Examples of Generalized Functions

The following results are based on material from I.M. Gel'fand and G.E. Shilov, *Generalized Functions, Volume 1: Properties and Operations* (Academic Press, New York, NY, 1964), and D.S. Jones, *The Theory of Generalised Functions*, 2nd edition (Cambridge University Press, Cambridge, UK, 1982).

We begin by introducing the Heavyside step function,

$$\Theta(x) = \begin{cases} 1\,, & \text{for } x > 0\,, \\ 0\,, & \text{for } x < 0\,. \end{cases} \qquad \text{and} \qquad \Theta(-x) \equiv 1 - \Theta(x) = \begin{cases} 0\,, & \text{for } x > 0\,, \\ 1\,, & \text{for } x < 0\,. \end{cases}\,, \tag{1}$$

A specific value of $\Theta(x)$ at $x = 0$ is not specified. The delta function can then be defined as

$$\delta(x) = \frac{d}{dx}\Theta(x)\,.$$

The delta function $\delta(x)$ is not a function at all; instead it is a generalized function that only makes formal sense when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin, and then integrated over a range of x containing the origin. We shall also assume that $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ in order that integrals evaluated over the entire real line are convergent. It then follows that surface terms at $x = \pm\infty$, which arise when integrating by parts, vanish. The allowed set of functions $f(x)$ forms the space of test functions. In the case of the delta function, we have

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{\infty} \frac{d\Theta}{dx} f(x)\, dx = f(x)\Theta(x)\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Theta(x)\frac{df}{dx}\, dx = -f(x)\bigg|_0^{\infty} = f(0)\,.$$

By a similar computation, one can verify that the generalized function, $x\delta(x) = 0$.

The generalized function, $\mathrm{P}(1/x)$, can be defined by the following equation,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x}\, dx = \int_0^{\infty} \frac{f(x) - f(-x)}{x}\, dx\,, \tag{2}$$

where $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x| \to \infty$. In the Appendix, we show that eq. (2) is equivalent to the definition of the Cauchy principal value,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\, dx}{x} \equiv \lim_{\delta \to 0}\left\{\int_{-\infty}^{-\delta} \frac{f(x)\, dx}{x} + \int_{\delta}^{\infty} \frac{f(x)\, dx}{x}\right\}\,, \tag{3}$$

where $\delta > 0$. Gel'fand and Shilov define two additional functions, x_+^{-1} and x_-^{-1} via

$$x_+^{-1} = \Theta(x)\frac{1}{x} = \begin{cases} x^{-1}\,, & \text{for } x > 0\,, \\ 0\,, & \text{for } x < 0\,. \end{cases} \tag{4}$$

$$x_-^{-1} = -\Theta(-x)\frac{1}{x} = \begin{cases} 0\,, & \text{for } x > 0\,, \\ |x|^{-1}\,, & \text{for } x < 0\,. \end{cases} \tag{5}$$

As in the case of $\mathrm{P}(1/x)$, we would like to extend the definition of x_+^{-1} and x_-^{-1} such that they yield finite results when integrated against a test function over the real axis. The corresponding generalized functions are defined by,[1]

$$\frac{1}{x_+} = \lim_{\mu\to 0}\left\{\Theta(x)\frac{1}{x^{1-\mu}} - \frac{1}{\mu}\delta(x)\right\}, \tag{6}$$

$$\frac{1}{x_-} = \lim_{\mu\to 0}\left\{\Theta(-x)\frac{1}{(-x)^{1-\mu}} - \frac{1}{\mu}\delta(x)\right\}, \tag{7}$$

where μ is a real and positive infinitesimal. It then follows that

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{f(x)}{x_+}\,dx &= \lim_{\mu\to 0}\left\{\int_0^\infty \frac{f(x)}{x^{1-\mu}}\,dx - \frac{1}{\mu}\int_{-\infty}^{\infty}\delta(x)f(x)\,dx\right\}\\
&= \lim_{\mu\to 0}\left\{\int_0^1 \frac{f(x)}{x^{1-\mu}}\,dx - \frac{1}{\mu}f(0)\right\} + \int_1^\infty \frac{f(x)}{x}\,dx\\
&= \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \int_1^\infty \frac{f(x)}{x}\,dx + \lim_{\mu\to 0}\left\{f(0)\int_0^1 \frac{dx}{x^{1-\mu}}\,dx - \frac{1}{\mu}f(0)\right\}. \qquad (8)
\end{aligned}$$

