The Sokhotski-Plemelj Formula

1. The Sokhotski-Plemelj formula

The Sokhotski-Plemelj formula is a relation between the following generalized functions (also called *distributions*),

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm i\epsilon}=\mathrm{P}\frac{1}{x}\mp i\pi\delta(x)\,, \tag{1}$$

where $\epsilon>0$ is an infinitesimal real quantity. This identity formally makes sense only when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin, and then integrated over a range of x containing the origin. We shall also assume that $f(x)\to 0$ sufficiently fast as $x\to\pm\infty$ in order that integrals evaluated over the entire real line are convergent. Moreover, all surface terms at $\pm\infty$ that arise when integrating by parts are assumed to vanish.

To establish eq. (1), we shall prove that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x\pm i\epsilon}=\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x}\mp i\pi f(0)\,, \tag{2}$$

where the Cauchy principal value integral is defined as:

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x}\equiv\lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\,dx}{x}+\int_{\delta}^{\infty}\frac{f(x)\,dx}{x}\right\}\,, \tag{3}$$

assuming $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x|\to\infty$.

In these notes, I will provide three different derivations of eq. (2). The first derivation is a mathematically non-rigorous proof of eq. (2), which should at least provide some insight into the origin of this result. A more rigorous derivation starts with a contour integral in the complex plane,

$$\int_C\frac{f(z)\,dz}{z}\,.$$

By defining C appropriately, we will obtain two different expressions for this integral. Setting the two resulting expressions equal yields eq. (2) with the upper sign. Complex conjugating this result yields eq. (2) with the lower sign. Finally, an elegant third proof makes direct use of the theory of distributions. Finally, a useful check is to consider the Fourier transform of eq. (1), as discussed in Appendix A.

Note that eq. (1) can be generalized as follows,

$$\lim_{\varepsilon\to 0}\frac{1}{x-x_0\pm i\varepsilon}=\mathrm{P}\frac{1}{x-x_0}\mp i\pi\delta(x-x_0)\,, \tag{4}$$

where

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0}\equiv\lim_{\delta\to 0}\left\{\int_{-\infty}^{x_0-\delta}\frac{f(x)\,dx}{x-x_0}+\int_{x_0+\delta}^{\infty}\frac{f(x)\,dx}{x-x_0}\right\}\,. \tag{5}$$

The corresponding generalization of eq. (2) is straightforward. Note that eq. (2) and its generalization involve integration along the real axis. These ideas generalize further to the so-called Cauchy type integrals as shown in Appendix B, and yield the Plemelj formulae of complex variables theory.

2. A non-rigorous derivation of the Sokhotski-Plemelj formula

We begin with the identity,

$$\frac{1}{x \pm i\epsilon} = \frac{x \mp i\epsilon}{x^2+\epsilon^2}\,,$$

where ϵ is a positive infinitesimal quantity. Thus, for any smooth function that is non-singular in a neighborhood of the origin,

$$\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x \pm i\epsilon} = \int_{-\infty}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} \mp i\epsilon \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x^2+\epsilon^2}\,. \tag{6}$$

The first integral on the right had side of eq. (6),

$$\int_{-\infty}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} = \int_{-\infty}^{-\delta} \frac{x f(x)\,dx}{x^2+\epsilon^2} + \int_{\delta}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} + \int_{-\delta}^{\delta} \frac{x f(x)\,dx}{x^2+\epsilon^2}\,. \tag{7}$$

In the first two integrals on the right hand side of eq. (7), it is safe to take the limit $\epsilon \to 0$. In the third integral on the right hand side of eq. (7), if δ is small enough, then we can approximate $f(x) \simeq f(0)$ for values of $|x| < \delta$. Hence, eq. (7) yields,

$$\int_{-\infty}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} = \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta} \frac{f(x)\,dx}{x} + \int_{\delta}^{\infty} \frac{f(x)\,dx}{x}\right\} + f(0)\int_{-\delta}^{\delta} \frac{x\,dx}{x^2+\epsilon^2}\,. \tag{8}$$

However,

$$\int_{-\delta}^{\delta} \frac{x\,dx}{x^2+\epsilon^2} = 0\,,$$

since the integrand is an odd function of x that is being integrated symmetrically about the origin, and

