Physics 195a
Course Notes
Path Integrals: An Example – Solutions to Exercises
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1 Exercises

1. Show that the **B** field is as given in Eqn. 6, and that the vector potential is as given in Eqn. 7, up to gauge transformations.

   Solution: We wish to show that:
$$\mathbf{B} = \begin{cases} 0 & \text{outside solenoid} \\ B_0\mathbf{e}_z & \text{inside solenoid,} \end{cases} \tag{1}$$
   Use the form of Maxwell's equations which states:
$$\oint \mathbf{B}\cdot d\ell = 4\pi I_{\text{enclosed}}. \tag{2}$$
   Consider a circle of radius r, centered on the solenoid and perpendicular to the z axis, where r may be larger or smaller than the solenoid radius. In either case the enclosed current is zero, and hence $B_\phi = 0$ both inside and outside the solenoid (using the circular symmetry to do the line integral of the magnetic field). etc.

2. Verify that the wave function in Eqn. 17 satisfies the Schrödinger equation.

   Solution: The Schrödinger equation is
$$i\partial_t\psi_\Phi = \frac{1}{2m}(-i\nabla - q\mathbf{A})^2\psi \tag{3}$$
   We are given left and right solutions that satisfy the Schrödinger equation with $\mathbf{A} = 0$:
$$\begin{aligned} i\partial_t\psi_\Phi &= i\partial_t\psi_\ell e^{iqs_\ell} + i\partial_t\psi_r e^{iqs_r} \\ &= -\frac{1}{2m}e^{iqs_\ell}\nabla^2\psi_\ell - \frac{1}{2m}e^{iqs_r}\nabla^2\psi_r. \end{aligned} \tag{4}$$
   Consider now
$$\begin{aligned} (-i\nabla - q\mathbf{A})^2\psi_\ell e^{iqs_\ell} &= (-i\nabla - q\mathbf{A})\cdot\left[(-i\nabla\psi_\ell)e^{iqs_\ell} + \psi_\ell(q\nabla s_\ell)e^{iqs_\ell} - q\mathbf{A}\psi_\ell e^{iqs_\ell}\right] \\ &= (-i\nabla - q\mathbf{A})\cdot(-i\nabla\psi_\ell)e^{iqs_\ell} \\ &= -\nabla^2\psi_\ell e^{iqs_\ell}. \end{aligned} \tag{5}$$

Repeat for ψ_r and plug into the Schrödinger equation.

3. A number of assumptions have been made, possibly implicitly in the discussion of this effect.

   (a) Critique the discussion, pointing out areas where the argument may break down.

   (b) Resolve the problematic areas in your critique, or else demonstrate that the argument really does break down.

   Solution:

4. We have discussed the interesting Aharonov-Bohm effect. Let us continue a bit further the thinking in this example.

   (a) Consider again the path integral in the vicinity of the long thin solenoid. In particular, consider a path which starts at $\boldsymbol{x}$, loops around the solenoid, and returns to $\boldsymbol{x}$. Since the $\boldsymbol{B}$ and $\boldsymbol{E}$ fields are zero everywhere in the region of the path, the only effect on the particle's wave function in traversing this path is a phase shift, and the amount of phase shift depends on the magnetic flux in the solenoid, as we discussed in the note. Suppose we are interested in a particle with charge of magnitude e (*e.g.*, an electron). Show that the magnetic flux Φ in the solenoid must be quantized, and give the possible values that Φ can have.

   Solution: We know that the total change in phase of the wave function in traversing a loop is given by:

$$\begin{aligned}\Delta\theta &= e\oint A\cdot ds.\\ &= e\Phi,\end{aligned}$$

   where the second equality holds if the loop encloses the solenoid. Single-valuedness for the wave function imposes the constraint that $\Delta\theta$ must be an integral muliple of 2π. Hence, we can only have:

$$e\Phi = 2\pi n, \qquad \text{where } n = \ldots, -2, -1, 0, 1, 2, \ldots.$$

   BUT SEE BELOW!!

