbol{\cdot}\vec{\boldsymbol{u}})\,\vec{\boldsymbol{\beta}}_w - \gamma_w\,\vec{\boldsymbol{\beta}}_w\,u^0\,. \tag{6}$$

[1]We exclude massless particles, where $v=c$ and $\gamma=\infty$, in which case the velocity four-vector does not exist. In particular, the proper time, which is defined via $c^2d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2\gamma^{-2}dt^2$, is not well defined since there is no reference frame relative to which a massless particle is at rest.

[2]Note that we have employed a subscript $w$ on $\vec{\boldsymbol{\beta}}_w$ and $\gamma_w$ to distinguish these quantities from $\vec{\boldsymbol{\beta}}\equiv\vec{\boldsymbol{v}}/c$ and $\gamma\equiv(1-|\vec{\boldsymbol{\beta}}|^2)^{-1/2}$, which appear in the definition of the four vector $u^\mu=(u^0\,;\,\vec{\boldsymbol{u}})=(\gamma c\,;\,\gamma\vec{\boldsymbol{v}})$.

Dividing these two equations yields:

$$\frac{\vec{\boldsymbol{u}}'}{u'^0} = \frac{1}{u^0 - \vec{\boldsymbol{\beta}}_w\!\cdot\!\vec{\boldsymbol{u}}}\left[\frac{\vec{\boldsymbol{u}}}{\gamma_w} + \frac{(\gamma_w - 1)}{\gamma_w|\vec{\boldsymbol{\beta}}_w|^2}(\vec{\boldsymbol{\beta}}_w\!\cdot\!\vec{\boldsymbol{u}})\,\vec{\boldsymbol{\beta}}_w - \vec{\boldsymbol{\beta}}_w\,u^0\right]. \tag{7}$$

Substituting $u^0 = \gamma c$, $\vec{\boldsymbol{u}} = \gamma\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{u}}'/u'^0 = \vec{\boldsymbol{v}}'/c$ in eq. (7), we arrive at:

$$\vec{\boldsymbol{v}}' = \frac{1}{1 - \dfrac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}}{c^2}}\left[\frac{\vec{\boldsymbol{v}}}{\gamma_w} + \frac{(\gamma_w - 1)}{|\vec{\boldsymbol{w}}|^2\gamma_w}\,(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}})\,\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}\right]. \tag{8}$$

This result can be rewritten as:

$$\boxed{\vec{\boldsymbol{v}}' = \frac{1}{1 - \dfrac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}}{c^2}}\left[\frac{1}{\gamma_w}\left(\vec{\boldsymbol{v}} - \frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2}\,\vec{\boldsymbol{w}}\right) - \left(1 - \frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2}\right)\vec{\boldsymbol{w}}\right].} \tag{9}$$

This is the law of addition of velocities. As a check, if $\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}$, then the particle in reference frame $K$ moves at the same velocity as reference frame $K'$ moves with respect to $K$. This means that $\boldsymbol{v}' = \vec{\boldsymbol{0}}$ as viewed by an observer in $K'$, in agreement with eq. (9).

In the simple case where $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are parallel, it follows that:[3]

$$\vec{\boldsymbol{v}} = \left(\frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2}\right)\vec{\boldsymbol{w}}\,. \tag{10}$$

In this case, eq. (9) simplifies immediately to:

$$\vec{\boldsymbol{v}}' = \frac{\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}}{1 - \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}/c^2}\,. \tag{11}$$

Note that the non-relativistic limit we can take $c \to \infty$ (equivalently, all velocities are much smaller that $c$) which implies that $\gamma_w \simeq 1$, in which case eqs. (9) and (11) both reduce to the expected result, $\vec{\boldsymbol{v}}' = \vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}$.

## 2. The momentum four-vector

The momentum four-vector (also called the four-momentum) is related in a simple way to the velocity four-vector:

$$p^\mu = mu^\mu = (E/c\,;\,\vec{\boldsymbol{p}})\,, \tag{12}$$

where [using eq. (1)]

$$\vec{\boldsymbol{p}} = \gamma m\vec{\boldsymbol{v}}\,, \tag{13}$$

$$E = \gamma mc^2\,. \tag{14}$$

Note that by dividing these two equations, one deduces an expression for the particle velocity:

$$\vec{\boldsymbol{v}} = \frac{\vec{\boldsymbol{p}}\,c^2}{E}\,. \tag{15}$$

[3] If $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are parallel, then $\vec{\boldsymbol{v}} = k\vec{\boldsymbol{w}}$ for some constant $k$. Taking the dot product of both sides of this equation with $\vec{\boldsymbol{w}}$ yields $k = \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}/|\vec{\boldsymbol{w}}|^2$, which establishes eq. (10).

Again, it must be emphasized that $\vec{\boldsymbol{v}}$, which appears both explicitly and implicitly in the factors of $\gamma$ in eqs. (13)–(15) corresponds to the velocity of the particle. Thus, in the rest frame of the particle, $\vec{\boldsymbol{v}} = 0$ and $\gamma = 1$, which implies that $p^\mu = mc(1\,;\,\vec{\boldsymbol{0}}\,)$.

