The Addition Theorem of Spherical Harmonics

The addition theorem for spherical harmonics states that

$$P_\ell(\cos\theta) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\boldsymbol{\hat{n}'})Y^*_{\ell m}(\boldsymbol{\hat{n}''})\,, \qquad \text{where } \cos\theta \equiv \boldsymbol{\hat{n}'\cdot\hat{n}''}. \tag{1}$$

The standard proof of this theorem can be found on pp. 110–111 of John David Jackson, *Classical Electrodynamics*, 3rd Edition (John Wiley & Sons, Inc., Hoboken, NJ, 1999). In this note, I shall provide an alternative proof that makes use of the theory of angular momentum operators in quantum mechanics.

An (actively) rotated state vector, corresponding to a state with no internal spin degrees of freedom, is denoted by

$$|\psi\rangle_R \equiv U[R]|\psi\rangle\,, \tag{2}$$

where $R\equiv R(\chi\boldsymbol{\hat{u}})$ is a counterclockwise rotation by an angle χ about a fixed axis along the unit vector $\boldsymbol{\hat{u}}$, and $U[R]=e^{-i\chi\boldsymbol{\hat{u}\cdot}\vec{\boldsymbol{L}}/\hbar}$ is the corresponding unitary operator that acts on quantum states of the Hilbert space. Likewise, the coordinate basis ket $|\vec{\boldsymbol{x}}\rangle$ can also be rotated,

$$|\vec{\boldsymbol{x}}'\rangle = U[R]|\vec{\boldsymbol{x}}\rangle\,, \qquad \text{where } \vec{\boldsymbol{x}}'=R\vec{\boldsymbol{x}}\,. \tag{3}$$

We also define unit vectors, $\boldsymbol{\hat{n}}\equiv\vec{\boldsymbol{x}}/|\vec{\boldsymbol{x}}|$ and $\boldsymbol{\hat{n}'}\equiv\vec{\boldsymbol{x}}'/|\vec{\boldsymbol{x}}'|$, where $\boldsymbol{\hat{n}'}=R\boldsymbol{\hat{n}}$. With respect to a fixed z-axis, $\boldsymbol{\hat{n}}$ points in a direction with polar angle θ and azimuthal angle ϕ, and $\boldsymbol{\hat{n}'}$ points in a direction with polar angle θ' and azimuthal angle ϕ'.

Since $U[R]$ is a unitary operator, we can write,

$$\psi(\vec{\boldsymbol{x}}) = \langle\vec{\boldsymbol{x}}\,|\,\psi\rangle = \langle\vec{\boldsymbol{x}}|\,U^\dagger[R]\,U[R]\,|\psi\rangle = \langle\vec{\boldsymbol{x}}'\,|\,\psi\rangle_R = \psi_R(\vec{\boldsymbol{x}}')\,,$$

after employing eqs. (2) and (3). That is,[1]

$$\psi_R(\vec{\boldsymbol{x}}') = \psi(\vec{\boldsymbol{x}}) = \psi(R^{-1}\vec{\boldsymbol{x}}')\,. \tag{4}$$

Consider the state vector $|\psi\rangle = |\ell\, m\rangle$, which is a simultaneous eigenstate of $\vec{\boldsymbol{L}}^{\boldsymbol{2}}$ and L_z with corresponding eigenvalues $\hbar^2\ell(\ell+1)$ and $\hbar m$, respectively. Then, using eq. (2),

$$\psi_R(\vec{\boldsymbol{x}}') = \langle\vec{\boldsymbol{x}}'\,|\,\psi\rangle_R = \langle\vec{\boldsymbol{x}}'|\,U[R]\,|\ell\, m\rangle = \sum_{\ell=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\langle\vec{\boldsymbol{x}}'\,|\,\ell'\, m'\rangle\langle\ell'\, m'|\,U[R]\,|\ell\, m\rangle\,. \tag{5}$$

Note that $D^{(\ell)}_{m'm}(R)\,\delta_{\ell\ell'}\equiv\langle\ell'\, m'|\,U[R]\,|\ell\, m\rangle$, since $[\vec{\boldsymbol{L}},L_i]=0$ implies that the matrix elements of $\vec{\boldsymbol{L}}$ (as well as any function of $\vec{\boldsymbol{L}}$) are diagonal in ℓ. In addition, $Y_{\ell'm'}(\boldsymbol{\hat{n}'})=\langle\vec{\boldsymbol{x}}'\,|\,\ell'\, m'\rangle$.[2] Hence, eq. (5) yields

$$\psi_R(\vec{\boldsymbol{x}}') = \sum_{m'=-\ell}^{\ell} Y_{\ell m'}(\boldsymbol{\hat{n}'})D^{(\ell)}_{m'm}[R]\,. \tag{6}$$

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[1]Since eq. (4) is true for any $\vec{\boldsymbol{x}}'$ (which can be treated as a dummy variable), we are free to drop the prime superscript and write $\psi_R(\vec{\boldsymbol{x}})=\psi(R^{-1}\vec{\boldsymbol{x}})$.

[2]Since θ' and ϕ' are the polar and azimuthal angles of the unit vector $\boldsymbol{\hat{n}'}$, we denote $Y_{\ell'm'}(\boldsymbol{\hat{n}'})\equiv Y_{\ell'm'}(\theta',\phi')$.

