Properties of the Wigner d-matrices

The matrix elements of Wigner's small d-matrix are defined by,

$$d^{(j)}_{m'm}(\theta) \equiv \langle j\, m'|e^{-i\theta J_y/\hbar}|j\, m\rangle\,,$$

where j is a non-negative half-integer and $m, m' = -j, -j+1, \ldots, j-1\,,\, j$.

Property 1: $d^{(j)}_{m'm}(\theta)$ are real for all values of m, m' and j.

Recall the explicit form for the matrix elements of the angular momentum operator J_y obtained in class,

$$\langle jm'|J_y|jm\rangle = -\tfrac{1}{2}i\hbar\biggl[\delta_{m',m+1}\bigl[(j-m)(j+m+1)\bigr]^{1/2} - \delta_{m',m-1}\bigl[(j+m)(j-m+1)\bigr]^{1/2}\biggr]\,. \quad (1)$$

Thus the matrix elements of iJ_y are real. It then follows that the matrix elements of $e^{-i\theta J_y/\hbar}$ are also real.

Property 2: $d^{(j)}_{m'm}(-\theta) = d^{(j)}_{mm'}(\theta)$ for all values of m, m' and j.

First, note that $e^{-i\theta J_y/\hbar}$ is a unitary operator. From property 1, it follows that $e^{-i\theta J_y/\hbar}$ is real. Hence, $e^{-i\theta J_y/\hbar}$ is a real orthogonal operator, which implies that its matrix representation satisfies $d^{\mathsf{T}}(\theta)d(\theta) = I$, where I is the identity matrix, or equivalently, $d^{\mathsf{T}}(\theta) = d^{-1}(\theta)$. Moreover, the inverse of $e^{-i\theta J_y/\hbar}$ is given by $e^{i\theta J_y/\hbar}$, which yields $d^{-1}(\theta) = d(-\theta)$. Hence, we conclude that

$$d^{\mathsf{T}}(\theta) = d(-\theta)\,.$$

The matrix elements of this equation correspond to Property 2 above.

Property 3: $d^{(j)}_{m'm}(\theta) = d^{(j)}_{-m,-m'}(\theta)$ for all values of m, m' and j.

This property immediately follows from eq. (1), since J_y is invariant under the interchange $m \to -m'$, $m' \to -m$. In particular, under this interchange,

$$\langle j,-m|J_y|j,-m'\rangle = -\tfrac{1}{2}i\hbar\biggl[\delta_{-m,-m'+1}\bigl[(j+m')(j-m'+1)\bigr]^{1/2} - \delta_{-m,-m'-1}\bigl[(j-m')(j+m'+1)\bigr]^{1/2}\biggr]\,.$$

In the first term above, the Kronecker delta imposes $m' = m+1$, and in the second term above, the Kronecker delta imposes $m' = m-1$. Hence,

$$\langle j,-m|J_y|j,-m'\rangle = -\tfrac{1}{2}i\hbar\biggl[\delta_{-m,-m'+1}\bigl[(j+m+1)(j-m)\bigr]^{1/2} - \delta_{-m,-m'-1}\bigl[(j-m+1)(j+m)\bigr]^{1/2}\biggr]\,. \quad (2)$$

Noting that $\delta_{m',m+1} = \delta_{-m,-m'+1}$ and $\delta_{m',m-1} = \delta_{-m,-m'-1}$, it follows from eqs. (1) and (2) that,

$$\langle jm'|J_y|jm\rangle = \langle j,-m|J_y|j,-m'\rangle\,.$$

Upon exponentiation, Property 3 is confirmed.

Property 4: $d^{(j)}_{m'm}(\theta) = (-1)^{m-m'} d^{(j)}_{mm'}(\theta)$ for all values of m, m' and j.

From the definition of the raising and lowering operators, $J_\pm \equiv J_x \pm iJ_y$, it follows that

$$e^{-i\theta J_y/\hbar} = e^{-\theta(J_+ - J_-)/2} .$$

It is easy to check that the matrix representation of J_y is antisymmetric. Starting from eq. (1),

$$\begin{aligned}
\langle jm|J_y|jm'\rangle &= -\tfrac{1}{2}i\hbar\Big[\delta_{m,m'+1}\big[(j-m')(j+m'+1)\big]^{1/2} - \delta_{m,m'-1}\big[(j+m')(j-m'+1)\big]^{1/2}\Big] \\
&= -\tfrac{1}{2}i\hbar\Big[\delta_{m',m-1}\big[(j-m+1)(j+m)\big]^{1/2} - \delta_{m',m+1}\big[(j+m+1)(j-m)\big]^{1/2}\Big] \\
&= -\langle j,m'|J_y|j,m\rangle .
\end{aligned}$$

