Integral representation of the Heavyside step function

The Heavyside step function is defined as,

$$\Theta(k) = \begin{cases} 1\,, & \text{if } k > 0\ , \\ 0\,, & \text{if } k < 0\,. \end{cases} \tag{1}$$

Although the value of $\Theta(k)$ is not defined at $k = 0$, we shall nevertheless demand that

$$\Theta(k) + \Theta(-k) = 1\,, \tag{2}$$

should be satisfied for all real values of k, *including* the origin, $k = 0$. The Heavyside step function is related to the Dirac delta function by differentiation,

$$\delta(k) = \frac{d\Theta(k)}{dk}\,. \tag{3}$$

The goal of these notes is to express the step function as a Fourier transform,

$$\Theta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} f(x)\, dx\,, \tag{4}$$

where the function $f(x)$ is to be determined.[1]

The function $f(x)$ is determined by the inverse Fourier transform,

$$f(x) = \int_{-\infty}^{\infty} e^{-ikx}\, \Theta(k)\, dk\,. \tag{5}$$

This integral is not well defined. However it can be re-interpreted in *the sense of distributions.* What this phrase really means is that quantities are treated as generalized functions (also called distributions), which make sense only when integrated against test functions that are smooth, regular, and vanish sufficiently fast at $\pm\infty$. We can evaluate the $f(x)$ using the following trick. Note that

$$1 = \int_{-\infty}^{\infty} e^{-ikx}\, \delta(k)\, dk = \int_{-\infty}^{\infty} e^{-ikx}\, \frac{d\Theta(k)}{dk}\, dk\,. \tag{6}$$

We now integrate by parts. We can set the surface term to zero by employing

$$\lim_{k\to\pm\infty} e^{-ikx} = 0\,, \tag{7}$$

[1]These notes are based on a derivation given on p. 151 of Ram P. Kanwal, *Generalized Functions: Theory and Applications*, 3rd edition (Birkhäuser, Boston, 2004).

where the limit is interpreted in the sense of distributions (as mention in the class handout, *The Riemann-Lebesgue Lemma*). It then follows that

$$1 = -\int_{-\infty}^{\infty} \Theta(k) \frac{d}{dk} e^{-ikx}\, dk = ix \int_{-\infty}^{\infty} \Theta(k) e^{-ikx}\, dk = ixf(x)\,. \tag{8}$$

To solve eq. (8), let us define $h(x) \equiv if(x)$ and consider the equation

$$xh(x) = 1\,. \tag{9}$$

The solution to this equation for $x \neq 0$ is clearly $h(x) = 1/x$. But, how should we deal with $x = 0$? The answer is again to appeal to generalized functions. That is, eq. (9) should be interpreted as

$$\int_{-\infty}^{\infty} xh(x)g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx\,, \tag{10}$$

for any smooth regular test function $g(x)$ that vanishes sufficiently fast at $\pm\infty$.

The most general solution to the inhomogeneous equation, $xh(x) = 1$, must be of the form,

$$h(x) = h_p(x) + h_h(x)\,, \tag{11}$$

where $h_p(x)$ is a particular solution that satisfies $xh_p(x) = 1$ and $h_h(x)$ is the solution to the homogeneous equation, $xh_h(x) = 0$. I claim that one choice for the particular solution to eq. (9) is,

$$h_p(x) = \mathrm{P}\frac{1}{x}\,, \tag{12}$$

where P indicates the Cauchy principal value prescription when integrated against a test function, $g(x)$,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{g(x)\, dx}{x} \equiv \lim_{\delta \to 0} \left\{ \int_{-\infty}^{-\delta} \frac{g(x)\, dx}{x} + \int_{\delta}^{\infty} \frac{g(x)\, dx}{x} \right\}\,, \tag{13}$$

with $\delta > 0$.

Let us check that $h(x) = h_p(x)$ given by eq. (12) provides a solution to eq. (10). It is sufficient to observe that,

$$\mathrm{P}\int_{-\infty}^{\infty} x\,\frac{1}{x}\, g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx\,, \tag{14}$$

where the P symbol can be dropped on the right hand side of eq. (14) since the corresponding integral is now well defined. Hence, it follows that (in the sense of distributions),

$$x\,\mathrm{P}\frac{1}{x} = 1\,, \tag{15}$$

and eq. (12) is verified.