(b) Wait a minute!!! Did you just show that there is no Aharonov-Bohm effect? We know from experiment that the effect is real. So, if you did what I expected you to do in part (a), there is a problem. Discuss!

Solution: We may see a hint of the problem by noticing that the answer for the quantum of magnetic flux obtained in part (a) seems to depend on the charge of the particle being used to probe the vector potential. If we use a different charge, we'll get a different quantum – the answer isn't intrinsic to the solenoid alone.

We'll discuss this further in class. It'll motivate us into a discussion of Berry's phase.

(c) The BCS theory for superconductivity assumes that the basic "charge carrier" in a superconductor is a pair of electrons (a "Cooper pair"). The Meissner effect for a (Type I) superconductor is that when such a material is placed in a magnetic field, and then cooled below a critical temperature, the magnetic field is excluded from the superconductor. Suppose that there is a small non-superconducting region traversing the superconductor, in which magnetic flux may be "trapped" as the material is cooled below the critical temperature. Ignoring part (b) above, what values do you expect to be possible for the trapped flux? [This effect has been experimentally observed.] What is the value (in Tesla-m^2) of the smallest non-zero flux value. You may find the following conversion constant handy:

$$1 = 0.3 \text{ Tesla-m/GeV}, \tag{6}$$

where the "0.3" is more precisely the speed of light in nanometers/second.

Solution: In this case, the "basic" charge carriers have charge magnitude $2e$, and hence the quantization condition in part (a) becomes:

$$\Phi_n = \frac{2\pi n}{2e}, \qquad \text{where } n = \ldots, -2, -1, 0, 1, 2, \ldots.$$

Let's determine the value of the flux for $n = 1$. We have to be a bit careful about our units here. If we were to blindly say $e = \sqrt{\alpha}$, we would get:

$$
\begin{aligned}
\Phi_1 &= \frac{\pi}{e} = \frac{\pi}{\sqrt{1/137}} 0.3\,\text{T-m/GeV} \times 10^{-3}\,\text{GeV/MeV} \times 197\,\text{MeV-fm} \times 10^{-15}\,\text{m/fm} \\
&\approx 2.2 \times 10^{-15}\ \text{T-m}^2. \quad \text{(WRONG)} \qquad (7)
\end{aligned}
$$

But this isn't quite right for the MKS system. Actually,

$$\frac{e^2}{4\pi\epsilon_0} = \alpha, \qquad (8)$$

or, since $\epsilon_0 = 1/\mu_0$, and $\mu_0 = 4\pi \times 10^{-7}$ N/A^2,

$$e^2 = \frac{10^7}{137}\ \text{A}^2/\text{N}. \qquad (9)$$

Thus,

$$\Phi_1 = \frac{\pi}{e} = \pi\sqrt{137 \times 10^7\,\text{N/A}^2}. \qquad (10)$$

So what is $\sqrt{\text{N/A}^2}$, in T-m^2 (or Webers)? Well,

$$
\begin{aligned}
\sqrt{\frac{\text{N}}{\text{A}^2}} &= \sqrt{\frac{\text{kg-m/s}^2}{(\text{kg/T-s}^2)^2}} \\
&= \text{T-m}^2\sqrt{\frac{\text{s}^2}{\text{m}^3\text{kg}}} \\
&= \text{T-m}^2\sqrt{\frac{1}{\text{m-J}}} \\
&= \text{T-m}^2\sqrt{\frac{1}{\text{m-J}} 1.602 \times 10^{-19}\frac{\text{J}}{\text{eV}} 197 \times 10^{-9}\,\text{eV-m}} \\
&= 1.78 \times 10^{-13}\ \text{T-m}^2. \qquad (11)
\end{aligned}
$$