Furthermore, the mass $m$ is a scalar quantity (which is Lorentz invariant); it corresponds to the rest energy of the particle divided by $c^2$. This also follows from the observation[4] that the Lorentz invariant scalar $p_\mu p^\mu = m^2c^2$. Finally, by noting that

$$p^2 \equiv g_{\mu\nu}p^\mu p^\nu = (p^0)^2 - |\vec{\boldsymbol{p}}|^2 = m^2c^2\,, \tag{16}$$

and inserting $p^0 = E/c$, one obtains an expression for the relativistic energy:

$$E^2 = c^2|\vec{\boldsymbol{p}}|^2 + m^2c^4\,. \tag{17}$$

Taking the square root, and expanding out resulting expression in the limit of $|\vec{\boldsymbol{v}}| \ll c$ yields:

$$E \simeq mc^2 + \frac{|\vec{\boldsymbol{p}}|^2}{2m}\,, \tag{18}$$

which we recognize as the sum of the rest energy and the non-relativistic kinetic energy. More generally, the relativistic energy can be written as $E = mc^2 + T$, which defines the relativistic kinetic energy as:

$$T = \sqrt{c^2|\vec{\boldsymbol{p}}|^2 + m^2c^4} - mc^2\,. \tag{19}$$

The above results apply to massive particles. In the case of a massless particle ($m = 0$), although the velocity four-vector is undefined, the momentum four-vector exists and satisfies $p^2 = 0$. That is, one can formally take the limit of $m \to 0$ such that the product $p^\mu = mu^\mu$ is meaningful. Eqs. (16) and (17) are still valid so that,

$$p^2 = g_{\mu\nu}p^\mu p^\nu = 0 \quad \Longrightarrow \quad E = c|\vec{\boldsymbol{p}}|\,. \tag{20}$$

We see that $p^\mu$ is a lightlike vector that can be written as $p^\mu = |\vec{\boldsymbol{p}}|(1\,;\,\hat{\boldsymbol{p}})$, where $\hat{\boldsymbol{p}}$ is a unit vector that points in the direction of the momentum three-vector.

## 3. The force and acceleration four-vectors

The relation between the three-vector force and the three-vector momentum remains valid in special relativity,

$$\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt}\,.$$

Using eq. (13), it follows that for a massive particle ($m \neq 0$),

$$\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt} = \frac{d}{dt}\big(\gamma m\vec{\boldsymbol{v}}\big) = \gamma m\frac{d\vec{\boldsymbol{v}}}{dt} + m\vec{\boldsymbol{v}}\,\frac{d\gamma}{dt}\,. \tag{21}$$

[4] Since Lorentz scalars do not depend on the reference frame, I may compute it in any frame. By choosing the rest frame of the particle, the computation is trivial.

Using

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{v}}}{c^2}\right)^{-1/2} = \frac{\gamma^3}{c^2}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\,, \tag{22}$$

it follows that

$$\boxed{\vec{\boldsymbol{F}} = \gamma m\left[\frac{d\vec{\boldsymbol{v}}}{dt} + \frac{\gamma^2}{c^2}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right)\vec{\boldsymbol{v}}\right]\,.} \tag{23}$$

which is the relativistic generalization of Newton's second law for a massive particle. Note that eq. (23) can be rewritten as follows:

$$\boxed{\vec{\boldsymbol{F}} = \gamma^3 m\left\{\frac{d\vec{\boldsymbol{v}}}{dt} + \frac{1}{c^2}\left[\vec{\boldsymbol{v}}\times\left(\vec{\boldsymbol{v}}\times\frac{d\vec{\boldsymbol{v}}}{dt}\right)\right]\right\}\,.} \tag{24}$$

Expanding out the triple cross product and using $\gamma^2 = (1 - v^2/c^2)^{-1}$, we recover the result of eq. (23).

Two special cases are noteworthy:

1. $\vec{\boldsymbol{v}} \parallel d\vec{\boldsymbol{v}}/dt$ (linear motion).

In this case, $\vec{\boldsymbol{v}}\times(d\vec{\boldsymbol{v}}/dt) = 0$. Plugging this result into eq. (24) yields

$$\vec{\boldsymbol{F}} = \gamma^3 m\frac{d\vec{\boldsymbol{v}}}{dt}\,, \qquad \text{for linear motion.} \tag{25}$$

One can also derive eq. (25) by noting that if $\vec{\boldsymbol{v}}$ is parallel to $\vec{\boldsymbol{v}}/dt$ then it follows that

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \kappa\vec{\boldsymbol{v}}\,, \tag{26}$$

for some constant $\kappa$. Taking the dot product of eq. (26) with $\vec{\boldsymbol{v}}/v^2$ (where $v^2 \equiv \vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{v}}$ is the squared magnitude of the velocity three-vector), one obtains $\kappa = [\vec{\boldsymbol{v}}\boldsymbol{\cdot}d\vec{\boldsymbol{v}}/dt]/v^2$. Inserting this result back into eq. (26) yields:

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{\vec{\boldsymbol{v}}}{v^2}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right)\,. \tag{27}$$

Using this result in eq. (23) and noting that $1 + v^2\gamma^2/c^2 = \gamma^2$, we end up with eq. (25), as expected.

2. $\vec{\boldsymbol{v}} \perp d\vec{\boldsymbol{v}}/dt$ (circular motion).