In the last step above, we wrote $f(x) = f(x) - f(0) + f(0)$ and took the $\mu \to 0$ limit in the first term, which is allowed since the corresponding integral converges for any smooth function $f(x)$ that vanishes as $x \to \pm\infty$. Finally, the last two terms in eq. (8) cancel exactly, and we end up with a well-defined and finite result,

$$\int_{-\infty}^{\infty}\frac{f(x)}{x_+}\,dx = \int_0^\infty \frac{f(x)-f(0)\Theta(1-x)}{x}\,dx = \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \int_1^\infty \frac{f(x)}{x}\,dx\,. \tag{9}$$

A similar computation yields,

$$\int_{-\infty}^{\infty}\frac{f(x)}{x_-}\,dx = \int_0^\infty \frac{f(-x)-f(0)\Theta(1-x)}{x}\,dx = \int_0^1 \frac{f(-x)-f(0)}{x}\,dx + \int_1^\infty \frac{f(-x)}{x}\,dx\,. \tag{10}$$

It is easy to verify that the generalized functions $1/x_\pm$ satisfy,

$$\mathrm{P}\frac{1}{x} = \frac{1}{x_+} - \frac{1}{x_-}\,. \tag{11}$$

Note that eq. (11) is true for $x \neq 0$ in light of the definitions of $x_\pm$ given by eqs. (4) and (5). In addition, one can check that subtracting eq. (10) from eq. (9) yields eq. (2). One can also use eq. (11) to obtain another definition of $\mathrm{P}(1/x)$ by employing eqs. (6) and (7),

$$\mathrm{P}\frac{1}{x} = \lim_{\mu\to 0}\left\{\Theta(x)\frac{1}{x^{1-\mu}} - \Theta(-x)\frac{1}{(-x)^{1-\mu}}\right\}. \tag{12}$$

[1]The conditions, $xx_+^{-1} = \Theta(x)$ and $xx_-^{-1} = -\Theta(-x)$, do not yield unique generalized functions. For example, $x\big[x_+^{-1} + C\delta(x)\big] = \Theta(x)$ for any constant C. The definitions given in eqs. (6) and (7) are taken from D.S. Jones, *The Theory of Generalised Functions*, op. cit., and are motivated by the desire that the integrals given in eqs. (9) and (10) should be well-defined and finite. Some books write $\mathrm{Pf}(1/x_\pm)$ in eqs. (6) and (7) to distinguish these generalized functions from the ones defined in eqs. (4) and (5) [where Pf stands for *pseudofunction*], but following Gel'fand and Shilov (and Jones) we will not do so in these notes.

In light of $|x| = -x$ for $x < 0$, it follows that

$$\Theta(x)\frac{1}{x^{1-\mu}} - \Theta(-x)\frac{1}{(-x)^{1-\mu}} = \big[\Theta(x) - \Theta(-x)\big]\frac{1}{|x|^{1-\mu}} = \frac{\operatorname{sgn}(x)}{|x|^{1-\mu}}\,, \tag{13}$$

where $\operatorname{sgn}(x)$ is the sign function,

$$\operatorname{sgn}(x) = \Theta(x) - \Theta(-x) = \begin{cases} +1\,, & \text{for } x > 0, \\ -1\,, & \text{for } x < 0. \end{cases} \tag{14}$$

Hence, eq. (12) yields

$$\mathrm{P}\frac{1}{x} = \lim_{\mu\to 0}\frac{\operatorname{sgn}(x)}{|x|^{1-\mu}}\,. \tag{15}$$

It is a simple exercise to verify that multiplying the right hand side of eq. (15) by $f(x)$ and integrating over the real line yields eq. (2),

$$\begin{aligned} \mathrm{P}\int\frac{f(x)}{x}\,dx &= \lim_{\mu\to 0}\int_{-\infty}^{\infty}\frac{f(x)}{|x|^{1-\mu}}\operatorname{sgn}(x)\,dx = \lim_{\mu\to 0}\left\{\int_0^{\infty}\frac{f(x)}{x^{1-\mu}}\,dx - \int_{-\infty}^{0}\frac{f(x)}{x^{1-\mu}}\,dx\right\} \\ &= \lim_{\mu\to 0}\int_0^{\infty}\frac{f(x)-f(-x)}{x^{1-\mu}}\,dx = \int_0^{\infty}\frac{f(x)-f(-x)}{x}\,dx\,, \end{aligned} \tag{16}$$

after changing the integration variable $x \to -x$ in the second integral in the penultimate step above. In the final step, we can set $\mu = 0$ since the resulting integral is well-defined and finite, under the assumption that $f(x)$ is a smooth function that vanishes as $x \to \pm\infty$.

