INSTRUCTIONS: This is a take home exam. You may refer to the course textbook by Sakurai and Napolitano, the class handouts and any second quantum mechanics textbook of your choosing. (If you do consult a second text, please indicate which one you used.) You may also make use of any material that is contained on the course website. Any reference for integrals or other mathematical facts, and any personal handwritten notes are also OK. However, you should *not* collaborate with anyone else. The point value of each problem is indicated in the square brackets below; each part is worth 10 points.

Please note that part (c) of Problem 3 is for extra credit (and is thus optional). Do *not* attempt to solve this part of the problem until you have completed the rest of the exam. Completed exams should be delivered to my campus mailbox by the end of the day on Thursday February 22, 2018.

There is no need to rederive results that have been previously obtained in the textbook, the class notes or the class handouts. But if you make use of any previously derived result, please cite the source of the result.

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1. [40] Consider a spin-$\frac{1}{2}$ particle placed in a uniform magnetic field,

$$\vec{\boldsymbol{B}} = \frac{B_0}{\sqrt{2}}\big(\hat{\boldsymbol{x}} + \hat{\boldsymbol{z}}\big)\,.$$

The dynamics of the particle are governed by the Hamiltonian, $H = -\gamma\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}$, where $\vec{\boldsymbol{S}} = \frac{1}{2}\hbar\vec{\boldsymbol{\sigma}}$ is the vector spin operator for spin-$\frac{1}{2}$ particles, the $\vec{\boldsymbol{\sigma}} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices, and γ is a positive constant.

(a) Calculate the matrix elements of the Hamiltonian with respect to the $\{|+\rangle\,, |-\rangle\}$ basis, where $S_z\,|\pm\rangle = \pm\frac{1}{2}\hbar\,|\pm\rangle$.

(b) Calculate the eigenvalues and the normalized eigenvectors of H.

(c) At time $t = 0$, the system is in the state $|-\rangle$, and the energy is measured. What are the possible outcomes of the energy measurement and with what probabilities?

(d) Evaluate the state vector $|\psi(t)\rangle$ at time t, assuming the initial condition specified in part (c). Evaluate the expectation value of S_x at time t.

2. [30] Consider an ensemble of one-dimensional harmonic oscillators in thermal equilibrium with a heat bath at temperature T. Define $\beta \equiv 1/kT$, where k is Boltzmann's constant. You may assume that $\beta > 0$. The density matrix that describes such a system is given by

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}\,, \tag{1}$$

where H is the harmonic oscillator Hamiltonian operator.

(a) Evaluate the matrix elements of ρ in the oscillator energy basis.

(b) Compute $\mathrm{Tr}\,\rho^2$ and determine whether ρ represents a pure or mixed state. Discuss the limit of $\beta \to \infty$, and indicate whether the quantum state in this limit is pure or mixed. Explain your conclusions.

(c) Does ρ, defined in eq. (1) [and evaluated in the Schrödinger representation], vary with time? Explain your answer.

3. [30] A particle of mass m moves in a one-dimensional double-well potential,

$$V(x) = -g\big[\delta(x-a) + \delta(x+a)\big]\,. \tag{2}$$

where a and g are real positive parameters.

(a) Derive a transcendental equation for the energy eigenvalues of the system. Show explicitly that the corresponding energy eigenfunctions are either parity even or parity odd.

(b) Using the result of part (a), determine the number of bound states. What is the parity of the ground state energy eigenfunction?

HINT: In analyzing the transcendental equation obtained in part (a), employ a graphical technique to ascertain the possible solutions.

(c) [*EXTRA CREDIT*] Estimate the energy splitting between the first excited bound state and the ground state in the limit of large a.

(d) Consider the scattering problem for an incident wave of energy $E = \hbar^2 k^2/(2m)$ subject to the potential given in eq. (2). Calculate the transmission coefficient, $T = |S_{21}|^2$, where S_{21} is the 21 element of the S-matrix. Find a transcendental equation whose solutions yield the scattering energies at which resonance behavior occurs. Finally, determine the location of the poles of $S_{21}(E)$ and compare with the results of part (a).

HINT: A successful completion of part (d) will require some careful algebraic manipulations. In determining the transcendental equation for the values of k corresponding to the resonance condition, use appropriate trigonometric identities to express the resulting equation in terms of $\sin 2ka$ and $\cos 2ka$.