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} \equiv \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta} \frac{f(x)\,dx}{x} + \int_{\delta}^{\infty} \frac{f(x)\,dx}{x}\right\}\,,$$

defines the principal value integral. Hence, eq. (8) yields

$$\int_{-\infty}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x}\,. \tag{9}$$

Next, we consider the second integral on the right hand side of eq. (6). Since ϵ is an infinitesimal quantity, the only significant contribution from

$$\epsilon\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x^2+\epsilon^2}$$

can come from the integration region where $x \simeq 0$, where the integrand behaves like ϵ^{-2}. Thus, we can again approximate $f(x) \simeq f(0)$, in which case we obtain

$$\epsilon \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x^2+\epsilon^2} \simeq \epsilon f(0) \int_{-\infty}^{\infty} \frac{dx}{x^2+\epsilon^2} = \pi f(0)\,, \tag{10}$$

where we have made use of

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+\epsilon^2} = \frac{1}{\epsilon} \tan^{-1}(x/\epsilon)\Big|_{-\infty}^{\infty} = \frac{\pi}{\epsilon}\,.$$

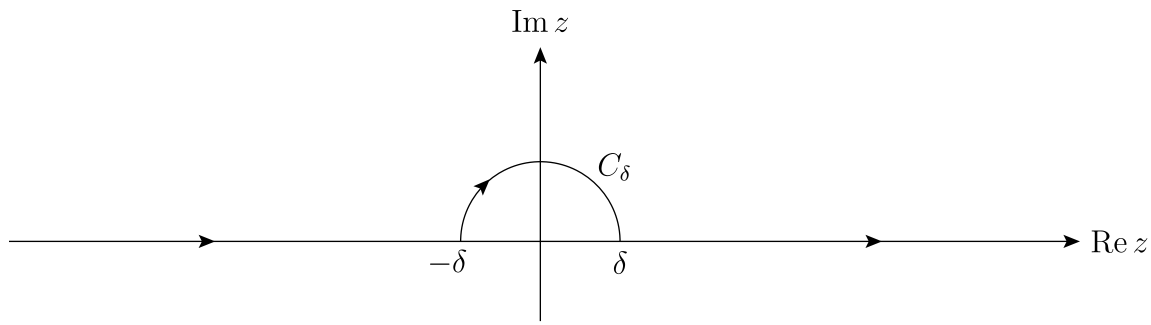
Using the results of eqs. (9) and (10), we see that eq. (6) yields,

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x \pm i\epsilon} = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} \mp i\pi f(0)\,, \tag{11}$$

which establishes eq. (2).

2. A more rigorous derivation of the Sokhotski-Plemelj formula

We consider the following path of integration in the complex plane, denoted by C, shown below.



That is, C is the contour along the real axis from $-\infty$ to $-\delta$, followed by a semicircular path C_δ (of radius δ), followed by the contour along the real axis from δ to ∞. The infinitesimal quantity δ is assumed to be positive. Then

$$\int_C \frac{f(x)}{x}\,dx = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} + \int_{C_\delta} \frac{f(x)}{x}\,dx\,, \tag{12}$$

where the principal value integral is defined in eq. (3). In the limit of $\delta \to 0$, we can approximate $f(x) \simeq f(0)$ in the last integral on the right hand side of eq. (12). Noting that the contour C_δ can be parameterized as $x = \delta e^{i\theta}$ for $0 \leq \theta \leq \pi$, we end up with

$$\lim_{\delta\to 0} \int_{C_\delta} \frac{f(x)}{x}\,dx = f(0) \lim_{\delta\to 0} \int_{\pi}^{0} \frac{i\delta e^{i\theta}}{\delta e^{i\theta}}\,d\theta = -i\pi f(0)\,.$$

Hence,

$$\int_C \frac{f(x)}{x}\,dx = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} - i\pi f(0)\,. \tag{13}$$

We can also evaluate the left hand side of eq. (13) by deforming the contour C to a contour C' that consists of a straight line that runs from $-\infty + i\varepsilon$ to $\infty + i\varepsilon$, where ε is a positive infinitesimal (of the same order of magnitude as δ). Assuming that $f(x)$ has no singularities in an infinitesimal neighborhood around the real axis, we are free to deform the contour C into C' without changing the value of the integral. It follows that

$$\int_C \frac{f(x)}{x}\, dx = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{f(x)}{x}\, dx = \int_{-\infty}^{\infty} \frac{f(y+i\varepsilon)}{y+i\varepsilon}\, dy\,, \tag{14}$$

where in the last step we have made a change of the integration variable.

