DUE: THURSDAY, FEBRUARY 1, 2018

1. At time $t = 0$, the wave function of a free particle moving in a one-dimension is given by,

$$\psi(x,0) = N \int_{-\infty}^{+\infty} e^{-|k|/k_0} e^{ikx}\, dk \ ,$$

where N and k_0 are real positive constants.

(a) What is the probability that a measurement of the momentum performed at time $t = 0$ will yield a result between $-p_1$ and p_1? How does the probability change if the measurement is performed instead at time t? Explain your result.

(b) What is the form of the (position space) wave packet at time $t = 0$? Calculate the product $\Delta X \Delta P$ at time $t = 0$. Describe qualitatively the subsequent evolution of the wave packet.

2. (a) Consider a quantum mechanical ensemble characterized by a density matrix ρ. Suppose that the system is governed by a Hamiltonian H (which may be time dependent). Show that the time evolution of ρ (in the Schrödinger picture) is given by:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}\,[H, \rho]\,.$$

(b) Let $U(t, t_0)$ be the time evolution operator. Find a general expression for $\rho(t)$ in terms of $\rho(t_0)$ and $U(t, t_0)$.

(c) Prove that $\operatorname{Tr} \rho^2$ is time-independent. Hence, show that a pure state cannot evolve into a mixed state.

3. A particle in one dimension is subject to a potential energy $V(x) = -fx$, where $f > 0$.

(a) Write Ehrenfest's theorem for the mean values of the position $\langle X\rangle$ and the momentum $\langle P\rangle$ of the particle. Integrate these equations and compare with the classical motion.

(b) Show that the root-mean-square deviation ΔP does not vary with time.

(c) Write the Schrödinger equation in the p-representation and deduce a relation between $\frac{\partial}{\partial t}|\langle p|\psi(t)\rangle|^2$ and $\frac{\partial}{\partial p}|\langle p|\psi(t)\rangle|^2$. Solve the equation thus obtained and give a physical interpretation.

(d) Write the Schrödinger equation in the x-representation. What are the energy eigenfunctions? (This will take a little research on your part in the area of special functions.)

(e) Is the energy spectrum continuous or discrete? What is the behavior of the energy eigenfunctions as $|x| \to \infty$? If the potential were replaced by $V(x) = f|x|$, how would your answer change?

4. Consider a particle, whose Hamiltonian is given by $H = \vec{\boldsymbol{P}}^2/(2m) + V(\vec{\boldsymbol{X}})$, in three dimensions. By calculating the commutator, $[\vec{\boldsymbol{X}}\cdot\vec{\boldsymbol{P}}\,,\,H]$, derive the quantum Virial Theorem,

$$\frac{d}{dt}\left\langle \vec{\boldsymbol{X}}\cdot\vec{\boldsymbol{P}}\right\rangle = \left\langle \frac{\vec{\boldsymbol{P}}^2}{m}\right\rangle - \left\langle \vec{\boldsymbol{X}}\cdot\vec{\boldsymbol{\nabla}}V\right\rangle . \tag{1}$$

To identify eq. (1) with the quantum mechanical analog of the Virial Theorem, it is essential that the left-hand side of eq. (1) vanish. Under what condition does this happen?

5. In this problem, you are asked to derive the Feynman-Hellmann Theorem.

(a) If the Hamiltonian $H(\lambda)$ depends on a real parameter λ, i.e., $H(\lambda)\,|\psi\rangle = E(\lambda)\,|\psi\rangle$, then show that:

$$\frac{\partial E}{\partial \lambda} = \left\langle \psi \left| \frac{\partial H}{\partial \lambda} \right| \psi \right\rangle .$$

[*HINT*: Evaluate $\frac{\partial}{\partial\lambda}\langle\psi|\psi\rangle$.]

(b) Consider the one-dimensional problem with the Hamiltonian:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\,,$$

where V is independent of the parameter m. Suppose one finds that this Hamiltonian possesses a particular energy eigenstate with energy eigenvalue E. Describe the behavior of E as m decreases.

6. Consider the operator:

$$a = \frac{m\omega X + iP}{\sqrt{2m\hbar\omega}}\;.$$

where X and P are the position and momentum operators, respectively.

(a) Let $|z\rangle$ be an eigenvector of a with eigenvalue z. This state is called a *coherent state*. Compute $\langle x|z\rangle$. Using this result, show that $\langle z|z'\rangle \neq 0$. Why does the lack of orthogonality of these states not violate any of our quantum mechanics postulates?

(b) Consider the operator a in the context of the one-dimensional harmonic oscillator. Compute $\langle n|z\rangle$, where $|n\rangle$ is the nth energy eigenstate. (Assume that $|z\rangle$ is normalized to unity.) Given a coherent state $|z\rangle$, find the most probable value of n (and corresponding energy E).

(c) Prove that the normalized coherent state can be written as, $|z\rangle = e^{-|z|^2/2}e^{za^\dagger}\,|0\rangle$.

(d) Prove that the coherent state is a state of minimum uncertainty, i.e., $\Delta X\Delta P = \frac{1}{2}\hbar$.

(e) Consider coherent states in which the parameter z is a real positive number much larger than 1. Evaluate the expectation value of the quantum Hamiltonian with respect to a coherent state with $|z| \gg 1$. In what way is this state a good approximation to the classical limit of the harmonic oscillator?