Hence,

$$\Phi_1 \approx 2.07 \times 10^{-15}\ \text{T-m}^2. \qquad (12)$$

(d) How can we reconcile the answer to part (c), which turns out to be a correct result (even if the derivation might be flawed), with your discussion in part (b)? Let's examine the superconducting case more carefully. Let us suppose we have a ring of superconducting material. We assume a model for superconductivity in which the superconducting electrons are paired, and the resulting pairs are in a "Bose-condensate". Well, this precedes our discussion on identical particles, but we essentially mean that the pairs are all in the same quantum state. We may write our wave function for the superconducting pairs in the form:

$$\psi(\mathbf{x}) = \sqrt{\frac{\rho_s}{2}} e^{i\theta(\mathbf{x})}, \tag{13}$$

where ρ_s is the number density of superconducting electrons, and θ is a position-dependent phase. Note that we have normalized our wave function so that its absolute square gives the density of Cooper pairs. Find an expression for $\frac{\rho_s}{2}\mathbf{v}$, the Cooper pair number current density. Use this with the expression for the canonical momentum of a Cooper pair in a magnetic field (vector potential $\mathbf{A}$) to arrive at an expression for the electromagnetic current density of the superconducting electrons.

Now consider the following scenario: We apply an external magnetic field with the superconductor above its critical temperature (that is, not in a superconducting state). We then cool this system down below the critical temperature. We want to know what we can say about any magnetic flux which is trapped in the hole in the superconductor. Consider a contour in the interior of the superconductor, much further from the surfaces than any penetration depths. By considering an integral around this contour, see what you can say about the allowed values of flux through the hole.

Solution: We first look for an expression for the Cooper pair number current density. The Schrödinger equation for a free particle is:

$$i\partial_t \psi(\mathbf{x}, t) = -\frac{\nabla^2}{2m}\psi(\mathbf{x}, t), \tag{14}$$

where m is the mass of the electron. Multiply this equation by ψ^*. Take the complex conjugate of the Schrödinger equation, and multiply by ψ. Finally, take the difference between the two results, to obtain:

$$\psi^* i\partial_t \psi - \psi(-i)\partial_t \psi^* = -\psi^* \frac{\nabla^2}{2m}\psi - (-)\psi\frac{\nabla^2}{2m}\psi^* \tag{15}$$

$$
\begin{aligned}
i\partial_t|\psi|^2 &= \frac{1}{2m}\left(-\psi^*\nabla^2\psi + \psi\nabla^2\psi^*\right) && (16)\\
i\partial_t\frac{\rho_s}{2} &= \frac{1}{2m}\nabla\cdot\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right). && (17)
\end{aligned}
$$

This is of the form of a continuity equation ($\partial_t Q + \nabla\cdot\mathbf{J} = 0$), and we read off the Cooper pair number current density:

$$
\frac{\rho_s}{2}\mathbf{v} = \frac{i}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right), \tag{18}
$$

where v is the Cooper pair speed (assumed to be non-relativistic, of course).

Now for the canonical momentum of a Cooper pair. For this purpose, we see that a Cooper pair is a "particle" of charge $-2e$ and mass $2m$. The canonical momentum is

$$
\mathbf{p} = -i\nabla, \tag{19}
$$

and is related to the kinematic momentum $2m\mathbf{v}$ in a magnetic field by

$$
\mathbf{p} = 2m\mathbf{v} + 2e\mathbf{A}. \tag{20}
$$

We take expectation values to find:

$$
\langle\mathbf{p}\rangle = \nabla\theta = 2m\langle\mathbf{v}\rangle + 2e\mathbf{A}. \tag{21}
$$

The electromagnetic current density carried by Cooper pairs is thus:

$$
\begin{aligned}
\langle\mathbf{J_{em}}\rangle &= 2e\frac{\rho_s}{2}\langle\mathbf{v}\rangle && (22)\\
&= \frac{e^2\rho_s}{m}\left(\frac{1}{2e}\nabla\theta - \mathbf{A}\right). && (23)
\end{aligned}
$$

