

Airy functions and their properties

Airy functions are defined as solutions to the following differential equation:

$$\frac{d^2\psi}{dz^2} - z\psi(z) = 0.$$

The two independent solutions are the Airy functions of the first and second kind:

$$\psi(z) = C_1 \text{Ai}(z) + C_2 \text{Bi}(z),$$

where C_1 and C_2 are constant coefficients to be determined by the boundary conditions of the problem. The Airy functions are related to Bessel functions of one-third order:

$$\begin{aligned} \text{Ai}(z) &= \frac{z^{1/2}}{3} \left[I_{-1/3} \left(\frac{2z^{3/2}}{3} \right) - I_{1/3} \left(\frac{2z^{3/2}}{3} \right) \right], \\ \text{Bi}(z) &= \left(\frac{z}{3} \right)^{1/2} \left[I_{-1/3} \left(\frac{2z^{3/2}}{3} \right) + I_{1/3} \left(\frac{2z^{3/2}}{3} \right) \right], \end{aligned}$$

where $I_\nu(z)$ is the Bessel function of imaginary argument.

The most important properties of the Airy functions for our purposes are the following asymptotic expansions, valid in the limit of real $z \rightarrow \infty$:

$$\begin{aligned} \text{Ai}(z) &\sim \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right) \\ \text{Ai}(-z) &\sim \frac{1}{\sqrt{\pi}} z^{-1/4} \cos\left(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\right) \\ \text{Bi}(z) &\sim \frac{1}{\sqrt{\pi}} z^{-1/4} \exp\left(+\frac{2}{3}z^{3/2}\right) \\ \text{Bi}(-z) &\sim \frac{-1}{\sqrt{\pi}} z^{-1/4} \sin\left(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\right) \end{aligned}$$

REFERENCE:

N.N. Lebedev, *Special Functions and Their Applications* (Dover Publications Inc., Mineola, NY, 1972).