Moreover, the matrix elements of J_y are nonzero if and only if $|m - m'| = 1$. Hence, if follows that for any positive integer k,

1. The matrix representation of J_y^{2k} is symmetric and its matrix elements vanish if $|m-m'|$ is an odd integer.

2. The matrix representation of J_y^{2k+1} is antisymmetric and its matrix elements vanish if $|m-m'|$ is an even integer.

The matrix exponential, $e^{-i\theta J_y/\hbar}$, is defined via its Taylor series. In light of the two results obtained above, the validity of Property 4 follows.

Property 5: $d^{(j)}_{mm'}(2\pi) = (-1)^{2j}\,\delta_{mm'}$ for all values of m, m' and j.

We begin by using Theorem 1, which states that if R is a rotation by an angle θ about a fixed axis $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{u}}' = R\hat{\boldsymbol{u}}$, then

$$U[R]\,\vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{u}}\,U^\dagger[R] = \vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{u}}' . \tag{3}$$

The proof of Theorem 1 is given in Appendix A. Exponentiating eq. (3) yields

$$U[R]\,e^{-i\beta\vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{u}}/\hbar}\,U^\dagger[R] = e^{-i\beta\vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{u}}'/\hbar} . \tag{4}$$

In eq. (4), choose $\theta = 2\pi$, $\hat{\boldsymbol{u}} = \hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{u}}' = \hat{\boldsymbol{y}}$. Then,

$$U[R]\,e^{-2\pi iJ_z/\hbar}\,U^\dagger[R] = e^{-2\pi iJ_y/\hbar} . \tag{5}$$

We will now evaluate eq. (5) by multiplying on the left by $\langle jm|$ and on the right by $|jm'\rangle$. We first observe that

$$\langle jm|e^{-2\pi iJ_x/\hbar}|jm'\rangle = e^{-2\pi im'}\,\delta_{mm'} = e^{-2\pi ij}e^{2\pi i(j-m')}\,\delta_{mm'} = e^{-2\pi ij}\,\delta_{mm'} = (-1)^{2j}\,\delta_{mm'} ,$$

after noting that $j - m'$ is an integer so that $e^{2\pi i(j-m')} = 1$. In the final step above, we wrote $e^{-2\pi ij} = (e^{-i\pi})^{2j} = (-1)^{2j}$. Hence, multiplying on the left by $\langle jm|$ and on the right by $|jm'\rangle$ on the left hand side of eq. (5) yields,

$$\begin{aligned}
\langle jm|U[R]\, e^{-2\pi iJ_z/\hbar}\, U^\dagger[R]|jm'\rangle &= \sum_{m_1}\sum_{m_2}\langle jm|U[R]|jm_1\rangle\langle jm_1|e^{-2\pi iJ_z/\hbar}|jm_2\rangle\langle jm_2|U^\dagger[R]|jm'\rangle \\
&= (-1)^{2j}\sum_{m_1}\sum_{m_2} D^{(j)}_{mm_1}[R] D^{(j)\dagger}_{m_2m'}[R]\delta_{m_1m_2} \\
&= (-1)^{2j}\big(D^{(j)}[R)]D^{(j)\dagger}[R]\big)_{mm'} = (-1)^{2j}\,\delta_{mm'}\,,
\end{aligned}$$

since $D^{(j)}[R]$ is the *unitary* matrix representation of the operator $U[R]$, which implies that $D^{(j)}[R)]D^{(j)\dagger}[R]\big) = \mathbf{I}$.

Finally, multiplying on the left by $\langle jm|$ and on the right by $|jm'\rangle$ on the right hand side of eq. (5) yields,

$$\langle jm|e^{-2\pi iJ_y/\hbar}|jm'\rangle = d^{(j)}_{mm'}(2\pi)\,.$$

Hence, we have proved Property 5. In summary, we have established that

$$d^{(j)}(2\pi) = \begin{cases} \mathbf{I}\,, & \text{for integral values, } j = 0, 1, 2, \ldots \\ -\mathbf{I}\,, & \text{for half-integral values, } j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \end{cases}$$

That is, a 2π rotation returns the wave function of a boson to its original value, whereas one must perform a 4π rotation to return the wave function of a fermion to its original value.