We now turn to the most general solution to the homogeneous equation,

$$xh_h(x) = 0\,. \tag{16}$$

We shall solve eq. (16) using a Fourier transform technique. We first write

$$h_h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} q(k)\, dk\,. \tag{17}$$

Inverting the Fourier transform yields

$$q(k) = \int_{-\infty}^{\infty} e^{-ikx} h_h(x)\, dx\,. \tag{18}$$

We now take the derivative of $q(k)$ with respect to k to obtain,

$$\frac{dq}{dk} = -i\int_{-\infty}^{\infty} e^{-ikx} x h_h(x)\, dx = 0\,. \tag{19}$$

where we have used eq. (16) in the final step. The most general solution to the differential equation $dq/dk = 0$ is $q(k) = C$, where C is an arbitrary constant.[2] Inserting this solution back into eq. (17), we end up with

$$h_h(x) = \frac{C}{2\pi}\int_{-\infty}^{\infty} e^{ikx} q(k)\, dk = C\,\delta(x)\,, \tag{20}$$

after employing the integral representation of the delta function,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, dk\,. \tag{21}$$

It is simple to check the validity of eq. (20). In particular,

$$\int_{-\infty}^{\infty} x\delta(x) g(x)\, dx = xg(x)\bigg|_{x=0} = 0\,, \tag{22}$$

where again we have used the fact that $g(x)$ is a smooth regular function. It then follows that (in the sense of distributions),

$$x\delta(x) = 0\,. \tag{23}$$

Hence, $xh_h(x) = Cx\delta(x) = 0$ as required. Combining the results of eqs. (12) and (20), one obtains the most general solution of eq. (9),

$$h(x) = h_p(x) + h_h(x) = \mathrm{P}\frac{1}{x} + C\delta(x)\,. \tag{24}$$

Returning to eq. (8), it follows in light of eq. (24) that

$$h(x) = if(x) = i\int_{-\infty}^{\infty} \Theta(k) e^{-ikx}\, dk = \mathrm{P}\frac{1}{x} + C\delta(x)\,, \tag{25}$$

[2]This statement is trivial if solutions are restricted to the space of ordinary functions. Nevertheless, $q(k) = C$ is still the unique solution of $dq/dk = 0$ even if the solution space is expanded to included generalized functions. A proof of this assertion can be found on pp. 39–41 of I.M. Gel'fand and G.E. Shilov, *Generalized Functions, Volume 1: Properties and Operations* (Academic Press, New York, NY, 1964).

where the constant C is still yet to be determined. To fix the constant C we proceed as follows. Replacing $x \to -x$ in eq. (25) yields,

$$-\mathrm{P}\frac{1}{x} + C\delta(x) = i\int_{-\infty}^{\infty} \Theta(k)e^{ikx}\,dk = i\int_{-\infty}^{\infty} \Theta(-k)e^{-ikx}\,dk\,, \tag{26}$$

after noting that $\delta(-x) = \delta(x)$ and changing the integration variable from k to $-k$. Adding eqs. (25) and (26) and using eq. (2), we end up with

$$2C\delta(x) = i\int_{-\infty}^{\infty} \big[\Theta(k) + \Theta(-k)\big]e^{-ikx}\,dk = i\int_{-\infty}^{\infty} e^{-ikx}\,dk\,. \tag{27}$$

Finally, using the integral representation of the delta function [cf. eq. (21)], we conclude that $C = i\pi$. We now insert this value of C into eq. (25) and employ the Sokhotski-Plemelj formula [cf. the class handout entitled, *The Sokhotski-Plemelj Formula*],

$$\lim_{\varepsilon\to 0} \frac{1}{x - i\epsilon} = \mathrm{P}\frac{1}{x} + i\pi\delta(x)\,. \tag{28}$$

The end result is

$$if(x) = i\int_{-\infty}^{\infty} \Theta(k)e^{-ikx}\,dk = \lim_{\varepsilon\to 0} \frac{1}{x - i\varepsilon}\,. \tag{29}$$

Returning to eq. (4), we conclude that[3]

$$\Theta(k) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x - i\varepsilon}\,dx\,. \tag{30}$$

