1. Lorentz invariant phase space

The n-particle Lorentz invariant phase space element is defined by

$$d\,\mathrm{Lips}(p_1, p_2, \ldots, p_n) \equiv \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3\,2E_i}\,,$$

where $p_i = (E_i\,;\,\vec{\boldsymbol{p}})$ and $E_i = (|\vec{\boldsymbol{p}}|^2 + m_i^2)^{1/2}$. The corresponding n-particle phase space integral is given by

$$R_n(s) \equiv \int d\,\mathrm{Lips}(p_1, p_2, \ldots, p_n)(2\pi)^4\delta\left(p - \sum_i p_i\right)\,, \tag{1}$$

where s is the Lorentz invariant quantity,

$$s \equiv p^2\,. \tag{2}$$

In these notes, we consider explicitly the case of $n = 2$. In this case, we must evaluate,

$$R_2(s) = (2\pi)^{-2}\int \frac{d^3p_1}{2E_1}\,\frac{d^3p_2}{2E_2}\,\delta^4(p - p_1 - p_2)\,. \tag{3}$$

The first step is to make use of the identity given by eq. (38) in Solution Set 1,

$$\frac{1}{2E}\,\delta(p_0 - E) = \delta(p^2 - m^2)\,\Theta(p_0)\,,$$

where $p^2 = p_0^2 - |\vec{\boldsymbol{p}}|^2$ is the square of the momentum four-vector and $\Theta(x)$ is the step function,

$$\Theta(x) \equiv \begin{cases} 1 & x > 0\,, \\ 0 & x < 0\,. \end{cases} \tag{4}$$

It then follows that,

$$\int \frac{d^3p}{2E} = \int d^4p\,\frac{1}{2E}\delta(p_0 - E) = \int d^4p\,\delta(p^2 - m^2)\Theta(p_0)\,, \tag{5}$$

after writing $d^4p = d^3p\,dp_0$. Using eq. (5), we can rewrite eq. (3) as

$$\begin{aligned} R_2(s) &= (2\pi)^{-2}\int \frac{d^3p_1}{2E_1}\,d^4p_2\,\delta(p_2^2 - m_2^2)\Theta(p_{20})\delta^4(p - p_1 - p_2) \\ &= (2\pi)^{-2}\int \frac{d^3p_1}{2E_1}\,\delta\big((p - p_1)^2 - m_2^2\big)\Theta\big((p - p_1)_0\big)\,, \end{aligned} \tag{6}$$

after using the four-dimensional delta function to integrate over p_2.

Since $R_2(s)$ is a Lorentz invariant quantity, we can evaluate it in any frame. We shall choose the center-of-momentum frame, where

$$p_1 = (E_1\,;\,\vec{\boldsymbol{p}}_{\rm CM})\,, \qquad p_2 = (E_2\,;\,-\vec{\boldsymbol{p}}_{\rm CM})\,, \qquad p = (\sqrt{s}\,;\,\vec{\boldsymbol{0}})\,. \tag{7}$$

Noting that $\sqrt{s} = E_1 + E_2$ due to the four-momentum conserving delta function which sets $p = p_1 + p_2$, and the relativistic energies given by $E_1^2 = |\vec{\boldsymbol{p}}_{\rm CM}|^2 + m_1^2$, and $E_2^2 = |\vec{\boldsymbol{p}}_{\rm CM}|^2 + m_2^2$ (the so-called *mass-shell conditions*), it is straightforward to derive the following results,