Next, we consider the function,

$$\frac{1}{|x|} = \big[\Theta(x) - \Theta(-x)\big]\frac{1}{x} = \begin{cases} x^{-1}\,, & \text{for } x > 0\,, \\ -x^{-1}\,, & \text{for } x < 0\,. \end{cases} \tag{17}$$

We again propose to extend the definition of $|x|^{-1}$ such that it yields a finite result when integrated against a test function over the real axis. The corresponding generalized function, denoted by $\mathrm{Pf}(1/|x|)$, is defined by,[2]

$$\mathrm{Pf}\frac{1}{|x|} = \lim_{\mu\to 0}\left\{\frac{1}{|x|^{1-\mu}} - \frac{2}{\mu}\delta(x)\right\}. \tag{18}$$

In order to see the relation between $\mathrm{Pf}(1/|x|)$ and the generalized functions $x_{\pm}^{-1}$, we employ $\Theta(x) + \Theta(-x) = 1$ [cf. eq. (1)] to obtain the identity,

$$\frac{1}{|x|^{1-\mu}} = \frac{1}{|x|^{1-\mu}}\big[\Theta(x) + \Theta(-x)\big] = \Theta(x)\frac{1}{x^{1-\mu}} + \Theta(-x)\frac{1}{(-x)^{1-\mu}}\,. \tag{19}$$

[2]A generalized function that coincides with $1/|x|$ for $x \neq 0$ not unique; see footnote 1. One particularly convenient choice is $\mathrm{Pf}(1/|x|)$, which in the notation of Gel'fand and Shilov (and Jones) is denoted as $1/|x|$. However, in these notes, we prefer to keep the Pf (pseudofunction) symbol in this case.

Hence, one can re-express eq. (18) as,

$$\mathrm{Pf}\frac{1}{|x|} = \lim_{\mu\to 0}\left\{\Theta(x)\frac{1}{x^{1-\mu}} + \Theta(-x)\frac{1}{(-x)^{1-\mu}} - \frac{2}{\mu}\delta(x)\right\}. \tag{20}$$

Comparing with eqs. (6) and (7), we see that

$$\mathrm{Pf}\frac{1}{|x|} = \frac{1}{x_+} + \frac{1}{x_-}\,. \tag{21}$$

Note that eq. (21) is clearly true for $x \neq 0$ in light of the initial definitions of $x_\pm^{-1}$ and $|x|^{-1}$ given by eqs. (4), (5) and (17).

It is convenient to rewrite eq. (10) by changing the integration variable $x \to -x$ and writing $|x| = -x$ for $x < 0$, which yields

$$\int_{-\infty}^{\infty}\frac{f(x)}{x_-}\,dx = -\int_{-\infty}^{0}\frac{f(x)-f(0)\Theta(1+x)}{x}\,dx = \int_{-\infty}^{-1}\frac{f(x)}{|x|}\,dx + \int_{-1}^{0}\frac{f(x)-f(0)}{|x|}\,dx\,. \tag{22}$$

Using eqs. (9) and (22), it then follows that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{Pf}\frac{1}{|x|}\,dx \equiv \int_{-\infty}^{-1}\frac{f(x)}{|x|}\,dx + \int_{-1}^{1}\frac{f(x)-f(0)}{|x|}\,dk + \int_{1}^{\infty}\frac{f(x)}{|x|}\,dx\,, \tag{23}$$

which is a well-defined and finite result, assuming $f(x)$ is smooth and vanishes as $x \to \pm\infty$.

A few more generalized functions are of interest. First, $\lim_{\varepsilon\to 0} 1/(x \pm i\varepsilon)$ is related to other known generalized functions via the Sokhotski-Plemelj formula,

$$\lim_{\varepsilon\to 0}\frac{1}{x \pm i\varepsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x)\,, \tag{24}$$

where $\varepsilon > 0$ is an infinitesimal real quantity. Perhaps less well known is the related formula,[3]

$$\lim_{\varepsilon\to 0}\frac{1}{x \pm i\varepsilon}\ln(x \pm i\varepsilon) = \mathrm{P}\frac{1}{x}\ln|x| - \mp i\pi\frac{1}{x_-} + \tfrac{1}{2}\pi^2\delta(x)\,, \tag{25}$$

In eq. (25), x is a real variable and the logarithm is given by its principal value,

$$\lim_{\varepsilon\to 0}\ln(x \pm i\varepsilon) = \ln|x| \pm i\pi\Theta(-x)\,. \tag{26}$$

