DUE: TUESDAY, MARCH 6, 2018

CLASS SCHEDULE ALERT: No classes will be held on Tuesday February 20 and Thursday February 22. There will be a a make-up class on Monday February 26 starting at 5 pm in ISB 165.

1. Here is a clever operator method for solving the two-dimensional harmonic oscillator. Consider the Hamiltonian of the two-dimensional harmonic oscillator:

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + \tfrac{1}{2}m\omega^2(X^2 + Y^2) \ .$$

Define the operators:

$$\begin{aligned} K_1 &= \frac{1}{4m\omega}\,(P_x^2 - P_y^2) + \tfrac{1}{4}m\omega(X^2 - Y^2)\,, \\ K_2 &= \frac{1}{2m\omega}\,P_x P_y + \tfrac{1}{2}m\omega XY\,, \\ K_3 &= \tfrac{1}{2}(XP_y - YP_x) \end{aligned}$$

Note that $K_3 = \frac{1}{2}L_z$.

(a) Compute the commutation relations $[K_i\,,\,K_j]$ and $[K_i\,,\,H]$.

(b) Obtain an expression for H in terms of $\vec{\boldsymbol{K}}^2 = K_1^2 + K_2^2 + K_3^2$.

(c) What are the possible eigenvalues of $\vec{\boldsymbol{K}}^2$ and K_3?

(d) Using the results of parts (b) and (c), determine the possible energy eigenvalues for the Hamiltonian and compute the degeneracy of each level. Explain the degeneracy in terms of the symmetries of the problem.

2. In this problem, we use algebraic methods to deduce that the eigenvalues of L_z are integers. Hence, the orbital angular momentum quantum number ℓ must be a non-negative integer (i.e., half-integer values must be rejected).

(a) Taking inspiration from the algebraic solution to the harmonic oscillator, introduce creation and annihilation operators $a_j^\dagger$ and a_j respectively, where j labels the three possible directions in three-dimensional space. The position and momentum operators are defined via

$$X_j \equiv \sqrt{\frac{\hbar}{2}}\left(a_j + a_j^\dagger\right)\,, \qquad P_j \equiv -i\sqrt{\frac{\hbar}{2}}\left(a_j - a_j^\dagger\right)\,.$$

Compute the operator L_z in terms of these creation and annihilation operators.

(b) Show that by means of a linear transformation on a_j and $a_j^\dagger$, L_z can be expressed in terms of new annihilation and creation operators b_1, b_2 and their hermitian conjugates as follows:

$$L_z = \hbar(b_2^\dagger b_2 - b_1^\dagger b_1)\,.$$

(c) Show that the eigenvalues of L_z must be integers.

3. (a) Generalize the result of problem 6(a) of Problem Set 1. Prove that $e^A B e^{-A}$ can be expressed as an infinite series,

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A\,,\,[A, B]] + \frac{1}{3!}\Big[A\,,\,[A\,,\,[A, B]]\Big] + \ldots\,,$$

without making any assumptions about the commutation properties of $\big[A, B\big]$ and A.

(b) Using the result of part (a), verify that:

$$e^{-i\theta J_y/\hbar} J_z\, e^{i\theta J_y/\hbar} = J_z\cos\theta + J_x\sin\theta\,.$$

(c) Using the result of part (b), prove that

$$(J_x \pm iJ_y)e^{-i\theta J_y/\hbar} = \frac{1}{\sin\theta}\, e^{-i\theta J_y/\hbar} J_z - \cot\theta J_z\, e^{-i\theta J_y/\hbar} \mp \hbar\frac{\partial}{\partial\theta}\, e^{-i\theta J_y/\hbar}$$

(d) Define:

$$d^{(j)}_{mm'}(\theta) \equiv \langle j\, m|\, e^{-i\theta J_y/\hbar}\, |j\, m'\rangle\,,$$

where j is a non-negative half-integer and $m, m' = -j, -j+1, \ldots, j-1\,,\, j$. Using the result of part (c), obtain the following recursion relation,

$$d^{(j)}_{m\pm 1,m'}(\theta) = [(j \pm m + 1)(j \mp m)]^{-1/2}\left(\frac{m'}{\sin\theta} - m\cot\theta \pm \frac{\partial}{\partial\theta}\right) d^{(j)}_{mm'}(\theta)\,. \qquad (1)$$

(e) The spherical harmonics are related to the $d^{(j)}_{mm'}(\theta)$ by the following relation,

$$Y_{\ell m}(\theta, \phi) = \left(\frac{2\ell + 1}{4\pi}\right)^{1/2} d^{(\ell)}_{m0}(\theta)e^{im\phi}\,, \qquad (2)$$

where ℓ is a non-negative integer and $m = -\ell, -\ell+1, \ldots, \ell-1\,,\, \ell$. To see that eq. (2) is plausible, consider eq. (1) after setting $m' = 0$ and taking $j = \ell$ to be an integer. Then, derive a similar relation for the spherical harmonics $Y_{\ell m}(\theta, \phi)$. Interpret the resulting relation in terms of the raising and lowering operators $L_\pm$.

4. The wave function of a particle subjected to a spherically symmetric potential $V(r)$ is given by:

$$\psi(\vec{\boldsymbol{x}}) = (x + y + 3z) f(r)\,,$$

where $r \equiv |\vec{\boldsymbol{x}}|$.

(a) Show that $\psi(\vec{\boldsymbol{x}})$, when expressed as a function of spherical coordinates, can be written as a linear combination of spherical harmonics.

(b) Is ψ an eigenfunction of $\vec{\boldsymbol{L}}^2$? If so, what is the ℓ-value? If not, what are the possible values of ℓ that can be obtained when $\vec{\boldsymbol{L}}^2$ is measured?

(c) What are the probabilities for the particle to be found in various m states? Consider all allowed values of m and check that the probabilities adds up to one.

(d) Suppose it is known somehow that $\psi(\vec{\boldsymbol{x}})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

5. Suppose a half integer ℓ-value, say $\frac{1}{2}$, were allowed for orbital angular momentum. From,

$$L_+ Y_{1/2,1/2}(\theta,\phi) = 0\,,$$

we may deduce, as usual, that

$$Y_{1/2,1/2}(\theta,\phi) \propto e^{i\phi/2}\sqrt{\sin\theta}\,.$$

Now try to construct $Y_{1/2,-1/2}(\theta,\phi)$ by (i) applying L_- to $Y_{1/2,1/2}(\theta,\phi)$; and (ii) using $L_- Y_{1/2,-1/2}(\theta,\phi) = 0$. Show that the two procedures lead to contradictory results.[1]

6. Solve for the lowest energy state in a square well in two and three dimensions with zero angular momentum (i.e., $\ell = 0$):

$$V(r) = \begin{cases} -V_0\,, & \text{if } r < a, \\ 0\,, & \text{if } r > a, \end{cases}$$

where V_0 is positive. In the case of three dimensions, find the minimum value of V_0 which is necessary in order that there be at least one bound state. This is in contrast to the situation in one dimension where there is always binding no matter how small V_0 is. In two dimensions, is the situation analogous to the three dimensional case or to the one dimensional case?

[1] The results of problem 2 and problem 5 provide independent arguments against half-integer ℓ-values for orbital angular momentum.