DUE: THURSDAY, MAY 10, 2012

MIDTERM ALERT: The midterm exam is a take-home exam that will be given out in class on Thursday May 10, 2012. This exam must returned no later than 6 pm on Friday May 11. Completed exams should be placed in my physics department mailbox.

While working on the exam, you may refer to Shankar and Baym and any third quantum mechanics textbook of your choosing. (If you do consult a third text, please indicate which one you used.) Any reference for integrals or other mathematical facts, and any personal handwritten notes are also OK. You are also free to consult any of the class handouts, including the solution sets. However, you may *not* collaborate with anyone else. The exam will the first five topics of the course syllabus (and the material covered on first three problem sets of this course).

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1. A system of three unperturbed states consisting of a degenerate pair of states of energy E_1 and a non-degenerate state of energy E_2 is subsequently perturbed, and is represented by the Hamiltonian matrix:

$$\begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix}, \tag{1}$$

where $E_2 > E_1$. The quantities a and b are to be regarded as perturbations that are of the same order but small compared with $E_2 - E_1$.

(a) Use second order non–degenerate perturbation theory to calculate the perturbed eigenvalues. Is this procedure correct?

(b) Use second order degenerate perturbation theory to calculate the perturbed eigenvalues.

(c) Calculate the eigenvalues exactly and compare with the results of parts (a) and (b).

2. Positronium is a bound state of two spin-1/2 particles: an electron (e^-) and a positron (e^+). Consider the Hamiltonian for the system, where we focus only on the spin degrees of freedom. In the presence of a uniform external magnetic field, we may take:

$$H = A(1 - \boldsymbol{\vec{\sigma}_1}\cdot\boldsymbol{\vec{\sigma}_2}) + \mu_B(\boldsymbol{\vec{\sigma}_1} - \boldsymbol{\vec{\sigma}_2})\cdot\boldsymbol{\vec{B}}\,,$$

where A is a constant and μ_B is the Bohr magneton. The labels 1 and 2 refer to the electron and positron respectively.

(a) In zero magnetic field, a transition is observed to occur from the $^3S_1$ state to the $^1S_0$ state (which is the ground state). The emitted photon is observed to have a frequency of 2×10^5 MHz. Evaluate the constant A in units of eV.

(b) Now turn on the magnetic field. Assume it points in the z-direction. Treating the magnetic field as a perturbation, compute the energy eigenvalues to second order in B and the energy eigenstates to first order in B.

(c) Repeat the calculation of part (b), but now solve the problem exactly. Expand out your solutions in a power series in B, and verify that the results of part (b) are indeed correct.

3. Consider the hydrogen atom in the $n = 2$ state. It is placed in a uniform magnetic field B. The Hamiltonian for the hydrogen atom in the non-relativistic limit is a sum of the Coulomb interaction with minimal coupling to the electromagnetic field, the fine structure and the hyperfine structure contributions. That is,

$$H = H_{\rm C} + H_{\rm FS} + H_{\rm HFS}\,.$$

In this problem the hyperfine structure will be ignored. The Coulomb Hamiltonian with minimal coupling to the electromagnetic field is given by:

$$H_{\rm C} = \frac{\vec{\boldsymbol{p}}^{\,2}}{2m} - \frac{e^2}{r} + \frac{e}{2m_e c}\vec{\boldsymbol{B}}\boldsymbol{\cdot}(\vec{\boldsymbol{L}} + 2\vec{\boldsymbol{S}}) + \frac{e^2}{8m_e c^2}(\vec{\boldsymbol{B}} \times \vec{\boldsymbol{r}})^2\,,$$

and the fine structure Hamiltonian is given by:

$$H_{\rm FS} = m_e c^2 - \frac{\vec{\boldsymbol{p}}^4}{8m_e^3 c^2} + \frac{1}{2m_e^2 c^2 r}\frac{dV}{dr}\vec{\boldsymbol{L}}\boldsymbol{\cdot}\vec{\boldsymbol{S}} + \frac{\hbar^2}{8m_e^2 c^2}\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}} V(\vec{\boldsymbol{r}})\,.$$

Here $\vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{S}}$ are the orbital and spin angular momentum of the electron, respectively.

(a) Using first order degenerate perturbation theory, compute the energy levels as a function of B, assuming that the contributions due to the fine structure and due to the term in the Hamiltonian that is linear in B are roughly of equal strength. You may neglect the term in the Hamiltonian that is quadratic in B. Also, neglect the spin of the proton and the associated hyperfine structure.

HINT: Treat both $H_{\rm FS}$ and the term of $H_{\rm C}$ that depends linearly in $\vec{\boldsymbol{B}}$ field as perturbations. For the latter, you can employ the first order energy splitting due to the fine structure Hamiltonian obtained in class [cf. eq. (17.3.22) on p. 469 of Shankar],

$$E^{(1)}_{\rm FS} = -\frac{m_e c^2 \alpha^2}{2n^2} \cdot \frac{\alpha^2}{n}\left(\frac{1}{j + 1/2} - \frac{3}{4n}\right),$$

where j is the total angular momentum. In addition, the following Clebsch-Gordon coefficients may be useful:

$$\langle \ell \,, m - m_s \,; \tfrac{1}{2} \; m_s \,|\, \ell + \tfrac{1}{2} \,, m \rangle = \left[\frac{\ell + 2mm_s + \frac{1}{2}}{2\ell + 1} \right]^{1/2} ,$$

$$\langle \ell \,, m - m_s \,; \tfrac{1}{2} \; m_s \,|\, \ell - \tfrac{1}{2} \,, m \rangle = (-1)^{m_s + \frac{1}{2}} \left[\frac{\ell - 2mm_s + \frac{1}{2}}{2\ell + 1} \right]^{1/2} .$$

(b) Compute the limits of large and small B.