Since ε is infinitesimal, we can approximate $f(y+i\varepsilon) \simeq f(y)$.[1] Thus, after relabeling the integration variable y as x, eq. (14) yields

$$\int_C \frac{f(x)}{x}\, dx = \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\, dx\,. \tag{15}$$

Inserting this result back into eq. (13) yields

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\, dx = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x} - i\pi f(0)\,. \tag{16}$$

Eq. (16) is also valid if $f(x)$ is replaced by $f^*(x)$. We can then take the complex conjugate of the resulting equation. The end result is[2]

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x\pm i\varepsilon}\, dx = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x} \mp i\pi f(0)\,,$$

in agreement with eq. (11).

3. An elegant derivation of the Sokhotski-Plemelj formula

Starting from the definition of the Cauchy principal value given in eq. (3), we integrate by parts to obtain

$$\int_{-\infty}^{-\delta} \frac{f(x)}{x}\, dx = f(x)\ln|x|\Big|_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} f'(x)\ln|x|\, dx = f(-\epsilon)\ln\epsilon - \int_{-\infty}^{-\delta} f'(x)\ln|x|\, dx\,,$$

$$\int_{\delta}^{\infty} \frac{f(x)}{x}\, dx = f(x)\ln|x|\Big|_{-\infty}^{-\delta} - \int_{\delta}^{\infty} f'(x)\ln|x|\, dx = -f(\epsilon)\ln\epsilon - \int_{\delta}^{\infty} f'(x)\ln|x|\, dx\,,$$

[1]More precisely, we can expand $f(y+i\varepsilon)$ in a Taylor series about $\varepsilon = 0$ to obtain $f(y+i\varepsilon) = f(y) + \mathcal{O}(\varepsilon)$. At the end of the calculation, we may take $\varepsilon \to 0$, in which case the $\mathcal{O}(\varepsilon)$ terms vanish.

[2]Alternatively, we can repeat the above derivation where the contour C_δ is replaced by a semicircle of radius δ in the lower half complex plane, which yields eq. (13) with $-i$ replaced by i. Finally, after deforming the contour of integration to a new contour that consists of a straight line that runs from $-\infty - i\varepsilon$ to $\infty - i\varepsilon$, one obtains eq. (15) with i replaced by $-i$.

where $f'(x) \equiv df/dx$ and we have assumed that $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ so that the surface terms at $\pm\infty$ vanish. Hence,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} = \lim_{\delta\to 0}\left\{\left[f(-\delta) - f(\delta)\right]\ln\delta - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx\right\}. \tag{17}$$

Since $f(x)$ is differentiable and well behaved, we can define

$$g(x) \equiv \int_0^1 f'(xt)dt = \frac{f(x) - f(0)}{x},$$

which implies that $g(x)$ is smooth and non-singular and

$$f(x) = f(0) + xg(x). \tag{18}$$

Inserting eq. (18) back into eq. (17) then yields

$$\begin{aligned}\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} &= \lim_{\delta\to 0}\left\{\left[-2g(x)\delta\ln\delta - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx\right\}\right. \\ &= -\int_{-\infty}^{\infty} f'(x)\ln|x|\,dx.\end{aligned}$$

Note that $\ln|x|$ is integrable at $x = 0$, so that the last integral is well-defined. Finally, we integrate by parts and drop the surface terms at $\pm\infty$ (under the usual assumption that $f'(x) \to 0$ sufficiently fast as $x \to \infty$). The end result is

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} = \int_{-\infty}^{\infty} f(x)\frac{d}{dx}\ln|x|\,dx.$$

