The Riemann-Lebesgue Lemma

The Riemann Lebesgue Lemma is one of the most important results of Fourier analysis and asymptotic analysis. It has many physics applications, especially in studies of wave phenomena. In this short note, I will provide a simple proof of the Riemann-Lebesgue lemma which will be adequate for most cases that arise in physical applications.

The simplest form of the Riemann-Lebesgue lemma states that for a function $f(x)$ for which the integral

$$\int_a^b |f(x)|\, dx < \infty\,, \tag{1}$$

where a and b are real numbers, we have

$$\lim_{k\to\infty} \int_a^b f(x)\, e^{ikx}\, dx = 0\,. \tag{2}$$

Sometimes, the result of eq. (2) appears in the form,

$$\lim_{k\to\infty} e^{ikx} = 0\,. \tag{3}$$

Of course eq. (3) makes no sense when interpreted as a standard limit in mathematical analysis. However, if one interprets the limit of eq. (3) in the *sense of distributions*, i.e. by treating e^{ikx} as a generalized function, then eq. (3) can be assigned a useful meaning.[1]

If we further assume that $f(x)$ has certain "nice" properties [e.g., a sufficient (but not necessary) condition is that $f(x)$ is continuously differentiable for $a \leq x \leq b$], then it follows that

$$\int_a^b f(x)\, e^{ikx}\, dx = \mathcal{O}\left(\frac{1}{k}\right)\,, \qquad \text{as } k \to \infty. \tag{4}$$

Moreover, eqs. (2) and (4) continue to hold if $a \to -\infty$ and/or $b \to \infty$, assuming that eq. (1) holds over the infinite interval.

We will present a proof of eq. (2) under the assumption that $f(x)$ is continuous over the closed interval $a \leq x \leq b$. The origin of eq. (2) in this case is not too difficult to understand. In the limit of $k \to \infty$, the factor e^{ikx} oscillates faster and faster such that $f(x)\, e^{ikx}$ averages out to zero over any finite region of x inside the interval.

[1]For further details, see J. Campos Ferreira, *Introduction to the Theory of Distributions* (Addison Wesley Longman Limited, Essex, UK, 1997).

If we can assume that $f(x)$ is N-times differentiable in the region $a \leq x \leq b$, one can derive eq. (4) simply by a repeated integration by parts. Namely,

$$\begin{aligned}\int_a^b f(x)\,e^{ikx}\,dx &= \frac{f(x)}{ik}\,e^{ikx}\bigg|_a^b - \frac{1}{ik}\int_a^b f'(x)\,e^{ikx}\,dx \\ &= \frac{e^{ikb}f(b) - e^{ika}f(a)}{ik} - \frac{f'(x)}{(ik)^2}\,e^{ikx}\bigg|_a^b + \frac{1}{(ik)^2}\int_a^b f''(x)\,e^{ikx}\,dx \\ &= \sum_{n=0}^{N-1}(-1)^n\,\frac{e^{ikb}f^{(n)}(b) - e^{ika}f^{(n)}(a)}{(ik)^{n+1}} + \mathcal{O}\left(\frac{1}{k^{N+1}}\right)\,, \qquad (5)\end{aligned}$$

where $f^{(0)}(x) \equiv f(x)$ and $f^{(n)}(x) \equiv d^n f/dx^n$. For example $f'(b)$ is equal to the first derivative of $f(x)$ evaluated at $x = b$, etc. Taking $N = 1$ in eq. (5), we see that the leading term that survives is of $\mathcal{O}(1/k)$ as asserted by eq. (4).