Property 6: $d^{(j)}_{m'm}(\pi) = (-1)^{j-m}\delta_{m,-m'}$ for all values of m, m' and j.

Starting from eq. (15), we choose $\vec{\boldsymbol{V}} = \vec{\boldsymbol{J}}$ and consider a rotation parameterized by $\hat{\boldsymbol{n}} = \hat{\boldsymbol{y}}$ and $\theta = \pi$. The corresponding 3×3 rotation matrix R is given by [cf. eq. (20) of the class handout entitled *Three Dimensional Rotation Matrices*]:

$$R(\hat{\boldsymbol{y}}, \pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

By choosing $\hat{\boldsymbol{u}}$ to be the unit vector that points in the x, y, and z directions, respectively, we obtain,

$$\exp(i\pi J_y/\hbar)J_x \exp(-i\pi J_y/\hbar) = -J_x\,, \tag{6}$$

$$\exp(i\pi J_y/\hbar)J_y \exp(-i\pi J_y/\hbar) = J_y\,, \tag{7}$$

$$\exp(i\pi J_y/\hbar)J_z \exp(-i\pi J_y/\hbar) = -J_z\,. \tag{8}$$

If we operate on the state $|j\, m\rangle$ with eq. (8), we obtain

$$J_z\Big[\exp(-i\pi J_y/\hbar)|j\, m\rangle\Big] = -\hbar m\Big[\exp(-i\pi J_y/\hbar)|j\, m\rangle\Big]\,.$$

Since $\vec{\boldsymbol{J}}^2$ and J_y commute, it follows that

$$\vec{\boldsymbol{J}}^2\Big[\exp(-i\pi J_y/\hbar)|j\,m\rangle\Big] = \hbar j(j+1)\Big[\exp(-i\pi J_y/\hbar)|j\,m\rangle\Big]\,.$$

Hence, we can conclude that[1]

$$\exp(-i\pi J_y/\hbar)|j\,m\rangle = e^{i\alpha(j,m)}|j\,,\,-m\rangle\,, \tag{9}$$

where $e^{i\alpha(j,m)}$ is a complex phase that can depend in principle on j and m.

To determine $e^{i\alpha(j,m)}$, we first note that eqs. (6) and (7) can be rewritten as

$$\exp(i\pi J_y/\hbar)J_\pm \exp(-i\pi J_y/\hbar) = -J_\mp\,, \tag{10}$$

where $J_\pm \equiv J_x \pm iJ_y$. Applying eq. (10) to the state $|j\,m\rangle$, and making use of eq. (9),

$$\begin{aligned} J_\pm \exp(-i\pi J_y/\hbar)|j\,m\rangle &= -\exp(-i\pi J_y/\hbar)J_\mp|j\,m\rangle \\ &= -\hbar\big[(j\pm m)(j\mp m+1)\big]^{1/2}\exp(-i\pi J_y/\hbar)|j\,,\,m\mp 1\rangle\,. \\ &= -\hbar\big[(j\pm m)(j\mp m+1)\big]^{1/2}e^{i\alpha(j,m\mp 1)}|j\,,\,-m\pm 1\rangle\,. \end{aligned}$$

However, eq. (9) also yields

$$J_\pm \exp(-i\pi J_y/\hbar)|j\,m\rangle = e^{i\alpha(j,m)}J_\pm|j\,,\,-m\rangle = \hbar\big[(j\pm m)(j\mp m+1)\big]^{1/2}e^{i\alpha(j,m)}|j\,,\,-m\pm 1\rangle\,.$$

Consequently, $e^{i\alpha(j,m)} = -e^{i\alpha(j,m\mp 1)}$. Given the value of $\alpha(j,j)$, one can obtain $\alpha(j,m)$ for $m=-j\,,\,-j+1\,,\,\ldots\,,\,j-1\,,\,j$,

$$e^{i\alpha(j,j-n)} = (-1)^n e^{i\alpha(j,j)}\,,\qquad \text{for } n=0,1,2,\ldots,2j\,.$$

Thus, writing $n=j-m$, it follows that

$$e^{i\alpha(j,m)} = (-1)^{j-m}e^{i\alpha(j,j)}\,,\qquad \text{for } m=-j\,,\,-j+1\,,\,\ldots\,,\,j-1\,,\,j\,.$$

Hence, we conclude that

$$\exp(-i\pi J_y/\hbar)|j\,m\rangle = e^{i\alpha(j,j)}\,(-1)^{j-m}|j\,,\,-m\rangle\,, \tag{11}$$

and

$$d^{(j)}_{m'm}(\pi) = \langle j\,m'|\exp(-i\pi J_y/\hbar)|j\,m\rangle = e^{i\alpha(j,j)}\,(-1)^{j-m}\delta_{m,-m'}\,. \tag{12}$$

In Appendix B, we demonstrate that $e^{i\alpha(j,j)}=1$ for all values of j. Hence,

$$d^{(j)}_{m'm}(\pi) = (-1)^{j-m}\delta_{m,-m'}\,,$$

which completes the proof.