That is, we have derived the generalized function identity,

$$\frac{d}{dx}\ln|x| = \mathrm{P}\frac{1}{x}. \tag{19}$$

We can employ eq. (19) to provide a very elegant derivation of eq. (1). We begin with the definition of the principal value of the complex logarithm,

$$\mathrm{Ln}\ z = \ln|z| + i\arg\ z,$$

where $\arg z$ is the principal value of the argument (or phase) of the complex number z, with the convention that $-\pi < \arg z \leq \pi$. In particular, for real x and a positive infinitesimal ϵ,

$$\lim_{\epsilon\to 0}\mathrm{Ln}(x \pm i\epsilon) = \ln|x| \pm i\pi\Theta(-x), \tag{20}$$

where $\Theta(x)$ is the Heaviside step function. Differentiating eq. (20) with respect to x immediately yields the Sokhotski-Plemelj formula,[3]

$$\lim_{\varepsilon\to 0}\frac{1}{x \pm i\epsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x), \tag{21}$$

where we have used eq. (19) and

$$\frac{d}{dx}\Theta(-x) = -\frac{d}{dx}\Theta(x) = -\delta(x).$$

[3]The derivative of the complex logarithm is $d\,\mathrm{Ln}\ z/dz = 1/z$ for $z \neq 0$.

Appendix A: Fourier transforms of distributions

Eqs. (19) and (21), which we repeat below

$$\frac{d}{dx}\ln|x| = \mathrm{P}\frac{1}{x}\,, \tag{22}$$

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon} = \mathrm{P}\frac{1}{x}\mp i\pi\delta(x)\,, \tag{23}$$

are only meaningful when multiplied by a test function $f(x)$ and integrated over a region of the real line that includes the point $x = 0$. In the theory of *tempered distributions*, test functions must be infinitely differentiable and vanish at $\pm\infty$ faster than any inverse power of x. Clearly, e^{ikx} does not satisfy this requirement for a test function. Nevertheless, one can define Fourier transforms of tempered distributions by using the well known property of the Fourier transform,

$$\int_{-\infty}^{\infty}\widetilde{f}(k)g(k)\,dk = \int_{-\infty}^{\infty} f(k)\widetilde{g}(k)\,dx\,, \tag{24}$$

where

$$\widetilde{f}(k) \equiv \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx\,.$$

If $f(x)$ is a tempered distribution and $g(x)$ is a test function, then if follows that $\widetilde{g}(x)$ exists and is well defined. The Fourier transform of $f(x)$, denoted by $\widetilde{g}(k)$, is *defined* via eq. (24).

One can now check the validity of eqs. (22) and (23) by computing their Fourier transforms. To compute the Fourier transform of eq. (22), we make use of the property of Fourier transforms that

$$\int_{-\infty}^{\infty}\frac{df(x)}{dx}e^{ikx}\,dx = -ik\widetilde{f}(k)\,.$$

Hence,

$$\int_{-\infty}^{\infty}\frac{d}{dx}\ln|x|\,e^{ikx}\,dx = -ik\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx\,. \tag{25}$$

The calculation of the right-hand side of eq. (25) is rather involved, since it only exists in the sense of distributions. One can show that[4]

$$\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx = -\pi\left[\mathrm{Pf}\frac{1}{|k|}+2\gamma\delta(k)\right]\,, \tag{26}$$

where γ is the Euler-Mascheroni constant, and the distribution $\mathrm{Pf}(1/|k|)$ is defined as

$$\int_{-\infty}^{\infty} f(k)\,\mathrm{Pf}\frac{1}{|k|}\,dk \equiv \int_{-\infty}^{-1}\frac{f(k)}{|k|}\,dk + \int_{-1}^{1}\frac{f(k)-f(0)}{|k|}\,dk + \int_{1}^{\infty}\frac{f(k)}{|k|}\,dk\,, \tag{27}$$

for any valid test function $f(k)$.

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[4]For example, see Ram P. Kanwal, *Generalized Functions: Theory and Applications*, Third edition (Birkhäuser, Boston, 2004) pp. 153–154 and pp. 160–161. There are two typographical errors on these pages. In eq. (6.4.33d), $1/u$ should be $1/|u|$ and in the last term in eq. (6.4.57), $-i(u-i0)^{-1}$ should be $+i(u-i0)^{-1}$. Eq. (26) is a consequence of the corrected eq. (6.4.57).

