FINAL EXAM INSTRUCTIONS: This is an open book exam. You are permitted to consult the textbooks of Shankar and Baym (and any third quantum mechanics text of your choosing), your handwritten notes, and any class handouts that are posted on the course website. One mathematical reference book is also permitted. No other consultations or collaborations are permitted during the exam. *In order to earn total credit for a problem solution, you must show all work involved in obtaining the solution.* However, you are *not* required to re-derive any formulae that you cite from the textbooks or the class handouts.

The exam consists of twelve individual parts (and one part extra credit at the very end), each of which is worth ten points. Use this information to manage your time appropriately during the exam.

1. Consider a positively-charged spin-1/2 particle in an external magnetic field, governed by the Hamiltonian:

$$H = H_0\,\mathbf{I} - \gamma\vec{\boldsymbol{B}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}\,,$$

where $\mathbf{I}$ is the identity operator in spin space, $\vec{\boldsymbol{S}}$ is the vector of spin-1/2 spin matrices, and γ is a constant (for a positively-charged particle, $\gamma > 0$). H_0 is spin-independent and is independent of the magnetic field $\vec{\boldsymbol{B}}$. For simplicity, assume that H_0 possesses exactly one eigenvalue, which is denoted by E.

(a) If the magnetic field is given by $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$ (where $B > 0$), determine the energy eigenstates and eigenvalues of H.

(b) Assume that the magnetic field is given by $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$ for time $t < 0$. The system is initially observed to be in a spin-up state. At $t = 0$, a time-dependent perturbation is added by modifying the magnetic field. The new magnetic field for $t > 0$ is given by:

$$\vec{\boldsymbol{B}} = b\,(\hat{\boldsymbol{x}}\cos\omega t - \hat{\boldsymbol{y}}\sin\omega t) + B\hat{\boldsymbol{z}}\,,$$

where $b > 0$. Using first-order time-dependent perturbation theory, derive an expression for the probability that the system will be found in a spin-down state at some later time $t = T$. For what range of values of ω is this result reliable?

2. Consider the scattering of spinless particles in an attractive exponential spherically symmetric potential:

$$V(r) = -V_0\exp(-r/r_0)\,,$$

with $V_0 > 0$. It is convenient to define two dimensionless variables for this problem: $\xi \equiv kr_0$ and $\eta \equiv 2mV_0r_0^2/\hbar^2$, where $\hbar^2k^2/(2m)$ is the energy of the incoming beam.

(a) Compute, the scattering amplitude and the differential and total cross sections, in the Born approximation, in terms of the variables ξ, η and r_0. Evaluate the total cross section in the low energy limit.

(b) Using the scattering amplitude obtained in part (a), calculate the s-wave and p-wave phase shifts. [*NOTE*: it is sufficient to evaluate $e^{i\delta_\ell}\sin\delta_\ell$ for $\ell = 0,\, 1$.]

<u>*HINT*</u>: Expand the Born approximated scattering amplitude in a partial wave expansion, and use the orthogonality of the Legendre polynomials to obtain expressions for $e^{i\delta_\ell}\sin\delta_\ell$ for $\ell = 0$ and $\ell = 1$ in terms of ξ and η.

(c) Using the results of part (b), compute both the s-wave and p-wave phase shifts in the low energy limit. Do you find the expected behavior at low energies?

(d) At low energies, the angular distribution of scattering is approximately given by

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta\,.$$

Using the results of parts (b) and (c), compute the leading behavior of B/A as $k \to 0$. Are your results consistent with the differential cross section obtained in part (a)?

3. Consider the hydrogen atom, where the fine structure and the Lamb shift are included, but the hyperfine structure is neglected. The three lowest energy states (in order of increasing energy) are: $1s_{1/2}$, $2p_{1/2}$, and $2s_{1/2}$, where the notation $n\ell_j$ is used to label the states. The latter two states are separated by the Lamb shift ($\nu = 1058$ MHz).

(a) Using selection rules, determine to which state the $2s_{1/2}$ state can decay via an $E1$ transition.

(b) Compute the $E1$ transition rate for the decay of the $2s_{1/2}$ state and determine the numerical value of the corresponding lifetime. Compare this result with the lifetime of the $2p$ state of hydrogen computed in class.

<u>*HINT*</u>: You can neglect the electron spin and treat this as a decay of the $2s$ state. However, the resulting transition rate will be a factor of three too large. Can you explain this discrepancy?

(c) Can the $2s_{1/2}$ state decay via an $E2$ transition? Explain.

(d) Using selection rules, determine to which state the $2s_{1/2}$ state can decay via an $M1$ transition. By using explicit wave functions, evaluate the matrix element of the magnetic dipole operator, $\langle f|\vec{\boldsymbol{\mu}}|i\rangle$, and show that the $M1$ transition rate vanishes.

<u>*HINT:*</u> Recall that $\vec{\boldsymbol{\mu}}$ depends on the electron spin operator. Thus, you will need to employ hydrogenic wave functions that depend on the electron spin.