(c) Sketch a picture of the energy levels found in part (a) as a function of B.

4. Consider the hydrogen atom in an excited $n = 2$ state, which is subjected to an external uniform electric field $\mathcal{E}$. Do *not* neglect the spin of the electron. Assume that the field $\mathcal{E}$ is sufficiently weak so that $e\mathcal{E}a_0$ is small compared to the fine structure, but such that the Lamb shift δ ($\delta = 1057$ MHz) cannot be neglected. That is, treat the problem as a two-level system consisting of the the $2S_{1/2}$ and $2P_{1/2}$ states of hydrogen. In particular, you may ignore the $2P_{3/2}$ state of hydrogen and the hyperfine interactions.

(a) Compute the Stark effect for the $2S_{1/2}$ and $2P_{1/2}$ states of hydrogen by solving the two-level system exactly.

HINT: When electron spin is included, the hydrogen atom energy eigenstates are two-component wave functions given (in the coordinate representation) by:

$$\psi(\boldsymbol{\vec{r}}) = R_{n\ell}(r) \mathcal{Y}_{jm}^{\ell \, \frac{1}{2}}(\theta, \phi) \,,$$

where $R_{n\ell}(r)$ is the radial wave function of the hydrogen atom, and the spin spherical harmonics, $\mathcal{Y}_{jm}^{\ell \, \frac{1}{2}}(\theta, \phi)$, are defined in the class handout entitled, *Clebsch-Gordon coefficients and the tensor spherical harmonics.*

(b) Show that for $e\mathcal{E}a_0 \ll h\delta$, the energy shifts due to the external electric field are quadratic in $\mathcal{E}$, whereas for $e\mathcal{E}a_0 \gg h\delta$, they are linear in $\mathcal{E}$. Determine the (perturbed) energy eigenstates in both limiting cases.[1]

(c) The critical field is defined as:

$$\mathcal{E}_c \equiv \frac{h\delta}{\sqrt{3} e a_0} \,,$$

where the factor of $\sqrt{3}$ is conventional. The linear or quadratic behavior of the energy shifts obtained in part (b) depend on the magnitude of $\mathcal{E}$ as compared to $\mathcal{E}_c$. Determine $\mathcal{E}_c$ in volts/cm.

[1] Since δ is a frequency rather than an angular frequency, Planck's constant h appears in this problem instead of the usual $\hbar \equiv h/(2\pi)$.

5. Consider the scattering of particles by the square well potential in three dimensions:

$$V(r) = \begin{cases} -V_0\,, & \text{for } r < a\,, \\ 0\,, & \text{for } r > a\,, \end{cases} \tag{2}$$

where V_0 is positive.

(a) Obtain the differential cross-section in the Born approximation.

(b) Using the results of part (a), evaluate the total cross-section in the limits of low and high energy. Specifically, show that at low energies, the cross section can be approximated by:

$$\sigma \simeq \sigma_0(1 + Ak^2)\,,$$

where the energy $E = \hbar^2 k^2/(2m)$. You should evaluate the constants σ_0 and A. In the high energy limit, show that

$$\sigma \simeq \frac{C}{k^2}\,,$$

where the constant C should be determined.

HINT: To determine the total cross-section in the high-energy limit, you should convert the integral to a manageable form before making any approximations. First integrate over the azimuthal angle. Then, change variables from $\cos\theta$ to

$$y = 2ka\sin(\theta/2) = ka[2(1 - \cos\theta)]^{1/2}\,,$$

and express the total cross-section as an integral over y. Now, you can evaluate the integral by taking the infinite energy limit. However, the resulting integral is somewhat tricky. So, I will help you out by providing the following result:

$$\int_0^\infty \frac{[j_n(y)]^2}{y^p}\,dy = \frac{2^{p-2}\,\Gamma\left(\frac{2n+1-p}{2}\right)\Gamma^2\left(\frac{p+1}{2}\right)}{\Gamma(p+1)\,\Gamma\left(\frac{2n+p+3}{2}\right)}\,, \qquad (-1 < \text{Re } p < 2n+1)\,.$$

where $j_n(y)$ is a spherical Bessel function and $\Gamma^2(z)$ is the square of the gamma function. Show that the integral you are trying to evaluate corresponds to a specific choice of n and p above. Then evaluate it.

(c) What is the range of validity of your answers to parts (a) and (b). Consider separately the limits of low and high energy.

(d) Using the results of part (a), it is possible to perform the integral exactly and obtain an expression for the total cross-section. Obtain the exact formula for the cross-section in the Born approximation. Then, check the low and high energy limits and confirm the results of part (b).