Inserting the result of eq. (26) into eq. (25) and using $k\delta(k)=0$ and[5]

$$k\left(\mathrm{Pf}\frac{1}{|k|}\right)=\frac{k}{|k|}=\mathrm{sgn}(k)\,, \tag{28}$$

the end result is given by,

$$\int_{-\infty}^{\infty}\frac{d}{dx}\ln|x|\,e^{ikx}\,dx=i\pi\,\mathrm{sgn}(k)\,. \tag{29}$$

Next, we consider

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x}\,dx=\mathrm{P}\int_{-\infty}^{\infty}\frac{\cos(kx)}{x}\,dx+i\mathrm{P}\int_{-\infty}^{\infty}\frac{\sin(kx)}{x}\,dx\,. \tag{30}$$

Since $\cos(kx)/x$ is an odd function of x (i.e., it changes sign under $x\to -x$), it immediately follows from the definition of the Cauchy principle value that

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{\cos(kx)}{x}\,dx=0\,. \tag{31}$$

Next, we observe that $\lim_{x\to 0}\sin(kx)/x=k$; that is, $\sin(kx)/x$ is regular at $x=0$. Thus,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{\sin(kx)}{x}\,dx=\int_{-\infty}^{\infty}\frac{\sin(kx)}{x}\,dx=\mathrm{sgn}(k)\int_{-\infty}^{\infty}\frac{\sin y}{y}\,dy=\pi\,\mathrm{sgn}(k)\,. \tag{32}$$

Note that the P symbol has no effect on the integral given by eq. (32), since the integrand is regular at $x=0$. The factor of $\mathrm{sgn}(k)$ arises after changing the integration variable, $y=kx$. When $k<0$, the integration limits must be reversed, which then leads to the extra minus sign. Inserting eqs. (31) and (32) into eq. (30) then yields,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x}\,dx=i\pi\,\mathrm{sgn}(k)\,. \tag{33}$$

In light of eqs. (29) and (33), we have verified that the Fourier transform of eq. (22) is satisfied.

Likewise, we can verify that the Fourier transform of eq. (23) is satisfied. The following result is required,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x\pm i\varepsilon}\,dx=\mp 2\pi i\Theta(\mp k)\,, \tag{34}$$

which was derived in Solution Set 1 in Physics 215. Then, using eq. (33) and employing the two identities, $\mathrm{sgn}(k)=\Theta(k)-\Theta(-k)$ and $1=\Theta(k)+\Theta(-k)$, it follows that the Fourier transform of eq. (23) is

$$\mp 2\pi i\Theta(\mp k)=i\pi\big[\Theta(k)-\Theta(-k)\big]\mp i\pi\big[\Theta(k)+\Theta(-k)\big]\,. \tag{35}$$

It is a simple matter to check that eq. (35) is satisfied for either choice of sign.

Since the Fourier transform of a tempered distribution and its inverse Fourier transform are unique, one can conclude that if the Fourier transforms of eqs. (22) and (23) are satisfied, then eqs. (22) and (23) are valid identities. Thus, the Fourier transform technique exhibited this Appendix provides a fourth independent derivation of the Sokhotski-Plemelj formula.

[5] When we multiply $\mathrm{Pf}(1/|k|)$ by k, the singularity at $k=0$ is canceled and the prescription indicated by eq. (27) is no longer required. Noting that $k/|k|$ is equal to sign of k for $k\neq 0$, we end up with eq. (28).

Appendix B: The Plemelj Formulae of Complex Variables Theory

The Sokhotski-Plemelj formula derived in these notes is in fact a special case of a more general result of the theory of complex variables, which is often referred to as the Plemelj formulae (and less often as the Sokhotski formulae). In this Appendix, I shall simply state the relevant results, with references for further details.

Consider the Cauchy type integral,

$$F(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z}\, dt\,, \tag{36}$$

where z and t are complex variables, C is a smooth curve (which may be an open or a closed contour) and $f(t)$ is a function defined on C that satisfies,

$$|f(t_2) - f(t_1) < A|t_2 - t_1|^\lambda\,, \tag{37}$$

for any two points t_1 and t_2 located on the contour C, where A and λ are positive numbers. Eq. (37) is called the Hölder condition.[6]

For values of $z \notin C$, $F(z)$ is an analytic function. For values of z on the contour C, the value of $F(z)$ is not well defined due to the singularity encountered in the integration along C. Nevertheless, $F(z)$ does have unique value that depends on how z approaches C. Indeed, there are two different possible boundary values of $F(z)$ depending on whether the contour C is approached from the left or right. We therefore introduce $F_+(z)$ and $F_-(z)$ where the former is the limit as z approaches C from the left and the latter is the limit as z approaches from the right. Here, left and right are defined with respect to the positive direction of the contour C.[7] The explicit results for $F_\pm(z)$ are given by the Plemelj formulae,

$$F_\pm(z) = \pm\tfrac{1}{2} f(z) + \frac{1}{2\pi i}\,\mathrm{P}\int_C \frac{f(t)}{t-z}\, dt\,, \quad \text{for } z \in \widetilde{C}\,, \tag{38}$$

and $\widetilde{C}$ consists of all points of C excluding its endpoints (in the case of a closed contour, there are no endpoints to exclude).