In this case $\vec{\boldsymbol{v}}\boldsymbol{\cdot}d\vec{\boldsymbol{v}}/dt = 0$. Plugging this result into eq. (23) yields

$$\vec{\boldsymbol{F}} = \gamma m\frac{d\vec{\boldsymbol{v}}}{dt}\,, \qquad \text{for circular motion.} \tag{28}$$

In older introductory books on relativity, the concept of "relativistic mass," defined as $m_R \equiv \gamma m$ was introduced. I suppose that this was motivated by eqs. (13) and (14) which could be written as $\vec{\boldsymbol{p}} = m_R\vec{\boldsymbol{v}}$ (which resembles the non-relativistic expression of momentum) and $E = m_R c^2$. However, as eqs. (23)–(28) make clear, $m_R$ is not a useful construct.[5] Indeed, there is nothing special about $\gamma m$ that would single out this choice for some definition of the relativistic mass. In the formalism presented in these notes, $m^2 = p^2/u^2$ is the ratio of two Lorentz-invariant scalars, and thus is itself Lorentz-invariant (and independent of the choice of reference frame). In the older introductory books on relativity, $m$ was called the rest mass. In the formalism presented in these notes, $m$ is a Lorentz invariant quantity that is an intrinsic property of the particle (like electric charge).

We still have not yet developed an appropriate expression for a four-vector that can be related to the force. To see how to do this, we first consider the dot product of eq. (24) with $\vec{\boldsymbol{v}}$, which immediately yields,

$$\vec{\boldsymbol{F}}\boldsymbol{\cdot}\vec{\boldsymbol{v}} = \gamma^3 m\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\,. \tag{29}$$

This should be compared with

$$\frac{dE}{dt} = \frac{d}{dt}\big(\gamma mc^2\big) = mc^2\frac{d\gamma}{dt} = \gamma^3 m\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\,,$$

after using eq. (22). Hence, the relation between the power $dE/dt$ and $\vec{\boldsymbol{F}}\boldsymbol{\cdot}\vec{\boldsymbol{v}}$,

$$\frac{dE}{dt} = \vec{\boldsymbol{F}}\boldsymbol{\cdot}\vec{\boldsymbol{v}}\,,$$

remains valid in special relativity, as long as we define the power to be the time rate of change of the relativistic energy. The above results motivate the introduction of *Minkowski four-vector* force,

$$K^\mu \equiv \frac{dp^\mu}{d\tau} = \left(\frac{\gamma\vec{\boldsymbol{F}}\boldsymbol{\cdot}\vec{\boldsymbol{v}}}{c}\,;\,\gamma\vec{\boldsymbol{F}}\right) = m\left(\frac{\gamma^4}{c}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\,;\,\gamma^2\frac{d\vec{\boldsymbol{v}}}{dt} + \frac{\gamma^4}{c^2}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right)\vec{\boldsymbol{v}}\right)\,, \tag{30}$$

where we have used $d\tau = \gamma^{-1}dt$ and then employed eqs. (23) and (29) to obtain the final form above.

Likewise, we can define the acceleration four-vector for a massive particle,[6]

$$\alpha^\mu \equiv \frac{du^\mu}{d\tau} = \left(c\frac{d\gamma}{d\tau}\,;\,\frac{d}{d\tau}\big(\gamma\vec{\boldsymbol{v}}\big)\right)\,. \tag{31}$$

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[5] This point has been emphasized in L.B. Okun, *The concept of mass*, Physics Today **42**(6), 31–36 (1989). This article is based on a longer paper that was published in Usp. Fiz. Nauk **158**, 511-530 (July 1989). More recently, Okun published a paper that traces the way Einstein formulated the relation between energy and mass in his work from 1905 to 1955. See L.B. Okun, *The Einstein formula: $E_0 = mc^2$. "Isn't the Lord laughing?"*, Physics-Uspekhi **51**(5), 513–527 (2008). Okun ended this paper with the following remark. "It is high time we stopped deceiving new generations of college and high school students by inculcating into them the conviction that mass increasing with increasing velocity is an experimental fact."

[6] As in the case of the velocity four-vector, the acceleration four-vector of a massless particle does not exist.

Using $d\tau = \gamma^{-1}dt$, it follows that

$$\alpha^\mu = \left(\gamma c\frac{d\gamma}{dt}\,;\,\gamma\frac{d}{dt}(\gamma\vec{\boldsymbol{v}})\right) = \left(\frac{\gamma^4}{c}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\,;\,\gamma^2\frac{d\vec{\boldsymbol{v}}}{dt} + \frac{\gamma^4}{c^2}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right)\vec{\boldsymbol{v}}\right).$$

The three-vector acceleration is defined as usual by $\vec{\boldsymbol{a}} \equiv d\vec{\boldsymbol{v}}/dt$. Then, one can rewrite the four-vector acceleration as[7]

$$\alpha^\mu = \left(\frac{\gamma^4}{c}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\,;\,\gamma^2\vec{\boldsymbol{a}} + \frac{\gamma^4}{c^2}(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})\vec{\boldsymbol{v}}\right) = \frac{\gamma^4}{c}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\left(1\,;\,\frac{\vec{\boldsymbol{v}}}{c}\right) + \gamma^2\big(0\,;\,\vec{\boldsymbol{a}}\big). \tag{32}$$

Employing eqs. (12) and (31), we can insert $p^\mu = mu^\mu$ into eq. (30) to obtain

$$K^\mu = m\frac{du^\mu}{d\tau} = m\alpha^\mu\,, \tag{33}$$

which is the four-vector version of Newton's second law.