<u>*NUMERICAL DATA:*</u> The rest energy of the electron is $mc^2 = 5.11 \times 10^5$ eV, and

$$\hbar = 6.58 \times 10^{-16}\ \text{eV}\cdot\text{sec}\,, \qquad a_0 \equiv \frac{\hbar^2}{me^2} = 5.29 \times 10^{-9}\ \text{cm}\,, \qquad \alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,.$$

4. Two electrons are in plane wave states in a box of volume V. The Hamiltonian governing this system is

$$H = \frac{\vec{\boldsymbol{p}}_1^{\,2}}{2m} + \frac{\vec{\boldsymbol{p}}_2^{\,2}}{2m} + \frac{e^2}{|\vec{\boldsymbol{x}}_1 - \vec{\boldsymbol{x}}_2|}\,,$$

where the last term above is a result of the Coulomb interactions of the electrons. The second-quantized Hamiltonian for this system in terms of creation and annihilation operators is given by

$$H = \sum_{\vec{\boldsymbol{p}},s} \frac{\vec{\boldsymbol{p}}^{\,2}}{2m}\, a^\dagger_{\vec{\boldsymbol{p}},s} a_{\vec{\boldsymbol{p}},s} + \tfrac{1}{2} \sum_{\vec{\boldsymbol{p}}',\vec{\boldsymbol{q}}',s'} \sum_{\vec{\boldsymbol{p}},\vec{\boldsymbol{q}},s} a^\dagger_{\vec{\boldsymbol{p}},s} a^\dagger_{\vec{\boldsymbol{p}}',s'} a_{\vec{\boldsymbol{q}}',s'} a_{\vec{\boldsymbol{q}},s} \left\langle \vec{\boldsymbol{p}}, \vec{\boldsymbol{p}}' \right| \mathcal{V} \left| \vec{\boldsymbol{q}}', \vec{\boldsymbol{q}} \right\rangle\,, \tag{1}$$

where the spin variables s and s' can take on two possible values ($\pm\frac{1}{2}$) and

$$\left\langle \vec{\boldsymbol{p}}, \vec{\boldsymbol{p}}' \right| \mathcal{V} \left| \vec{\boldsymbol{q}}', \vec{\boldsymbol{q}} \right\rangle = \frac{1}{V^2} \int d^3x\, d^3x'\, \frac{e^2}{|\vec{\boldsymbol{x}}_1 - \vec{\boldsymbol{x}}_2|}\, e^{-i(\vec{\boldsymbol{p}}-\vec{\boldsymbol{q}})\cdot\vec{\boldsymbol{x}}/\hbar}\, e^{-i(\vec{\boldsymbol{p}}'-\vec{\boldsymbol{q}}')\cdot\vec{\boldsymbol{x}}'/\hbar}\,.$$

A two-particle electron state is given by

$$\left|\vec{\boldsymbol{p}}, s\,;\, \vec{\boldsymbol{p}}', s'\right\rangle = a^\dagger_{\vec{\boldsymbol{p}},s} a^\dagger_{\vec{\boldsymbol{p}}',s'} \left|0\right\rangle\,, \tag{2}$$

where $|0\rangle$ is the state with no electrons.

(a) Compute the expectation value,

$$\left\langle \vec{\boldsymbol{p}}, s\,;\, \vec{\boldsymbol{p}}', s' \right| H \left| \vec{\boldsymbol{p}}, s\,;\, \vec{\boldsymbol{p}}', s' \right\rangle\,, \tag{3}$$

in the case where $e = 0$ (i.e. where the Coulomb interactions are switched off). Explain the behavior of your result in the case of $\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}'$ and $s = s'$.

HINT: In order to compute the expectation value above, insert eqs. (1) and (2) into eq. (3) [after setting $e = 0$] and then employ the anticommutation relations of the creation and annihilation operators.

(b) Treating the Coulomb interactions to first-order in perturbation theory, compute the energy *difference* of the parallel ($s = s'$) and antiparallel ($s \neq s'$) spin alignments of the two electrons. Express your answer as a volume integral over the box.

HINT: Express the first-order energy difference in the coordinate basis. To evaluate the resulting integrals, consider new variables $\vec{\boldsymbol{R}} \equiv \frac{1}{2}(\vec{\boldsymbol{x}} + \vec{\boldsymbol{x}}')$ and $\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'$. Integrating over the box of volume V, the integral over $\vec{\boldsymbol{R}}$ is trivial, and one is left with a volume integral over $\vec{\boldsymbol{r}}$. Note that your final answer will depend explicitly on the volume of the box.

(c) [*EXTRA CREDIT*] Calculate the energy difference ΔE of the parallel and antiparallel spin alignments by evaluating the volume integral obtained in part (b) assuming that $|\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}'|L \gg 1$, where L is the length of a side of the cubical box. (Under the latter assumption, the volume integral can be evaluated in the infinite volume limit.) Show that in this case ΔE behaves inversely with the volume V. How does ΔE depend on V in the limiting case of $\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}'$?