DUE: THURSDAY, APRIL 26, 2012

1. You are the creator of a wonderful theory that uniquely predicts the quark-antiquark potential to be:

$$V(\vec{\boldsymbol{r}}) = \frac{a|\vec{\boldsymbol{r}}|}{\hbar c}\,, \tag{1}$$

with $a = 0.24\ \mathrm{GeV}^2$. Note that $\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2$, where $\vec{\boldsymbol{r}}_1$ is the location of the quark and $\vec{\boldsymbol{r}}_2$ is the location of the antiquark.

However, before you can claim The Prize, there is a small formality: a comparison between theory and experiment.

(a) Using the potential specified in eq. (1), calculate the energy differences,[1]

$$\Delta E_n \equiv E_{ns} - E_{1s}\,,$$

for a few lowest lying s-states ($\ell = 0$) of the bound state of a charmed quark and charmed antiquark, called charmonium ($c\bar{c}$), and the lowest lying s-states of the bound state of a bottom quark and bottom antiquark, called bottomonium ($b\bar{b}$). Do the calculation in two ways:

(i) Use the WKB approximation.

(ii) Solve the problem exactly. (A table of zeros of Airy functions and other useful facts can be found in *Handbook of Mathematical Functions*, by Abramowitz and Stegun.)

How well does the WKB approximation do?

(b) Using $m_c = m_{\bar{c}} = 1.5\ \mathrm{GeV/c^2}$, and $m_b = m_{\bar{b}} = 4.5\ \mathrm{GeV/c^2}$ for the masses of the charmed and bottom quarks, respectively, compare the theoretical results obtained in part (a) with the experimental results for ΔE_2, ΔE_3, ΔE_4, which are respectively given by:

(i) for charmonium: 0.589 GeV, 0.942 GeV, 1.324 GeV.

(ii) for bottomonium: 0.563 GeV, 0.895 GeV, 1.119 GeV.

(Are you going to travel to Stockholm after all?)

[1] Here, n is the principal quantum number, where E_{1s} corresponds to the ground state energy.

2. Consider the one-dimensional problem of tunneling through a smooth potential barrier, $V(x)$. Let x_1 and x_2 be the two turning points, and let

$$\kappa(x) \equiv \frac{\sqrt{2m(V(x)-E)}}{\hbar}\,, \qquad \text{for} \quad x_1 < x < x_2\,,$$

where E is the energy of the particle. Define the quantity:

$$S \equiv \exp\left\{-\int_{x_1}^{x_2} \kappa(x')\,dx'\right\}\,.$$

(a) Show that in order for the WKB approximation to be valid, we must have $S \ll 1$.

(b) Show that the transmission coefficient in the WKB approximation is given by:

$$T = \left(\frac{S}{4} + \frac{1}{S}\right)^{-2}\,.$$

Assuming that the WKB approximation is valid, we can use $S \ll 1$ to approximate the above result by

$$T \simeq S^2 = \left[\exp\left\{-\int_{x_1}^{x_2} \kappa(x')\,dx'\right\}\right]^2\,.$$

<u>*HINT*</u>: Write separate formulae for the WKB approximate wave functions in the three regions: I ($x < x_1$), II ($x_1 < x < x_2$) and III ($x > x_2$). Assume an incident wave and a reflected wave in region I, and a transmitted wave in region III. The wave function in region II will consist of a growing and a decaying exponential, so you will need to derive two new connection formulae to connect the growing exponential in region II to the corresponding wave functions in region I and region III, respectively. Once you have the relevant connection formulae, then you can write down two expressions for the wave function in region II. Setting these two expressions equal, one can deduce an expression for the transmission coefficient.

3. Consider a charged particle (with charge q) whose motion is confined to a circle of radius R in the x–y plane, with its center at the origin. A thin magnetic flux tube of radius $r < R$ is located with its axis along the z-axis. The magnetic field is confined within the flux tube, and the total magnetic flux through the x–y plane is denoted by Φ. In particular, the charged particle moves in a region where there is no magnetic field. It is convenient to work in cylindrical coordinates (ρ, θ, z), where $x = \rho\cos\theta$ and $y = \rho\sin\theta$. In the region where there is no magnetic field, $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = 0$, which implies that

$$\vec{\boldsymbol{A}}(\rho,\theta,z) = \vec{\boldsymbol{\nabla}}\chi(\rho,\theta,z)\,. \tag{2}$$

(a) Noting that Stokes' theorem relates Φ to the line integral of $\vec{\boldsymbol{A}}$ taken along the circle of radius R, show that the choice,

$$\chi(\rho, \theta, z) = \frac{\Phi\theta}{2\pi}\,,$$

satisfies Stokes' theorem and the Coulomb gauge condition.

HINT: Insert the value of χ into eq. (2) and evaluate $\vec{\boldsymbol{A}}$. Show that the vector potential points in the $\hat{\boldsymbol{\theta}}$ direction.

(b) The wave function for the charged particle is only a function of θ (since $\rho = R$ and $z = 0$ are fixed due to the confined motion). Write down the time-independent Schrodinger equation for the charged particle wave function $\psi(\theta)$ in the cylindrical coordinate representation (simplifying your equation as much as possible).

(c) Solve the Schrodinger equation of part (b) for the energy eigenvalues and eigenfunctions. Show that the allowed energies depend on Φ even though the charged particle on the circle never encounters the magnetic field.

HINT: Show that the energy eigenstates are also eigenstates of $\partial/\partial\theta$.

4. The hydrogen atom is placed in a weak uniform electric field of strength $\mathcal{E}$ pointing in the z-direction. The Hamiltonian describing the system is given by:

$$H = \frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2 - \frac{e^2}{r} - e\mathcal{E}z\,.$$

Compute the ground state energy of the system using the variational technique. Use the trial wave function

$$\psi(\vec{\boldsymbol{r}}) = N(1 + q\mathcal{E}z)\psi_{100}(\vec{\boldsymbol{r}})\,,$$

where $\psi_{100}(\vec{\boldsymbol{r}})$ is the ground state wave function of the hydrogen atom (in the absence of an external electric field), q is the variational parameter, and N is chosen such that the trial wave function is properly normalized. Ignore all spin effects (*i.e.*, ignore fine and hyperfine splittings). Since the external electric field is assumed to be weak, simplify your computations by expanding in $\mathcal{E}$ and keeping only the leading term. In particular, show that the first correction to the ground state energy of hydrogen is proportional to $\mathcal{E}^2$.

5. Exercise 17.2.3 on p. 457 of Shankar.