DUE: THURSDAY, JANUARY 18, 2018

1. (a) Let ω_i $(i = 1, 2, ..., n)$ be the eigenvalues of the $n \times n$ matrix Ω.[1] Prove the following two results:

$$\det \Omega = \prod_{i=1}^{n} \omega_i \,, \qquad\qquad \mathrm{Tr}\, \Omega = \sum_{i=1}^{n} \omega_i \ .$$

Do not assume that Ω is a diagonalizable matrix.

HINT: See the class handout entitled, *The Characteristic Polynomial.*

(b) Show that:

$$\det \exp(\Omega) = \exp(\mathrm{Tr}\, \Omega) \ .$$

2. (a) Consider a differentiable function $f(x)$ such that $f(x_n) = 0$. Assume that the x_n are *isolated* zeros of the function $f(x)$. Under the assumption that $df/dx_n \neq 0$, where $df/dx_n \equiv (df/dx)_{x=x_n}$, show that:

$$\delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{|df/dx_n|} \,.$$

(b) Use this result of part (a) to obtain simplified expressions for $\delta(ax)$ and $\delta(x^2 - a^2)$, assuming that $a \neq 0$.

3. The δ–function may be defined as:

$$\delta(x - x') = \lim_{\epsilon \to 0} \ \frac{1}{\sqrt{\pi \epsilon^2}} \exp\left(-\frac{(x - x')^2}{\epsilon^2}\right) \ .$$

(a) Assuming that a is a real positive constant, verify that:

$$\delta(x - x') = \lim_{a \to 0} \ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik(x-x') - ak^2} \, dk \ .$$

(b) Let $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{ikx} \, dk$. Show that:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_{-\infty}^{+\infty} |a(k)|^2 \, dk \ .$$

(c) Interpret the result of part (b) in terms of Dirac bras and kets.

[1] Note that if there are degenerate eigenvalues, then they must be repeated according to their degeneracy in the list of eigenvalues ω_i.

4. The step function is defined as:

$$\Theta(k) = \begin{cases} 1\,, & \text{if } k > 0\,, \\ 0\,, & \text{if } k < 0\,. \end{cases}$$

(a) Using contour integration in the complex plane and the residue theorem, derive the following integral representation for $\Theta(k)$:

$$\Theta(k) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dx\, \frac{e^{ikx}}{x - i\varepsilon}\,,$$

where ε is a positive infinitesimal real number.

(b) Prove that:

$$\delta(k) = \frac{d}{dk}\Theta(k)\,.$$

(c) Consider the following important result in the theory of distributions:

$$\lim_{\varepsilon\to 0} \frac{1}{x \pm i\varepsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x)\,, \tag{1}$$

where ϵ is a positive infinitesimal and P denotes the Cauchy principal value. Formally, eq. (1) only makes sense when multiplied by a well-behaved function $F(x)$ and integrated from $-\infty$ to $+\infty$. (Here, well-behaved means that the resulting integrals are convergent.) In this case, the Cauchy principal value is defined by

$$\mathrm{P}\int_{-\infty}^{+\infty} \frac{1}{x} F(x)\, dx \equiv \lim_{a\to 0} \left\{ \int_{-\infty}^{-a} \frac{F(x)}{x}\, dx + \int_{a}^{+\infty} \frac{F(x)}{x}\, dx \right\}.$$

Using the result of part (a), show that the Fourier transform of eq. (1) is satisfied. That is, eq. (1) is valid when multiplied by the function $F(x) = e^{ikx}$ and integrated from $-\infty$ to $+\infty$.

HINT: In evaluating the principal value integral, write $e^{ikx} \equiv \cos kx + i \sin kx$.

5. Define the operators X and K such that $X\,|f\rangle = |xf\rangle$ and $K\,|f\rangle = -i\,|df/dx\rangle$. Then $[X, K] = iI$, where I is the identity operator in an infinite dimensional space.

(a) It appears that by using $\mathrm{Tr}(XK) = \mathrm{Tr}(KX)$, we obtain $0 = \infty$. Resolve the paradox.

(b) Show that if the functions F and G can be expressed as power series in their arguments, then:

$$[X, G(K)] = i\frac{dG}{dK} \qquad \text{and} \qquad [K, F(X)] = -i\frac{dF}{dX}\,.$$

(c) Let $|x\rangle$ be an eigenstate of X with eigenvalue x. Prove that $\exp(-iKa)\,|x\rangle$ is an eigenstate of X. What is the corresponding eigenvalue?

HINT: Use part (b), with $G(K) \equiv \exp(-iKa)$ and consider the action on $|x\rangle$.

6. (a) Show that if A commutes with $[A,B]$ then:

$$e^A B e^{-A} = B + [A,B]\ .$$

HINT: Compute $e^{aA}Be^{-aA}$ as a power series in the variable a, and set $a=1$ at the end of the computation.

(b) Suppose that A and B are two non-commuting operators such that their commutator $[A,B]$ commutes with both A and B. Prove that

$$\exp A\ \exp B = \exp\left(A + B + \tfrac{1}{2}[A,B]\right)\ .$$

7. In Chapter 1, Sakurai and Napolitano introduce the spin-$\frac{1}{2}$ operators S_x, S_y and S_z.

(a) With respect to the basis, $\{|+\rangle\ ,\ |-\rangle\}$, determine the explicit matrix representation for S_x, S_y and S_z. Denote the corresponding matrices by $\frac{1}{2}\hbar\sigma_i$ where $i=1,2,3$ corresponds to x, y, z, respectively. The three 2×2 matrices, σ_1, σ_2 and σ_3, are called the *Pauli matrices*.

(b) It is often convenient to write $\vec{\boldsymbol{\sigma}} = (\sigma_1\,,\,\sigma_2\,,\sigma_3)$. That is, $\vec{\boldsymbol{\sigma}}$ is a three dimensional "vector" whose "components" are the three 2×2 Pauli matrices. Prove the following identities involving the Pauli matrices:

(i) $\sigma_i\sigma_j = \delta_{ij}\mathbb{1}_{2\times 2} + i\epsilon_{ijk}\sigma_k$.

(ii) $(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}) = (\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}})\,\mathbb{1}_{2\times 2} + i\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})$.

(iii) $\exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2) = \mathbb{1}_{2\times 2}\cos\theta/2 - i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)$.

where $\mathbb{1}_{2\times 2}$ is the 2×2 identity matrix, and $\hat{\boldsymbol{n}}$ is a unit vector. Note that in (i) above, there is an implicit sum over the repeated index $k=1,2,3$.

8. Consider an arbitrary real 3×3 antisymmetric matrix A, whose matrix elements satisfy $A_{ij} = -A_{ji}$.

(a) Show that the matrix elements of A can be written in the form, $A_{ij} = \epsilon_{ijk}a_k$, where the a_k are real numbers, and there is an implicit sum over the repeated index $k=1,2,3$.

(b) Evaluate $\exp(A)$.