In eq. (38), we employ the the Cauchy principal value prescription to treat the singularity in the integrand. In this context, the principal value is a generalization of eq. (3),

$$\mathrm{P}\int_C \frac{f(t)}{t-z}\, dt = \lim_{\delta\to 0} \int_{C-C_\delta} \frac{f(t)}{t-z}\, dt\,, \tag{39}$$

where the contour C_δ consists of the part of C with length 2δ centered symmetrically around z, and $C - C_\delta$ is the contour C with the part C_δ removed.

If $f(t)$ is analytic on C, then the proof of eq. (38) is a straightforward generalization of the proof given in Section 2. However, the Plemelj formulae are more general and apply

[6]If $\lambda > 1$, then it follows that the derivative $f'(t)$ must vanish on C, in which case $f(t)$ is a constant. Thus, one typically assumes that $0 < \lambda \leq 1$.

[7]For example, for a closed counterclockwise contour C, $F_+(z)$ is given by the limit of $F(z)$ as z approaches C from the interior of the region bounded by C and $F_-(z)$ is given by the limit of $F(z)$ as z approaches C from the exterior of the region bounded by C.

to any function $f(t)$ that satisfies the Hölder condition on the contour C. In this case, the derivation of eq. (38) is more complicated. We have also sidestepped the case where z in eq. (38) is one of the endpoints of C (which is relevant if the contour C is open). The reader is referred to the references below for further details.

One can recast eq. (38) into another form that commonly appears in the literature,

$$F_+(z) - F_-(z) = f(z)\,, \tag{40}$$

$$F_+(z) + F_-(z) = \frac{1}{\pi i}\,\mathrm{P}\int_C \frac{f(t)}{t-z}\,dt\,, \tag{41}$$

for values of z located on all points of the contour C not coinciding with its endpoints. In particular, eq. (40) indicates that the function $F(z)$ defined in eq. (36), which is analytic for all complex values of $z \notin C$, has a discontinuous jump as z crosses the contour C. Moreover, the average of the two boundary values of $F(z)$ on C is given by eq. (36), where the singularity of the integrand is treated by the Cauchy principal value prescription.

Using the Plemelj formulae of complex variables theory, one can recover the results of Section 1 as follows. If C is a contour that runs along the real axis in the positive direction, then eq. (36) yields the boundary values, $F_\pm(z)$, of $F(z)$ as z approaches the real axis from above (i.e., from the left) or below (i.e., from the right), respectively,

$$F_\pm(z) = \lim_{\epsilon\to 0} F(z \pm i\epsilon) = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z\mp i\epsilon}\,dt\,, \tag{42}$$

where $\epsilon > 0$ is an infinitesimal real quantity. Hence, eqs. (38) and (42) yield

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f(t)}{t-z\mp i\epsilon}\,dt = \mathrm{P}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,dt \pm i\pi f(z)\,. \tag{43}$$

Eq. (43) is equivalent to the identity involving generalized functions given in eq. (4). As expected, setting $z = 0$ in eq. (43) reproduces eq. (2).

For further references on the material of Appendix B, see

1. Mark J. Ablowitz and Athanassios S. Fokas, *Complex Variables: Introduction and Applications* (Cambridge University Press, Cambridge, UK, 1997), Chapter 7.

2. A.I. Markushevich, *Theory of Functions of a Complex Variable*, Part I (AMS Chelsea Publishing, Providence, RI, 2005), Chapter 14.

3. F.D. Gakhov, *Boundary Value Problems* (Dover Publications, Inc., New York, NY, 1990), Chapters 1–5.

4. N.I. Muskhelishvili, *Singular Integral Equations: Boundary problems of functions theory and their applications to mathematical physics*, 2nd edition (Dover Publications, Inc., New York, NY, 2008), Chapter 2.

5. Alexander O. Gogolin, *Lectures on Complex Integration* (Springer International Publishing, Cham, Switzerland, 2014) Chapters 1.4 and 3.4.