One immediate consequence of $e^{i\alpha(j,j)}=1$ is that eq. (11) is now completely determined,

$$\exp(-i\pi J_y/\hbar)|j\,m\rangle = (-1)^{j-m}|j\,,\,-m\rangle\,. \tag{13}$$

[1] Since $|j\,m\rangle$ is a normalized state and $\exp(-i\pi J_y/\hbar)$ is a unitary operator, it follows that $\exp(-i\pi J_y/\hbar)|j\,m\rangle$ is also a normalized state, in which case the constant multiplying $|j\,,\,-m\rangle$ in eq. (9) must be a complex phase.

Eq. (13) plays an important role in the behavior of the angular momentum state $|jm\rangle$ under a time reversal transformation.[2]

An explicit form for the Wigner d-matrix

Julian Schwinger developed the connection between the algebra of angular momentum operators and the algebra of two independent harmonic oscillators, which you explored on problems 1 and 2 of Problem Set 4. Schwinger was then able to use this formalism to derive an explicit expression for $d^{(j)}_{m'm}(\theta)$. The derivation is given in Section 3.9 of Sakurai and Napolitano. For completeness, we provide the final expression here.

$$d^{(j)}_{m'm}(\theta) = \sum_{k=k_{\rm min}}^{k_{\rm max}} (-1)^{k-m+m'} \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j+m-k)!\,(j-k-m')!\,(k-m+m')\,!k!} \times \left(\cos\frac{\theta}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\theta}{2}\right)^{2k-m+m'}\,, \tag{14}$$

where k is an integer such that

$$k_{\rm min} = \min\bigl\{0\,,\, m-m'\bigr\} \geq -2j\,, \qquad k_{\rm max} = \max\bigl\{0\,,\, j+m\,,\, j-m'\bigr\} \leq 2j\,.$$

That is, in eq. (14) we sum over integer values of k such that none of the arguments of the factorials in the denominator are negative.

One can now use eq. (14) to check explicitly that Properties 1–6 are satisfied.

APPENDIX A: Proof of Theorem 1

Consider the unitary operator, $U[R] = e^{-i\theta\boldsymbol{\hat{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}/\hbar}$. For any vector operator, $\vec{\boldsymbol{V}} = (V_1, V_2, V_3)$,

$$U^\dagger[R]\, V_i\, U[R] = R_{ij} V_j\,, \tag{15}$$

where there is an implicit sum over the repeated index j. In eq. (15), R_{ij} are the matrix elements of the matrix R corresponding to a rotation by an angle θ about an axis $\boldsymbol{\hat{n}}$.[3]

Theorem 1: Suppose that R rotates a unit vector $\boldsymbol{\hat{u}}$ into $\boldsymbol{\hat{u}'}$; i.e., $\boldsymbol{\hat{u}'} = R\boldsymbol{\hat{u}}$. Then,

$$U[R]\,\vec{\boldsymbol{J}}\boldsymbol{\cdot}\boldsymbol{\hat{u}}\,U^\dagger[R] = \vec{\boldsymbol{J}}\boldsymbol{\cdot}\boldsymbol{\hat{u}'}\,. \tag{16}$$

Proof of Theorem 1: Since $\vec{\boldsymbol{J}}$ is a vector operator, it follows from eq. (15) that

$$U^\dagger[R]\, J_i\, U[R] = R_{ij} J_j\,. \tag{17}$$

[2]For details, see the class handout entitled *Implications of Time-Reversal Symmetry in Quantum Mechanics.*
[3]See, e.g., eq. (3.11.3) of Sakurai and Napolitano.