More generally, we can prove eq. (2) without the assumption $f(x)$ is differentiable in the interval. We do this by writing the integral

$$\mathcal{I}(k) = \int_a^b f(x)\,e^{ikx}\,dx\,,$$

in two different but equivalent ways:[2]

$$\mathcal{I}(k) = \int_a^{a+\pi/k} f(x)\,e^{ikx}\,dx + \int_{a+\pi/k}^b f(x)\,e^{ikx}\,dx \qquad (6)$$

and

$$\mathcal{I}(k) = \int_a^{b-\pi/k} f(x)\,e^{ikx}\,dx + \int_{b-\pi/k}^b f(x)\,e^{ikx}\,dx\,. \qquad (7)$$

By a change of variables, $x' = x - \pi/k$, it is straightforward to verify that

$$\int_{a+\pi/k}^b f(x)\,e^{ikx}\,dx = -\int_a^{b-\pi/k} f\left(x + \frac{\pi}{k}\right)\,e^{ikx}\,dx\,, \qquad (8)$$

after using $e^{-i\pi} = -1$ and dropping the primes from the x in the second integral. Thus, writing $\mathcal{I}$ as one half the sum of eqs. (6) and (7), and employing eq. (8), it follows that

$$\begin{aligned}\mathcal{I}(k) = \tfrac{1}{2}\int_a^{a+\pi/k} f(x)\,e^{ikx}\,dx \;+\; \tfrac{1}{2}\int_{b-\pi/k}^b f(x)\,e^{ikx}\,dx \\ +\,\tfrac{1}{2}\int_a^{b-\pi/k}\left[f(x) - f\left(x + \frac{\pi}{k}\right)\right]e^{ikx}\,dx\,. \qquad (9)\end{aligned}$$

[2]This proof is taken from Brian Davies, *Integral Transforms and Their Applications*, 3rd edition (Springer-Verlag, New York, 2002) pp. 39–40.

We not take the limit of $k \to \infty$. The mean value theorem of for integrals states that if $f(x)$ is continuous and bounded over a closed interval, $a \leq x \leq b$, then

$$\int_a^b f(x)\, dx = f(c)(b-a)\,,$$

for some real number c that lies in the interval $a \leq c \leq b$. Applying this to the first two integrals in eq. (9), we immediately conclude that

$$\int_a^{a+\pi/k} f(t)\, e^{ikx}\, dx = \mathcal{O}\left(\frac{1}{k}\right)\,, \qquad \int_{b-\pi/k}^{b} f(k)\, e^{ikx}\, dx = \mathcal{O}\left(\frac{1}{k}\right)\,,$$

which vanish in the limit of $k \to \infty$.

Finally, under the assumption that $f(x)$ is continuous at all points in the closed interval $a \leq x \leq b$, it follows that

$$\lim_{k\to\infty} \int_a^{b-\pi/tk} \left[f(x) - f\left(x + \frac{\pi}{k}\right)\right] e^{ikx}\, dx = 0\,. \tag{10}$$

This is true because a function that is continuous at all points in a closed, bounded interval is *uniformly* continuous over the interval.[3] Hence, one can make the integrand in eq. (10) arbitrarily small by choosing k sufficiently large. The limit of eq. (10) is thus established, and the Riemann-Lebesgue lemma stated in eq. (2) is proven.

Note that the argument above does *not* necessarily imply that

$$\int_a^{b-\pi/k} \left[f(x) - f\left(x + \frac{\pi}{k}\right)\right] e^{ikx} = \mathcal{O}\left(\frac{1}{k}\right)\,, \tag{11}$$

as $k \to \infty$. However, if the function $f(x)$ is continuously differentiable in the interval, then we can employ the mean value theorem for differentiable functions, which states that

$$f(b) - f(a) = f'(c)(b-a)\,, \qquad \text{for some } c \text{ between } a \text{ and } b\,.$$

It follows that

$$f(x) - f\left(x + \frac{\pi}{k}\right) = -\frac{\pi}{k} f'(x+c)\,, \qquad \text{for } 0 \leq c \leq \frac{\pi}{k}\,.$$

Hence, in this case eq. (11) does hold, in which case eq. (4) is satisfied.[4]

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[3]For further details, see e.g. David Bressoud, *A Radical Approach to Real Analysis*, 2nd edition (The Mathematics Association of America, Washington, DC, 2007) pp. 228–229.