An important property of the four-vector acceleration is

$$u_\mu\alpha^\mu = 0\,. \tag{34}$$

In light of eq. (33), the above result immediately yields, $u_\mu K^\mu = 0$. Eq. (34) can be proved as follows. Noting that $u^\mu u^\mu = c^2$ which is a constant, it follows that

$$0 = \frac{d}{d\tau}\big(u^\mu u^\mu\big) = 2u_\mu\alpha^\mu\,.$$

One can also derive the same result from eqs. (1) and (32),

$$u_\mu\alpha^\mu = \gamma^5\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}} - \gamma^3\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}} - \frac{\gamma^5 v^2}{c^2}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}} = \gamma^3\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\left(\gamma^2 - 1 - \frac{\gamma^2 v^2}{c^2}\right) = 0\,,$$

since $\gamma^2(1 - v^2/c^2) = 1$.

Moreover, after some algebra, one obtains,

$$\alpha_\mu\alpha^\mu = -\gamma^4\left[|\vec{\boldsymbol{a}}|^2 + \frac{\gamma^2}{c^2}\big(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\big)^2\right]. \tag{35}$$

Using the vector identity, $|\vec{\boldsymbol{v}}\times\vec{\boldsymbol{a}}|^2 = |\vec{\boldsymbol{v}}|^2|\,|\vec{\boldsymbol{a}}|^2 - (\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2$, we can rewrite eq. (35) as

$$\alpha_\mu\alpha^\mu = -\gamma^6\left[|\vec{\boldsymbol{a}}|^2 - \frac{|\vec{\boldsymbol{v}}\times\vec{\boldsymbol{a}}|^2}{c^2}\right] = -\gamma^6|\vec{\boldsymbol{a}}|^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right) \leq 0\,, \tag{36}$$

where the equality is satisfied if and only if $\alpha^\mu = 0$. That is, a nonzero acceleration four-vector is necessarily spacelike. This result is a special case of a more general result

[7]Most books simply use the notation $a^\mu$ for the four-vector acceleration. One disadvantage of this notation is that the space component of this four-vector, which would be denoted by $a^i$ is *not* the $i$th component of the vector $d\vec{\boldsymbol{v}}/dt$.

that if $a$ is a timelike vector and $a\cdot b = 0$, then either $b$ is spacelike or $b$ is the zero vector.[8] Since $u$ is timelike and $u\cdot\alpha = 0$, it follows that either $\alpha$ is spacelike or $\alpha = 0$.

The concept of constant acceleration must be reconsidered in special relativity. It clearly cannot mean that the three-vector $\vec{\boldsymbol{a}} = d\vec{\boldsymbol{v}}/dt$ is constant, since this would be a frame-dependent condition. Instead, constant acceleration means that the square of the four-vector acceleration, $\alpha_\mu\alpha^\mu$, is a constant. Clearly, the latter is a Lorentz-invariant condition. Since $\alpha_\mu\alpha^\mu \leq 0$, it is traditional to define,

$$g \equiv \sqrt{-\alpha_\mu\alpha^\mu}\,. \tag{37}$$

To motivate this definition, consider the case of $\vec{\boldsymbol{a}} \parallel \vec{\boldsymbol{v}}$, corresponding to linear motion. In this case, eq. (27) yields

$$v^2\vec{\boldsymbol{a}} = (\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})\vec{\boldsymbol{v}}\,, \qquad \text{for linear motion,} \tag{38}$$

where $v \equiv |\vec{\boldsymbol{v}}|$. Taking the magnitude of the three-vectors on both sides of eq. (38) yields $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = v|\vec{\boldsymbol{a}}|$. Inserting these results into eq. (32) yields after some simplification,

$$\alpha^\mu = \gamma^4\left(\frac{\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}}{c}\,;\,\vec{\boldsymbol{a}}\right)\,, \qquad \text{for linear motion.}$$

Using $|\vec{\boldsymbol{a}}|^2 = (\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2/v^2$, it then follows that

$$g^2 = -\alpha_\mu\alpha^\mu = \gamma^8|\vec{\boldsymbol{a}}|^2\left(1-\frac{v^2}{c^2}\right) = \gamma^6|\vec{\boldsymbol{a}}|^2\,. \tag{39}$$

That is, in the case of linear motion, constant linear acceleration in special relativity means that $g = \gamma^3|\vec{\boldsymbol{a}}|$ is constant. Note that in each instantaneous rest frame[9] of the accelerating particle, constant linear acceleration does indeed correspond to a constant vector $\vec{\boldsymbol{a}}$ as expected.

We can easily evaluate the velocity at time $t$ of a constantly linearly accelerating particle, which starts off at rest at $t = 0$. For example, if $\vec{\boldsymbol{v}}$ is parallel to $\vec{\boldsymbol{a}}$ then $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = v|\vec{\boldsymbol{a}}|$, and it follows that[10]

$$\frac{dv}{dt} = \frac{d}{dt}(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}})^{1/2} = \frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}}{v} = |\vec{\boldsymbol{a}}| = \gamma^{-3}g = \left(1-\frac{v^2}{c^2}\right)^{3/2} g\,,$$

where $g = \gamma^3|\vec{\boldsymbol{a}}|$ is constant. Integrating this equation subject to the boundary condition, $v(t=0) = 0$, yields

$$v(t) = \frac{gt}{\left(1+\dfrac{g^2t^2}{c^2}\right)^{1/2}}\,. \tag{40}$$

[8] To prove this more general result, one first shows that for any timelike vector $a$, there exists a reference frame such that $a^\mu = (a^0\,;\,\vec{\boldsymbol{0}})$. Given that $a\cdot b = 0$, it then follows that $b = (0\,;\,\vec{\boldsymbol{b}})$, in which case $b^2 = -|\vec{\boldsymbol{b}}|^2 \leq 0$. Since $b^2$ is a Lorentz scalar, it follows that $b^2 \leq 0$ in any reference frame and $b^2 = 0$ if and only if $b^\mu = 0$. That is, either $b$ is spacelike or $b$ is the zero vector.