To derive eq. (25), we first take the square of eq. (26),

$$\lim_{\varepsilon\to 0}\ln^2(x \pm i\varepsilon) = \ln^2|x| - \pi^2\Theta(-x) \pm 2\pi i\Theta(-x)\ln|x|\,, \tag{27}$$

where we have used the fact that $[\Theta(-x)]^2 = \Theta(-x)$. In order to derive eq. (25), we shall take the derivative of eq. (27) and divide by two. The derivative of the first term on the right hand side of eq. (27) is

$$\frac{d}{dx}\ln^2|x| = 2\ln|x|\frac{d}{dx}\ln|x| = 2\ln|x|\,\mathrm{P}\frac{1}{x}\,, \tag{28}$$

[3] As in the case of eq. (24), note that eq. (25) also yields two equations—one where the upper signs are employed and one with the lower signs are employed.

where we have used eq. (32) The derivative of the second term on the right hand side of eq. (27) is easily obtained after noting that

$$\frac{d}{dx}\Theta(-x) = -\frac{d}{dx}\Theta(x) = -\delta(x)\,.$$

The derivative of the third term on the right hand side of eq. (27) requires a little more effort. Under the assumption that the test function $f(x)$ is smooth and vanishes sufficiently fast as $x \to \pm\infty$, we evaluate (with the help of integration by parts),

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\frac{d}{dx}\bigl[\Theta(-x)\ln|x|\bigr]\,dx &= -\int_{-\infty}^{0}\frac{df}{dx}\ln(-x)\,dx = -\lim_{\delta\to 0}\int_{-\infty}^{-\delta}\frac{df}{dx}\ln(-x)\,dx \\
&= -\lim_{\delta\to 0}\left\{\ln(-x)f(x)\Big|_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx\right\} \\
&= -\lim_{\delta\to 0}\left\{\ln\delta\, f(\delta) - \int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx\right\} \\
&= -\lim_{\delta\to 0}\left\{f(0)\int_{-1}^{-\delta}\frac{dx}{x} - \int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx\right\} \\
&= \int_{-\infty}^{0}\frac{f(x)-f(0)\Theta(1+x)}{x}\,dx \\
&= -\int_{-\infty}^{\infty}\frac{f(x)}{x_-}\,dx\,. \qquad (29)
\end{aligned}$$

In the derivation above, we noted that $f(\delta) = f(0) + \delta f'(0) + \mathcal{O}(\delta^2)$ and made use of $\lim_{\delta\to 0}\delta\ln\delta = 0$. In the final step, we employed eq. (22). It then follows that

$$\frac{d}{dx}\bigl[\Theta(-x)\ln|x|\bigr] = -\frac{1}{x_-}\,. \qquad (30)$$

Although it is not required for this computation, it is instructive to note that a similar analysis (cf. Gel'fand and Shilov, *Generalized Functions*, op. cit., p. 25) yields,

$$\frac{d}{dx}\bigl[\Theta(x)\ln|x|\bigr] = \frac{1}{x_+}\,. \qquad (31)$$

We now add eqs. (30) and (31) and employ eq. (11) and $\Theta(x) + \Theta(-x) = 1$ to obtain,

$$\frac{d}{dx}\ln|x| = \mathrm{P}\frac{1}{x}\,. \qquad (32)$$

Likewise, if we subtract eq. (30) from eq. (31) and employ eqs. (14) and (21), we obtain,

$$\frac{d}{dx}\bigl[\mathrm{sgn}(x)\ln|x|\bigr] = \mathrm{Pf}\frac{1}{|x|}\,. \qquad (33)$$

In some books, eqs. (30)–(33) are employed as the definitions of the corresponding generalized functions, x_-^{-1}, x_+^{-1}, $\mathrm{P}(1/x)$, and $\mathrm{Pf}(1/|x|)$, respectively.

Finally, taking the the derivative of eq. (27) and dividing by two, we end up with

$$\lim_{\varepsilon\to 0}\frac{1}{x+i\varepsilon}\ln(x\pm i\varepsilon) = \mathrm{P}\frac{1}{x}\ln|x| + \tfrac{1}{2}\pi^2\delta(x) \mp \pi i\frac{1}{x_-}\,. \qquad (34)$$

Thus, eq. (25) has been established.[4]