In light of eq. (4), $\psi_R(\vec{\boldsymbol{x}}') = \psi(\vec{\boldsymbol{x}}) = \langle \vec{\boldsymbol{x}} \,|\, \ell\, m\rangle = Y_{\ell m}(\hat{\boldsymbol{n}})$. Hence, eq. (6) yields,

$$\boxed{Y_{\ell m}(\hat{\boldsymbol{n}}) = \sum_{m'=-\ell}^{\ell} Y_{\ell m'}(\hat{\boldsymbol{n}}') D^{(\ell)}_{m'm}[R]\,, \qquad \text{where } \hat{\boldsymbol{n}}' = R\hat{\boldsymbol{n}}.} \tag{7}$$

Note that an equivalent method for deriving eq. (7) starts from the observation that if $\hat{\boldsymbol{n}}' = R\hat{\boldsymbol{n}}$, then $|\hat{\boldsymbol{n}}'\rangle = U[R]|\hat{\boldsymbol{n}}\rangle$. It then follows that

$$\langle \hat{\boldsymbol{n}}'| = \langle \hat{\boldsymbol{n}}|\, U^\dagger[R]\,. \tag{8}$$

Since $U[R]$ is a unitary operator, we can write,

$$\langle \hat{\boldsymbol{n}} \,|\, \ell\, m\rangle = \langle \hat{\boldsymbol{n}}|\, U^\dagger[R]\, U[R]\,|\ell\, m\rangle = \langle \hat{\boldsymbol{n}}'|\, U[R]\,|\ell\, m\rangle = \sum_{\ell=0}^{\infty}\sum_{m'=-\ell'}^{\ell'} \langle \hat{\boldsymbol{n}}' \,|\, \ell'\, m'\rangle\langle \ell'\, m'|\, U[R]\,|\ell\, m\rangle$$

$$= \sum_{\ell=0}^{\infty}\sum_{m'=-\ell'}^{\ell'} \langle \hat{\boldsymbol{n}}' \,|\, \ell'\, m'\rangle\, D^{(\ell)}_{m'm}[R]\,\delta_{\ell\ell'} = \sum_{m'=-\ell}^{\ell} \langle \hat{\boldsymbol{n}}' \,|\, \ell\, m'\rangle\, D^{(\ell)}_{m'm}[R]\,,$$

which is equivalent to eq. (7).

The z axis points along the unit vector $\hat{\boldsymbol{z}}$. Given the rotation R that appears in eq. (7), we define a unit vector $\hat{\boldsymbol{n}}'' \equiv R\hat{\boldsymbol{z}}$. If the rotation $R = R(\alpha,\beta,\gamma)$ is parameterized by its Euler angles, α, β, γ, then $R(\alpha,\beta,\gamma) = R(\alpha\hat{\boldsymbol{z}})R(\beta\hat{\boldsymbol{y}})R(\gamma\hat{\boldsymbol{z}})$, where the three rotations are applied from right to left. Note that the direction of $\hat{\boldsymbol{n}}''$ does not depend on γ, since $R(\gamma\hat{\boldsymbol{z}})\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$. Hence, $\hat{\boldsymbol{n}}'' = R(\alpha,\beta,\gamma)\hat{\boldsymbol{z}} = R(\alpha\hat{\boldsymbol{z}})R(\beta\hat{\boldsymbol{y}})\hat{\boldsymbol{z}}$. That is, the unit vector $\hat{\boldsymbol{n}}''$ has polar angle β and azimuthal angle α with respect to the z-axis.

To obtain the addition theorem for spherical harmonics, we set $m = 0$ in eq. (7) and use the identities,[3]

$$Y_{\ell 0}(\hat{\boldsymbol{n}}) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)\,, \tag{9}$$

$$D^{(\ell)}_{m'0}(\alpha,\beta,\gamma) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m'}(\hat{\boldsymbol{n}}'')\,, \tag{10}$$

where θ is the polar angle of $\hat{\boldsymbol{n}}$ with respect to the z-axis. It then immediately follows that

$$P_\ell(\cos\theta) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\boldsymbol{n}}')Y^*_{\ell m}(\hat{\boldsymbol{n}}'')\,. \tag{11}$$

after relabeling m' with m. Finally, we use $\hat{\boldsymbol{n}}' = R\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'' = R\hat{\boldsymbol{z}}$ to obtain,[4]

$$\hat{\boldsymbol{n}}'\boldsymbol{\cdot}\hat{\boldsymbol{n}}'' = (R_{ij}n_j)(R_{ik}z_k) = \delta_{jk}n_jz_k = \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{z}} = \cos\theta\,. \tag{12}$$

Hence, we have reproduced the addition theorem of spherical harmonics given in eq. (1).

[3] A proof of eq. (10) appears in Appendix C of the class handout entitled, *Clebsch-Gordan coefficients and the tensor spherical harmonics*.

[4] In eq. (12), there is an implicit sum over the repeated indices. Since the rotation matrix R is orthogonal, it follows that $R_{ij}R_{ik} = R^{\mathsf{T}}_{ji}R_{ik} = (R^{\mathsf{T}}R)_{jk} = \delta_{jk}$.