[9] At any time $t$, the instantaneous rest frame corresponds to the inertial frame traveling at the same velocity $\vec{\boldsymbol{v}}$ as the accelerating particle at time $t$. For further details, see the Appendix A.

[10] In the case where $\vec{\boldsymbol{v}}$ is antiparallel to $\vec{\boldsymbol{a}}$, we have $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = -v|\vec{\boldsymbol{a}}|$, in which case $dv/dt$ is negative.

Note that the non-relativistic limit, $v(t) \simeq gt$, which is a good approximation when $gt \ll c$ (i.e., when the velocity is non-relativistic). Furthermore, in the limit of $t \to \infty$, we have

$$\lim_{t\to\infty} v(t) = c\,.$$

and $v < c$ for *all* finite times $t$. Thus, in special relativity, a particle that is constantly accelerating never reaches the speed of light in a finite amount of time!

Without loss of generality, we shall assume that both $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ point in the $\hat{\boldsymbol{x}}$ direction. Then, the coordinate $x(t)$ as a function of $t$ can be determined from eq. (40),

$$v = \frac{dx}{dt} = \frac{gt}{\left(1 + \dfrac{g^2t^2}{c^2}\right)^{1/2}}\,.$$

Integrating the above expression yields,

$$x(t) = x_0 + \frac{c^2}{g}\left[\left(1 + \frac{g^2t^2}{c^2}\right)^{1/2} - 1\right]\,, \tag{41}$$

where $x_0 \equiv x(t=0)$. One can check that in the non-relativistic limit where $gt \ll c$, eq. (41) reduces to $x(t) = x_0 + \frac{1}{2}gt^2$, as expected.

The coordinates $x$ and $t$ corresponding to the position and time measured in the accelerating reference frame. It is convenient to express these coordinates in terms of the proper time $\tau$. Using $d\tau = \gamma^{-1}dt$ where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$, it follows that

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \frac{g^2t^2}{c^2}\left(1 + \frac{g^2t^2}{c^2}\right)^{-1}} = \frac{1}{\sqrt{1 + \dfrac{g^2t^2}{c^2}}}\,,$$

after making use of eq. (40) for $v$. Hence,

$$\tau = \int_0^t \frac{dt}{\sqrt{1 + \dfrac{g^2t^2}{c^2}}} = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right)\,,$$

where we have applied a boundary condition that fixes $\tau = 0$ at $t = 0$. Hence,[11]

$$t = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right)\,. \tag{42}$$

Plugging the value of $t$ obtained in eq. (42) into eqs. (40) and (41) yields,

$$\frac{v}{c} = \tanh\left(\frac{g\tau}{c}\right)\,, \qquad x = x_0 - \frac{c^2}{g}\left[1 - \cosh\left(\frac{g\tau}{c}\right)\right]\,. \tag{43}$$

Due to the properties of the hyperbolic tangent function, it immediately follows that $-1 \leq v/c \leq 1$, which is consistent with the relativistic requirement that no object

[11]Note that in the non-relativistic limit (or equivalently, as $c \to \infty$), we obtain $t \simeq \tau$ as expected.

can exceed the speed of light. Moreover, if we choose the initial condition such that $x_0 = c^2/g$, we end up with the parametric equations,

$$ct = \frac{c^2}{g}\sinh\left(\frac{g\tau}{c}\right)\,, \qquad x = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right)\,. \tag{44}$$

It immediately follows that

$$x^2 - c^2t^2 = \frac{c^4}{g^2}\,, \tag{45}$$

which is the equation for a hyperbola in the $x$–$ct$ plane. This result explains why constant acceleration in relativity is often called *hyperbolic motion*.

## APPENDIX A: The comoving reference frame

Consider the velocity four vector $u^\mu = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}})$. We can boost to the rest frame using the boost matrix $\Lambda$. Explicitly,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\,\vec{\boldsymbol{v}}/c \\ -\gamma\,\vec{\boldsymbol{v}}/c & \delta^{ij} + (\gamma-1)\dfrac{v^i v^j}{v^2} \end{pmatrix}\,, \tag{46}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$, with $v \equiv |\vec{\boldsymbol{v}}|$. Thus, the rest frame velocity four vector is,

$$\begin{pmatrix} \gamma & -\gamma\,v^j/c \\ -\gamma\,v^i/c & \delta^{ij} + (\gamma-1)\dfrac{v^i v^j}{v^2} \end{pmatrix}\begin{pmatrix} \gamma\,c \\ \gamma\,v^j \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}\,. \tag{47}$$

Indeed, in the rest frame, $u^0 = (c\,;\,\vec{\boldsymbol{0}})$, as expected since in the rest frame the velocity of the particle is zero.

