DUE: TUESDAY, APRIL 17, 2012

1. Define the traceless symmetric second-rank Cartesian tensor,

$$T_{ij} = x_i x_j - \tfrac{1}{3} r^2 \delta_{ij}\,,$$

where $\vec{\boldsymbol{x}} \equiv (x_1, x_2, x_3)$ and $r^2 \equiv x_1^2 + x_2^2 + x_3^2$.

(a) Write T_{12}, T_{13}, and $T_{11} - T_{22}$ as linear combinations of the components of an irreducible spherical tensor of rank 2.

(b) The expectation value,

$$Q = e \left\langle \alpha, j, m = j \;\middle|\; 3x_3^2 - r^2 \;\middle|\; \alpha, j, m = j \right\rangle\,, \tag{1}$$

is known as the quadrupole moment. In eq. (1), α denotes other unspecified quantum numbers that characterize the state. Evaluate the matrix element,

$$e \left\langle \alpha, j, m' \;\middle|\; x_1^2 - x_2^2 \;\middle|\; \alpha, j, m = j \right\rangle\,,$$

where $m' = j, j-1, j-2, \ldots$, in terms of Q and the appropriate Clebsch-Gordon coefficients.

(c) Using the Wigner-Eckart theorem, prove that a spin-$\frac{1}{2}$ particle cannot possess a static quadrupole moment.

2. Exercise 15.3.3 on p. 420 of Shankar. [In previous printings, this is given as Exercise 15.3.4. This problem asks you to prove the projection theorem.]

3. In quantum mechanics, the time reversal operator Θ acting on a state produces a state that evolves backwards in time. That is, if we write the usual time evolution,

$$|\Psi(t)\rangle = e^{-iHt/\hbar}\,|\Psi(0)\rangle\,,$$

then

$$\Theta\,|\Psi(-t)\rangle = e^{-iHt/\hbar}\Theta\,|\Psi(0)\rangle\,.$$

As noted by Shankar, the complex conjugated one-component wave function satisfies the Schrodinger equation with $t \to -t$. It follows that

$$\Theta = UK\,,$$

where K is the complex conjugate operator and U is an arbitrary phase. For a $(2j+1)$-component wave function that describes a particle of spin j, then U is a

$(2j + 1) \times (2j + 1)$ unitary matrix. The complex conjugate operator is antiunitary; that is, it satisfies

$$\langle K\Psi | K\Phi\rangle = \langle \Psi|\Phi\rangle^* = \langle \Phi|\Psi\rangle\,.$$

Likewise, the time-reversal operator Θ is antiunitary.

(a) Since the angular momentum changes sign under time reversal, the quantum mechanical angular momentum operator $\vec{\boldsymbol{J}}$ must satisfy

$$\Theta\,\vec{\boldsymbol{J}}\,\Theta^{-1} = -\vec{\boldsymbol{J}}\,.$$

Using this result and the antiunitary nature of K, show that when acting on the $|jm\rangle$ basis,

$$\Theta = \exp(-i\pi J_y/\hbar)K\,,$$

up to an overall phase factor that can be chosen by convention to be unity.

(b) Show that

$$\Theta^2\,|jm\rangle = (-1)^{2j}\,|jm\rangle\ .$$

(c) An irreducible tensor is even or odd under time reversal if,

$$\Theta T^{(k)}_q \Theta^{-1} = \pm(-1)^q T^{(k)}_{-q}\,,$$

where even [odd] corresponds to the plus [minus] sign, respectively. Show that if the theory is time-reversal invariant, then the reduced matrix elements of such an operator must satisfy:

$$\left\langle \alpha\, j||T^{(k)}||\alpha\, j\right\rangle = \pm(-1)^k \left\langle \alpha\, j\|T^{(k)}\|\alpha\, j\right\rangle^*\ .$$

(d) The electric dipole operator is $\vec{\boldsymbol{\rho}} = e\vec{\boldsymbol{x}}$. Prove that if the neutron is observed to have a nonzero electric dipole moment, then both the parity and time reversal symmetries are separately violated.

HINT: Note that $\vec{\boldsymbol{x}}$ is odd under parity and even under time-reversal. For example, $\Theta\,\vec{\boldsymbol{x}}\,\Theta^{-1} = \vec{\boldsymbol{x}}$. Use the result of part (c) in showing that a nonzero electric dipole moment of an angular momentum eigenstate violates time reversal invariance.

4. Derive the following path integral representation of the n-point correlation function,

$$\langle x\,,\,t\,|\,T[X(t_1)X(t_2)\cdots X(t_n)]\,|x'\,,\,t'\rangle = \int \mathcal{D}[x(t)]\,x(t_1)x(t_2)\cdots x(t_n)\,e^{iS[x(t)]/\hbar}\,,$$

where T is the time-ordered product symbol, $S[x(t)]$ is the action [which depends on the path $x(t)$], $X(t)$ is the position operator in the Heisenberg picture, and $x(t)$ is the eigenvalue of $X(t)$ when acting on the position eigenstate $|x\,,\,t\rangle$. Assume that $t \geq t_i$ and $t' \leq t_i$ for all $i = 1, 2, \ldots, n$.

5. Exercise 8.6.1 on p. 234 of Shankar.