[4]We also obtain eq. (11) under slightly weaker conditions in which the function $f(x)$ satisfies the so-called *Lipschitz condition*, $|f(x) - f(y)| \leq M|x-y|$ for all x and y in the interval for some positive finite bound M. Indeed a Lipschitz continuous function is uniformly continuous (although the converse is not necessarily true). A Lipschitz continuous function need not be differentiable. On the other hand a differentiable function whose derivative is bounded on the interval satisfies the Lipschitz condition. Thus, to establish eq. (11) is is sufficient to require that $f(k)$ is Lipschitz continuous in the interval.

The extension to cases where $a \to -\infty$ and/or $b \to \infty$ is straightforward. For example, suppose that

$$\int_a^\infty |f(x)| dx < \infty \,. \tag{12}$$

Then, noting that one can write

$$\int_a^\infty f(x) e^{ikx}\, dx = \int_a^b f(x) e^{ikx}\, dx + \epsilon \,,$$

where

$$\epsilon < \int_b^\infty |f(x)| dx \,,$$

it follows [in light of eq. (12)] that it is always possible to make ϵ arbitrarily small by a suitable (finite) choice of b. Hence, we can use eq. (2) to conclude that

$$\lim_{k\to\infty} \int_a^\infty f(x)\, e^{ikx}\, dx = 0 \,. \tag{13}$$

Finally, it should be noted that taking the real and imaginary parts of eq. (2) under the assumption that $f(x)$ is a real valued function yields,

$$\lim_{k\to\infty} \int_a^b f(x) \sin kx\, dx = 0 \,, \tag{14}$$

$$\lim_{k\to\infty} \int_a^b f(x) \cos kx\, dx = 0 \,, \tag{15}$$

with a similar extension to cases where $a \to -\infty$ and/or $b \to \infty$. That is,

$$\lim_{k\to\infty} \cos kx = \lim_{k\to\infty} \sin kx = 0 \,, \tag{16}$$

where the limit is interpreted in the sense of distributions [cf. discussion below eq. (3)]. If we further assume that $f(x)$ has certain "nice" properties [cf. comments above eq. (4)], then the two integrals given in eqs. (14) and (15) behave as $\mathcal{O}(1/k)$ as $k \to \infty$.

By consulting a table of Fourier transforms,[5] one can see many examples of functions that satisfy eq. (1). In all cases, you will find that the corresponding Fourier transform satisfies eq. (2). It is interesting to look for cases that satisfy eq. (2) and not eq. (4). For example,

$$\int_0^\infty x^{\nu-1}\, e^{-ax}\, e^{ikx}\, dx = \frac{\Gamma(\nu)}{(a-ik)^\nu} \,, \qquad \text{for } a > 0 \text{ and } \mathrm{Re}\ \nu > 0 \,.$$

Indeed, eqs. (1) and (2) are satisfied for all Re $\nu > 0$, whereas eq. (4) is only satisfied when Re $\nu \geq 1$.

[5] See e.g., F. Oberhettinger, *Table of Fourier Transforms and Fourier Transforms of Distributions* (Springer-Verlag, Berlin, 1990).

Finally, we provide the following interesting example that contradicts eq. (14). Choosing $f(x) = \sin(x^2)$ and consulting I.S. Gradshteynn and I.M. Ryzhik, *Table of Integrals, Series and Products*, 8th edition, edited by Daniel Zwillinger (Academic Press, Elsevier, Amsterdam, 2015), we note the following result on p. 419 [formula 3.691 number 5],

$$\int_0^\infty \sin(x^2)\sin kx\,dx = \tfrac{1}{2}\sqrt{\pi}\cos\left(\frac{k^2+\pi}{4}\right)\,,$$

which does *not* vanish as $k \to \infty$. However, this is not surprising in light of the fact that eq. (12) [with $a = 0$] is not satisfied. Indeed, even though

$$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,,$$

is finite, it turns out that

$$\int_0^\infty |\sin(x^2)|dx$$

diverges. Thus, one of the key assumptions underlying the Riemann-Lebesgue lemma does not hold in this case.