Now for our scenario. The essential physical point, which makes this example different from the Aharonov-Bohm situation, is that, deep enough into the superconductor, the Cooper-pair current density is zero. We integrate around the contour to obtain:

$$
\frac{1}{2e}\oint_C \nabla\theta\cdot d\mathbf{s} = \oint_C \mathbf{A}\cdot d\mathbf{s} = \Phi. \tag{24}
$$

Single-valuedness of the wave function implies that

$$\Phi = \frac{1}{2e}\oint_C \nabla\theta \cdot d\mathbf{s} = \frac{1}{2e}2\pi k, \tag{25}$$

where k must be an integer.

(e) So far, no one has observed (at least not convincingly) a magnetic "charge", analogous to the electric charge. But there is nothing fundamental that seems to prevent us from modifying Maxwell's equations to accommodate the existence of such a "magnetic monopole". In particular, we may alter the divergence equation to:

$$\nabla \cdot \boldsymbol{B} = 4\pi\rho_M,$$

where ρ_M is the magnetic charge density.

Consider a magnetic monopole of strength e_M located at the origin. The $\boldsymbol{B}$-field due to this charge is simply:

$$\boldsymbol{B} = \frac{e_M}{r}\hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction. The $\hat{\boldsymbol{r}}$-component of the curl of the vector potential is:

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}.$$

A solution, as you should quickly convince yourself, is a vector potential in the ϕ direction:

$$A_\phi = e_M \frac{1-\cos\theta}{r\sin\theta}.$$

Unfortunately(?), this is singular at $\theta = \pi$, *i.e.*, on the negative z-axis. We can fix this by using this form everywhere except in a cone about $\theta = \pi$, *i.e.*, for $\theta \le \pi - \epsilon$, and use the alternate solution:

$$A'_\phi = e_M \frac{-1-\cos\theta}{r\sin\theta}$$

in the (overlapping) region $\theta \ge \epsilon$, thus covering the entire space. In the overlap region ($\epsilon \le \theta \le \pi - \epsilon$), either $\boldsymbol{A}$ or $\boldsymbol{A'}$ may be used, and must give the same result, *i.e.*, the two solutions are related by a gauge transformation – that is, they differ by the gradient of a scalar function.

Consider the effect of the vector potential on the wave function of an electron (charge $-e$). Invoke single-valuedness of the wave function, and determine the possible values of e_M that a magnetic charge can have. [This is sometimes called a "Dirac monopole".]

Solution: Consider the gauge transformation relating the two vector potentials in the overlap region:

$$\boldsymbol{A}' - \boldsymbol{A} = -\frac{2e_M}{r\sin\theta}\hat{\boldsymbol{e}}_\phi = \nabla\chi,$$

where χ is a scalar function. Since

$$\nabla\chi = \frac{\partial\chi}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial\chi}{\partial\theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi}\hat{\boldsymbol{e}}_\phi,$$

up to an unimportant constant, we thus have the gauge function

$$\chi = -2e_M\phi.$$

Under a gauge transformatin in $\boldsymbol{A}$, the wave function undergoes a corresponding transformation, in phase:

$$\psi \to \psi' = \psi\exp(ie\chi)$$

(Since $\int \boldsymbol{A}\cdot ds \to \int \boldsymbol{A}\cdot ds + \int \nabla\chi\cdot ds$, and $\int \nabla\chi\cdot ds = \chi$). Hence, we have:

$$\psi' = \psi\exp(-2iee_M\phi).$$

But this must be single-valued, giving the condition that

$$2ee_M = n, \qquad \text{where } n = \ldots, -2, -1, 0, 1, 2, \ldots.$$

Thus, the magnetic charge must be quantized in units of:

$$\frac{1}{2e} = \frac{137}{2}e.$$