It is amusing to revisit eq. (25) with $C = i\pi$, which yields the noteworthy result,

$$i\int_{-\infty}^{\infty} \Theta(k)e^{-ikx}\,dk = \mathrm{P}\frac{1}{x} + i\pi\delta(x)\,. \tag{31}$$

In particular, using the definition of the Heavyside step function, it follows that

$$\int_{0}^{\infty} e^{-ikx}\,dk = -i\,\mathrm{P}\frac{1}{x} + \pi\delta(x)\,. \tag{32}$$

Of course eq. (32) must be interpreted in the sense of distributions (since the integral does not converge in the usual sense[4]). Moreover, taking real and imaginary parts of eq. (32) yields,

$$\int_{0}^{\infty} \cos kx\,dk = \pi\delta(x)\,, \tag{33}$$

$$\int_{0}^{\infty} \sin kx\,dk = \mathrm{P}\frac{1}{x}\,. \tag{34}$$

[3]In problem 4(a) of Problem Set 1, you were given the right hand side of eq. (30) and asked to evaluate the integral using the residue theorem of complex analysis (with suitably chosen closed contours for the two cases of $k > 0$ and $k < 0$). The result of such a computation is, of course, $\Theta(k)$.

[4]Compare this with the integral in eq. (21), which does not converge in the usual sense, but nevertheless provides an integral representation of the delta function in the sense of distributions.

Once again, the integrals of eqs. (33) and (34) must be interpreted in the sense of distributions. For example, one can check that eq. (33) is a consequence of the integral representation of the delta function,

$$2\pi\delta(x) = \int_{-\infty}^{\infty} e^{ikx}\, dk = \int_{-\infty}^{0} e^{ikx}\, dk + \int_{0}^{\infty} e^{ikx}\, dk = \int_{0}^{\infty} \left[e^{ikx} + e^{-ikx}\right] dk = 2\int_{0}^{\infty} \cos kx\, dk\,, \tag{35}$$

after changing the integration variable, $k \to -k$, in the second integral above. Hence, we have recovered eq. (33).

One additional consequence of eq. (31) can be extracted if we invert the Fourier transform,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left[\mathrm{P}\frac{1}{x} + i\pi\delta(x)\right] dx = i\Theta(k)\,. \tag{36}$$

Using the Cauchy principal value prescription and integrating over the delta function yields

$$\frac{1}{2\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x}\, dx = i\left[\Theta(k) - \tfrac{1}{2}\right]. \tag{37}$$

It is convenient to introduce the sign function,

$$\mathrm{sgn}(k) = \Theta(k) - \Theta(-k) = \begin{cases} +1\,, & \text{for } k > 0, \\ -1\,, & \text{for } k < 0, \end{cases} \tag{38}$$

which satisfies

$$\frac{d}{dk}\,\mathrm{sgn}(k) = \frac{d}{dk}\left[\Theta(k) - \Theta(-k)\right] = 2\delta(k)\,, \tag{39}$$

in light of eq. (3). Using eq. (2), it follows that

$$\Theta(k) - \tfrac{1}{2} = \Theta(k) - \tfrac{1}{2}\left[\Theta(k) + \Theta(-k)\right] = \tfrac{1}{2}\left[\Theta(k) - \Theta(-k)\right] = \tfrac{1}{2}\,\mathrm{sgn}(k)\,. \tag{40}$$

Inserting this result into eq. (37) yields,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x}\, dx = i\pi\,\mathrm{sgn}(k)\,. \tag{41}$$

Taking the real and imaginary parts of eq. (41) yields,[5]

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{\cos kx}{x}\, dx = 0\,, \tag{42}$$

$$\int_{-\infty}^{\infty} \frac{\sin kx}{x}\, dx = \pi\,\mathrm{sgn}(k)\,. \tag{43}$$

Note that the vanishing of the integral in eq. (42) is due to the fact that the integrand is an odd function of x, which integrates to zero due to the Cauchy principal value prescription. The P symbol is not needed in eq. (43) since the corresponding integrand has a finite limit as $k \to 0$.

As one final check, let us take the derivative of eq. (41) with respect to k and employ eq. (39). This yields (once again) the integral representation of the delta function (where the P symbol can be dropped as the resulting integrand is not singular at $x = 0$).

[5]The result of eq. (43) is well-known and is often obtained using the residue theorem of complex analysis (and a suitably chosen closed contour).