In the class handout entitled, *The Sokhotski-Plemelj Formula*, the Fourier transform of $\ln|x|$ was recorded in eq. (26). However, this formula differs from the following one that is found in I.M. Gel'fand and G.E. Shilov, *Generalized Functions*, op. cit.,[5]

$$\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx = i\left\{\left[-\gamma+\tfrac{1}{2}i\pi-\ln(k+i\varepsilon)\right]\frac{1}{k+i\varepsilon}+\left[\gamma+\tfrac{1}{2}i\pi+\ln(k+i\varepsilon)\right]\frac{1}{k-i\varepsilon}\right\}, \qquad (35)$$

where γ is the Euler-Mascheroni constant. In eq. (35) and in what follows, we shall always assume the $\varepsilon\to 0$ limit without explicitly indicating the limit symbol. We can simplify eq. (35) as follows. First, we write

$$\begin{aligned}\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx = -i\gamma\left(\frac{1}{k+i\varepsilon}-\frac{1}{k-i\varepsilon}\right)-\frac{\pi}{2}\left(\frac{1}{k+i\varepsilon}+\frac{1}{k-i\varepsilon}\right)\\ -i\left\{\frac{1}{k+i\varepsilon}\ln(k+i\varepsilon)-\frac{1}{k-i\varepsilon}\ln(k-i\varepsilon)\right\}.\end{aligned} \qquad (36)$$

Using eqs. (24) and (25), we end up with

$$\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx = -2\pi\gamma\delta(k)-\pi\left(\mathrm{P}\frac{1}{k}+2\frac{1}{k_-}\right). \qquad (37)$$

Finally, employing eqs. (11) and (21) yields

$$\mathrm{P}\frac{1}{k}+2\frac{1}{k_-}=\mathrm{Pf}\frac{1}{|k|}\,. \qquad (38)$$

Inserting eq. (38) into eq. (37) yields,

$$\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx = -\pi\left[\mathrm{Pf}\frac{1}{|k|}+2\gamma\delta(k)\right], \qquad (39)$$

which reproduces eq. (26) of the class handout entitled, *The Sokhotski-Plemelj Formula*.

[4]Note that one cannot derive eq. (25) by simply multiplying eqs. (24) and (26). For an alternative proof of eq. (25), see pp. 97–98 of I.M. Gel'fand and G.E. Shilov, *Generalized Functions*, op. cit. pp. 96–98.

[5]Eq. (35) is also derived on pp. 160–161 of Ram P. Kanwal, *Generalized Functions: Theory and Applications*, 3rd edition (Birkhäuser, Boston, 2004). However, one must correct a typographical error in his eq. (6.4.57) where $-i(u-i0)^{-1}$ should be replaced by $+i(u-i0)^{-1}$. Kanwal also derives our eq. (39) above on pp. 153–154, although again one must correct a typographical error in his eq. (6.4.33d) where $1/u$ should be replaced by $1/|u|$

Appendix: The Cauchy Principle Value

In this Appendix, we verify that the definitions of the Cauchy principal value given in eqs. (2) and (3) are equivalent. We begin with eq. (2),

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx=\int_0^{\infty}\frac{f(x)-f(-x)}{x}\,dx\,. \tag{40}$$

For any positive real number δ, the following equation is an identity,

$$\int_0^{\infty}\frac{f(x)-f(-x)}{x}\,dx=\int_0^{\delta}\frac{f(x)-f(-x)}{x}\,dx+\int_{\delta}^{\infty}\frac{f(x)}{x}\,dx+\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx\,, \tag{41}$$

where the last integral on the right hand side of eq. (41) has been obtained after changing the integration variable, $x\to -x$. Since δ is an arbitrary positive number, we can take the limit as $\delta\to 0$ (from the positive side), which yields

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x}=\lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\,dx}{x}+\int_{\delta}^{\infty}\frac{f(x)\,dx}{x}+\int_0^{\delta}\frac{f(x)-f(-x)}{x}\,dx\right\}\,. \tag{42}$$

By assumption, the test function $f(x)$ is smooth and vanishes as $x\to\pm\infty$. Hence, one can Taylor expand $f(x)$ around the origin to obtain,

$$f(x)=f(0)+xf'(0)+\mathcal{O}(x^2)\,.$$

It follows that for $\delta\ll 1$,

$$\int_0^{\delta}\frac{f(x)-f(-x)}{x}\,dx=2f'(0)\delta+\mathcal{O}(\delta^3)\,, \tag{43}$$

which vanishes as $\delta\to 0$. Hence, eq. (42) yields,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x}=\lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\,dx}{x}+\int_{\delta}^{\infty}\frac{f(x)\,dx}{x}\right\}\,, \tag{44}$$

which establishes the equivalence of eqs. (2) and (3).