Multiplying eq. (17) by $\hat{u}'_i$, we obtain

$$U^\dagger[R]\,\vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{u}}'\,U[R] = R_{ij}\hat{u}'_i J_j\,. \tag{18}$$

But R is a real orthogonal matrix, which implies that $R^{\mathsf{T}} = R^{-1}$, Hence,

$$R_{ij}\hat{u}'_i = R^{\mathsf{T}}_{ji}\hat{u}'_i = (R^{-1}\hat{\boldsymbol{u}}')_j = \hat{u}_j\,, \tag{19}$$

where we used $\hat{\boldsymbol{u}} = R^{-1}\hat{\boldsymbol{u}}'$, as specified by Theorem 1. Inserting the result of eq. (19) back into eq. (18) yields,

$$U^\dagger[R]\,\vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{u}}'\,U[R] = \vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{u}}\,. \tag{20}$$

Finally, we shall make use of the fact that $U[R]$ is unitary, i.e., $U[R]\,U^\dagger[R] = \mathbf{I}$, where $\mathbf{I}$ is the identity operator. Hence, if one multiplies eq. (20) on the left by $U[R]$ and on the right by $U^\dagger[R]$, one obtains eq. (16).

APPENDIX B: Proof that $e^{i\alpha(j,j)} = 1$

We shall evaluate $e^{i\alpha(j,j)}$ by considering the Clebsch-Gordan series for the product of two d-functions [cf. eq. (14) of the class handout entitled *Clebsch-Gordan coefficients and the tensor spherical harmonics*]. Choosing the appropriate values for the relevant parameters,

$$\begin{aligned} d^{(j_1)}_{-j_1,j_1}(\pi)\,d^{(j_2)}_{-j_2,j_2}(\pi) = {}& \langle j_1 j_2\,;\,-j_1\,,\,-j_2\,|\,j_1 j_2\,;\,j_1+j_2\,,\,-(j_1+j_2)\rangle\langle j_1 j_2\,;\,j_1 j_2\,|\,j_1 j_2\,;\,j_1+j_2\,,\,j_1+j_2\rangle \\ & \times d^{(j_1+j_2)}_{-j_1-j_2\,,\,j_1+j_2}(\pi)\,. \end{aligned} \tag{21}$$

The Clebsch-Gordan factors above are trivial. In particular, the relation,

$$\langle j_1 j_2\,;\,j_1 j_2\,|\,j_1 j_2\,;\,j_1+j_2\,,\,j_1+j_2\rangle = 1$$

is the convention used to fix the phases of the Clebsch-Gordan coefficients. Moreover, one can show that the Clebsch-Gordan coefficients satisfy the following relation,

$$\langle j_1 j_2\,;\,m_1 m_2\,|\,j_1 j_2\,;\,jm\rangle = (-1)^{j_1+j_2-j}\langle j_1 j_2\,;\,-m_1,-m_2\,|\,j_1 j_2\,;\,j,-m\rangle\,.$$

It therefore follows that

$$\langle j_1 j_2\,;\,j_1 j_2\,|\,j_1 j_2\,;\,j_1+j_2\,,\,j_1+j_2\rangle = \langle j_1 j_2\,;\,-j_1\,,\,-j_2\,|\,j_1 j_2\,;\,j_1+j_2\,,\,-(j_1+j_2)\rangle = 1\,. \tag{22}$$

Hence, using eq. (12), we conclude that

$$e^{i\alpha(j_1,j_1)}\,e^{i\alpha(j_2,j_2)} = e^{i\alpha(j_1+j_2\,,\,j_1+j_2)}\,. \tag{23}$$

Finally, we consider the case of $j=\frac{1}{2}$. Then,[4]

$$d^{(1/2)}_{m'm}(\pi) = \langle \tfrac{1}{2}\,m'|\exp(-i\pi J_y/\hbar)|\tfrac{1}{2}\,m\rangle = e^{-i\pi\sigma_y/2} = -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{24}$$

This means that $d^{(1/2)}_{\frac{1}{2},-\frac{1}{2}}(\pi) = -1$. Comparing with eq. (12), it follows that $e^{i\alpha(\frac{1}{2},\frac{1}{2})} = 1$. In light of eq. (23), we conclude that

$$e^{i\alpha(j,j)} = 1\,, \quad \text{for all values of } j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \tag{25}$$

[4] In deriving eq. (24), we employed $\exp(-i\frac{1}{2}\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}) = \mathbf{I}\cos(\frac{1}{2}\theta) - i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,\sin(\frac{1}{2}\theta)$ [cf. part (b)-(iii) of problem 7 on Problem Set 1] with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{y}}$ and $\theta = \pi$.