Consider now the acceleration four vector given by eq. (32), which we repeat below,

$$\alpha^\mu = \left(\frac{\gamma^4}{c}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\,;\,\gamma^2\vec{\boldsymbol{a}} + \frac{\gamma^4}{c^2}(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})\vec{\boldsymbol{v}}\right) = \frac{\gamma^4}{c}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\left(1\,;\,\frac{\vec{\boldsymbol{v}}}{c}\right) + \gamma^2\big(0\,;\,\vec{\boldsymbol{a}}\big)\,. \tag{48}$$

Can we boost to the rest frame as we did in the case of the velocity four vector? Not quite, since as the particle accelerates along its worldline, its velocity is changing. If we apply a boost at one time to reach the rest frame of the particle, we would have to apply a different boost at another time. The best one can do is to define an instantaneous rest frame. That is, at time $t$, we define an inertial frame with a velocity $\vec{\boldsymbol{v}}(t)$ that coincides with the rest frame of the accelerating particle at time $t$. Considering the entire worldline, one then defines an infinite number of instantaneous rest frames, each of which coincides with a different inertial reference frame. At a given time $t$, the corresponding inertial frame that coincides with the instantaneous rest frame is called the *comoving frame*.

Given an accelerating particle, one can now define the acceleration three-vector measured in the momentarily comoving frame, which we denote by $\vec{\boldsymbol{a}}_{\rm co}$. In the comoving frame, the velocity four vector is $u^\mu = (c\,;\,\vec{\boldsymbol{0}})$. Since $u\cdot\alpha = 0$ [cf. eq. (34)], it follows that in the comoving frame, the acceleration four vector is given by

$$\alpha^\mu = (0\,;\,\vec{\boldsymbol{a}}_{\rm co})\,. \tag{49}$$

Let us check that $\vec{\boldsymbol{a}}_{\rm co}$ can be interpreted as the Newtonian acceleration of the particle that the comoving inertial observer would measure.[12] In the observer's reference frame, the velocity four vectors at the exact instant of comobility (call this time $t = 0$) and at an infinitesimal time interval $\Delta t$ later are given by,

$$u(t=0) = (c\,;\,\vec{\boldsymbol{0}})\,, \qquad\qquad u(\Delta t) = (\gamma_\Delta c\,;\,\gamma_\Delta \Delta\vec{\boldsymbol{v}})\,, \tag{50}$$

where $\gamma_\Delta \equiv [1-(\Delta\vec{\boldsymbol{v}}\cdot\Delta\vec{\boldsymbol{v}}/c^2]^{1/2}$. Since $d\tau = \gamma^{-1}dt$, it follows that $\Delta\tau/\Delta t \to 1$ as $t \to 0$. Moreover, $\Delta\vec{\boldsymbol{v}} = \mathcal{O}(\Delta t)$. Hence,

$$\alpha^\mu = \lim_{\Delta t\to 0}\,(0\,;\,\Delta\vec{\boldsymbol{v}}/\Delta t) = (0\,;\,\vec{\boldsymbol{a}}_{\rm co})\,. \tag{51}$$

To find $\vec{\boldsymbol{a}}_{\rm co}$ explicitly, we apply the method used in eq. (47) to the acceleration four vector. Using eq. (48),

$$\begin{pmatrix} \gamma & -\gamma\, v^j/c \\[6pt] -\gamma\, v^i/c & \delta^{ij} + (\gamma-1)\dfrac{v^i v^j}{v^2}\end{pmatrix}\begin{pmatrix} \dfrac{\gamma^4}{c}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} \\[6pt] \dfrac{\gamma^4}{c^2}(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}})v^j + \gamma^2 a^j\end{pmatrix} = \begin{pmatrix} 0 \\[6pt] \gamma^2\left[a^i + (\gamma-1)\dfrac{(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}})v^i}{v^2}\right]\end{pmatrix}. \tag{52}$$

Thus, we can identify,

$$\boxed{\vec{\boldsymbol{a}}_{\rm co} = \gamma^2\left[\vec{\boldsymbol{a}} + (\gamma-1)\frac{(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}})\vec{\boldsymbol{v}}}{v^2}\right]\,.} \tag{53}$$

Note that

$$\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}_{\rm co} = \gamma^3\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}\,. \tag{54}$$

Another common form for $\vec{\boldsymbol{a}}_{\rm co}$ makes use of the identity,

$$\frac{\gamma-1}{v^2} = \frac{\gamma^2}{c^2(\gamma+1)}\,. \tag{55}$$

Thus,

$$\vec{\boldsymbol{a}}_{\rm co} = \gamma^2\left[\vec{\boldsymbol{a}} + \frac{\gamma^2}{\gamma+1}\,\frac{(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}})\vec{\boldsymbol{v}}}{c^2}\right]\,. \tag{56}$$

[12]Here we following the illuminating discussion given in Anupam Garg, *Classical Electromagnetism in a Nutshell* (Princeton University Press, Princeton, NJ, 2012) pp. 538–539.

As a check of our computation, we work out the squared magnitude of $\vec{\boldsymbol{a}}_{\rm co}$. Using eq. (53),

$$\begin{aligned} |\vec{\boldsymbol{a}}_{\rm co}|^2 &= \gamma^4|\vec{\boldsymbol{a}}|^2 + \frac{\gamma^4(\gamma-1)^2}{v^2}(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2 + \frac{2\gamma^4(\gamma-1)}{v^2}(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2 \\ &= \gamma^4|\vec{\boldsymbol{a}}|^2 + \frac{\gamma^4(\gamma^2-1)}{v^2}(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2 \\ &= \gamma^4|\vec{\boldsymbol{a}}|^2 + \frac{\gamma^6(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2}{c^2}\,. \end{aligned} \tag{57}$$

Using $\gamma^{-2} = 1 - v^2/c^2$, we can rewrite eq. (57) in the following form:

$$\begin{aligned} |\vec{\boldsymbol{a}}_{\rm co}|^2 &= \gamma^6\left[|\vec{\boldsymbol{a}}|^2\left(1-\frac{v^2}{c^2}\right) + \frac{(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2}{c^2}\right] = \gamma^6\left[|\vec{\boldsymbol{a}}|^2 - \left(\frac{v^2|\vec{\boldsymbol{a}}|^2 - (\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2}{c^2}\right)\right] \\ &= \gamma^6\left[|\vec{\boldsymbol{a}}|^2 - \frac{|\vec{\boldsymbol{v}}\times\vec{\boldsymbol{a}}|^2}{c^2}\right]. \end{aligned} \tag{58}$$

An interesting application of eq. (58) is provided in Appendix B.

Recall eq. (37), which implies that $g^2 = -\alpha_\mu\alpha^\mu$. Eq. (36) yields,

$$g^2 = \gamma^6\left[|\vec{\boldsymbol{a}}|^2 - \frac{|\vec{\boldsymbol{v}}\times\vec{\boldsymbol{a}}|^2}{c^2}\right]. \tag{59}$$

Since $g^2$ is a Lorentz invariant, one can evaluate it in any frame (and obtain the same answer). In particular, in the comoving frame where $\alpha^\mu = (0\,;\,\vec{\boldsymbol{a}}_{\rm co})$, it follows that $g^2 = -\alpha_\mu\alpha^\mu = |\vec{\boldsymbol{a}}_{\rm co}|^2$. That is, eqs. (58) and (59) are in agreement. This serves as a consistency check and gives us confidence that eq. (53) is correct. Moreover, we see that constant acceleration in special relativity means a constant acceleration three-vector in the comoving frame of the accelerating particle, which is equivalent to a constant $g^2$, as asserted below eq. (39).

The form of eq. (53) may be somewhat surprising. After all, starting from the velocity four vector, $u^\mu = (\gamma c\,;\gamma\vec{\boldsymbol{v}})$, the rest frame is obtained by setting $\vec{\boldsymbol{v}} = 0$, in which case the rest frame velocity four vector $u^\mu = (c\,;\vec{\boldsymbol{0}})$ is obtained. One may have been tempted to say that starting from the acceleration four vector given in eq. (48), the rest frame acceleration four vector should be obtained by setting $\vec{\boldsymbol{v}} = 0$. However, this procedure yields $\alpha^\mu = (0\,;\,\vec{\boldsymbol{a}})$, which is clearly not the same as $\alpha^\mu = (0\,;\,\vec{\boldsymbol{a}}_{\rm co})$. This paradox is resolved by realizing that setting $\vec{\boldsymbol{v}} = 0$ only makes sense if there is a global rest frame—that is, a unique rest frame in which to observe the particle motion. For a particle traveling at constant velocity $\vec{\boldsymbol{v}}$, such a global rest frame exists, and the statements concerning the velocity four vector made at the beginning of this paragraph are valid. However, no global rest frame exists for an accelerating particle. The best we can do is to define the comoving frame, which corresponds to an instantaneous rest frame that is always changing along the particle trajectory. In particular, setting $\vec{\boldsymbol{v}} = 0$ in the expression for the acceleration four vector does not yield anything useful. Indeed, the correct analysis yields the comoving acceleration three-vector $\vec{\boldsymbol{a}}_{\rm co}$ exhibited in eq. (53), which has a nontrivial dependence on $\vec{\boldsymbol{a}} = d\vec{\boldsymbol{v}}/dt$.

It is instructive to apply the result of eq. (53) to two cases. First, if $\vec{\boldsymbol{a}} \parallel \vec{\boldsymbol{v}}$ (corresponding to linear motion), then we can use eq. (38) to write $v^2\vec{\boldsymbol{a}} = (\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}})\vec{\boldsymbol{v}}$. Inserting this result into eq. (53) yields $\vec{\boldsymbol{a}}_{\rm co} = \gamma^3\vec{\boldsymbol{a}}$. Thus, we have recovered the result of eq. (39), which was the starting point for our analysis of hyperbolic motion. Second, consider circular motion where $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{a}}$. In this case, we set $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = 0$ in eq. (53) to obtain $\vec{\boldsymbol{a}}_{\rm co} = \gamma^2\vec{\boldsymbol{a}}$. This result is consistent with eq. (59), since in this case, $g^2 = \gamma^4|\vec{\boldsymbol{a}}|^2$. In both special cases just considered, $\vec{\boldsymbol{a}}_{\rm co}$ is proportional to $\vec{\boldsymbol{a}}$. However, in the generic case, $\vec{\boldsymbol{a}}_{\rm co}$ is not parallel to $\vec{\boldsymbol{a}}$, since it has a component that points along the instantaneous velocity vector, $\vec{\boldsymbol{v}}$.

It is sometimes useful to express the acceleration four vector directly in terms of $\vec{\boldsymbol{a}}_{\rm co}$. Using eqs. (53) and (54), it follows that

$$\vec{\boldsymbol{a}} = \frac{1}{\gamma^3}\left[\gamma\vec{\boldsymbol{a}}_{\rm co} - \left(\frac{\gamma-1}{v^2}\right)(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}_{\rm co})\vec{\boldsymbol{v}}\right]\,. \tag{60}$$

Plugging in this result for $\vec{\boldsymbol{a}}$ into eq. (48) and making use of eq. (55) yields,

$$\alpha^\mu = \left(\frac{\gamma}{c}\,\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}_{\rm co}\,;\,\vec{\boldsymbol{a}}_{\rm co} + \frac{\gamma^2}{\gamma+1}\,\frac{(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}_{\rm co})\vec{\boldsymbol{v}}}{c^2}\right)\,. \tag{61}$$

One can check this last result by boosting the acceleration four vector in the instantaneous rest frame where $\alpha^\mu = (0\,;\,\vec{\boldsymbol{a}}_{\rm co})$ back to the original reference frame by using the inverse boost matrix $\Lambda^{-1}$ (which is obtained from $\Lambda$ by replacing $\vec{\boldsymbol{v}} \to -\vec{\boldsymbol{v}}$). That is,

$$\begin{pmatrix} \gamma & \gamma\, v^j/c \\ \\ \gamma\, v^i/c & \delta^{ij} + (\gamma-1)\dfrac{v^i v^j}{v^2}\end{pmatrix}\begin{pmatrix} 0 \\ \\ a^j_{\rm co}\end{pmatrix} = \begin{pmatrix} \dfrac{\gamma\,\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}_{\rm co}}{c} \\ \\ a^i_{\rm co} + \dfrac{\gamma-1}{v^2}\,(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}_{\rm co})v^i\end{pmatrix}\,. \tag{62}$$

Using eq. (55), we then recover the result of eq. (61), as expected.

Finally, one can perform a similar analysis involving the force four vector. In the comoving frame, the force acting on the accelerating particle in its instantaneous rest frame is $\vec{\boldsymbol{f}} = m\vec{\boldsymbol{a}}_{\rm co}$. Using eqs. (33) and (61), we obtain,[13]

$$K^\mu = m\alpha^\mu = \left(\frac{\gamma}{c}\,\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{f}}\,;\,\vec{\boldsymbol{f}} + \frac{\gamma^2}{\gamma+1}\,\frac{(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{f}})\vec{\boldsymbol{v}}}{c^2}\right)\,. \tag{63}$$

Just like the relation between $\vec{\boldsymbol{a}} = d\vec{\boldsymbol{v}}/dt$ and $\vec{\boldsymbol{a}}_{\rm co}$ is nontrivial, the same can be said for the relation between $\vec{\boldsymbol{F}} = d\vec{\boldsymbol{p}}/dt$ and $\vec{\boldsymbol{f}}$. Comparing eqs. (30) and (63) yields,

$$\vec{\boldsymbol{F}} = \gamma^{-1}\vec{\boldsymbol{f}} + \frac{\gamma}{\gamma+1}\,\frac{(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{f}})\vec{\boldsymbol{v}}}{c^2}\,. \tag{64}$$

Using eq. (55), one can quickly verify that

$$\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{F}} = \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{f}}\,. \tag{65}$$

[13]Eq. (63) appears in eq. (4.1.12) on p. 66 of Roman U. Sexl and Helmuth K. Urbantke, *Relativity, Groups, Particles—Special Relativity and Relativistic Symmetry in Fields and Particle Physics* (Springer-Verlag, Vienna, Austria, 2001).

Inverting the expression given in eq. (64) to obtain $\vec{\boldsymbol{f}}$ in terms of $\vec{\boldsymbol{F}}$, we end up with

$$\vec{\boldsymbol{f}} = \gamma\vec{\boldsymbol{F}} - \frac{\gamma^2}{\gamma+1}\,\frac{(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{F}})\vec{\boldsymbol{v}}}{c^2}\,. \tag{66}$$

That is, the force three-vector that acts in the instantaneous rest frame depends non-trivially on $\vec{\boldsymbol{F}} = d\vec{\boldsymbol{p}}/dt$.

We can perform one last check of our calculations by rewriting eq. (23) as

$$\vec{\boldsymbol{F}} = \gamma m\left[\vec{\boldsymbol{a}} + \frac{\gamma^2}{c^2}(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})\vec{\boldsymbol{v}}\right]\,, \qquad\qquad \vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{F}} = \gamma^3 m\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\,. \tag{67}$$

Plugging these results into eq. (66) yields,

$$\vec{\boldsymbol{f}} = \gamma^2 m\left[\vec{\boldsymbol{a}} + \frac{\gamma^2}{\gamma+1}\,\frac{(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})\vec{\boldsymbol{v}}}{c^2}\right] = m\vec{\boldsymbol{a}}_{\rm co}\,, \tag{68}$$

after employing eq. (56), as required.

## APPENDIX B: The Relativistic Larmour formula

Later on in this course, we will derive the Larmour formula for the power emitted by an accelerating point charge $q$ in the nonrelativistic limit,

$$P = \frac{2q^2|\vec{\boldsymbol{a}}|^2}{3c^3}\,, \tag{69}$$

where $\vec{\boldsymbol{a}}$ is the acceleration three-vector. Remarkably, one can derive the correct relativistic version of Larmour's formula using the results obtained in Appendix A.

Given an accelerating point charge, we can boost to the instantaneous (comoving) rest frame of the point charge. In this frame, eq. (69) is valid if we identify $\vec{\boldsymbol{a}}$ with the acceleration vector in the comoving rest frame, $\vec{\boldsymbol{a}}_{\rm co}$. That is,

$$P = \frac{2q^2|\vec{\boldsymbol{a}}_{\rm co}|^2}{3c^3}\,, \tag{70}$$

But now, it is a simple matter to employ eq. (58) to obtain,

$$P = \frac{2q^2\gamma^6\left[|\vec{\boldsymbol{a}}|^2 - |\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{a}}|^2\right]}{3c^3}\,, \tag{71}$$

where $\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{v}}/c$, which is the correct relativistic Larmour formula obtained by another method in class.