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}\,, \qquad E_2 = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}\,, \qquad |\vec{\boldsymbol{p}}_{\rm CM}| = \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, m_1^2, m_2^2)\,, \tag{8}$$

where $\lambda^{1/2}(s, m_1^2, m_2^2)$ is the square root of the *triangle function*, first introduced by Gunnar Källén. The triangle function can be written in many forms,

$$\begin{aligned}
\lambda(s, m_1^2, m_2^2) &= s + m_1^4 + m_2^4 - 2sm_1^2 - 2sm_2^2 - 2m_2^2m_2^2 \\
&= (s - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2 \\
&= \big[s - (m_1 + m_2)^2\big]\big[s - (m_1 - m_2)^2\big] \\
&= \big[m_1^2 - (\sqrt{s} + m_2)^2\big]\big[m_1^2 - (\sqrt{s} - m_2)^2\big] \\
&= \big[m_2^2 - (\sqrt{s} + m_1)^2\big]\big[m_2^2 - (\sqrt{s} - m_1)^2\big]\,.
\end{aligned} \tag{9}$$

Evaluating eq. (6) in the center-of-momentum frame, we note that

$$(p - p_1)^2 - m_2^2 = p^2 - 2p\!\cdot\! p_1 + p_1^2 - m_2^2 = s - 2\sqrt{s}E_1 + m_1^2 - m_2^2\,, \tag{10}$$

after using $p_1^2 = m_1^2$ and $p\!\cdot\! p_1 = \sqrt{s}E_1$ in light of eq. (7). Inserting the result of eq. (10) back into eq. (6) yields

$$R_2(s) = (2\pi)^{-2}\int \frac{d^3p_1}{2E_1}\,\delta\big(s - 2\sqrt{s}E_1 + m_1^2 - m_2^2\big)\Theta\big(\sqrt{s} - E_1\big)\,. \tag{11}$$

Using $E_1^2 = p_{\rm CM}^2 + m_1^2$ (where $p_{\rm CM}^2 \equiv |\vec{\boldsymbol{p}}_{\rm CM}|^2$). It follows that $E_1 dE_1 = p_{\rm CM}dp_{\rm CM}$. Hence, in the center-of-momentum frame,

$$d^3p_1 = p_{\rm CM}^2 dp_{\rm CM} d\Omega = p_{\rm CM}E_1 dE_1 d\Omega\,.$$

It follows that[1]

$$\begin{aligned}
R_2(s) &= \tfrac{1}{2}(2\pi)^{-2}\int p_{\rm CM} dE_1 d\Omega\,\delta\big(s - 2\sqrt{s}E_1 + m_1^2 - m_2^2\big)\Theta\big(\sqrt{s} - E_1\big) \\
&= \frac{p_{\rm CM}}{16\pi^2\sqrt{s}}\int d\Omega\,,
\end{aligned} \tag{12}$$

after using the delta function to integrate over E_1. Note that the theta function condition is

[1]Note that setting the argument of the delta function in eq. (12) to be zero yields the equation for E_1 obtained in eq. (8), as expected.

automatically satisfied since four-momentum conservation in the center-of-momentum frame implies that $\sqrt{s} = E_1 + E_2$ [cf. eq. (7)], and the energies E_1 and E_2 are non-negative.

Using eq. (8), we can also rewrite eq. (12) as

$$\boxed{R_2(s) = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{32\pi^2 s} \int d\Omega} \tag{13}$$

3. Decay rate for the two body decay of an unstable particle

We can now use eq. (12) to evaluate the decay rate $\Gamma = \tau^{-1}$ (where τ is the lifetime) of an unstable particle of mass $M = \sqrt{s}$ in its rest frame, which decays to two particles of masses m_1 and m_2, respectively. Starting from the general result

$$\Gamma = \frac{1}{2M} \sum_f \int d\,\mathrm{Lips}(p_1, p_2, \ldots, p_n)(2\pi)^4 \delta \left(p - \sum_i p_i \right) |\mathcal{M}_{fi}|^2 \,,$$

where f collectively refers to the final state particles. In the case of $n = 2$, it then follows that

$$\frac{d\Gamma}{d\Omega} = \frac{p_{\rm CM}}{32\pi^2 M^2} \sum_f |\mathcal{M}_{fi}|^2 \,.$$

Note that Ω refers to the solid angle of emitted particle 1 with respect to a fixed z-axis in the rest frame of the decaying particle. After summing over final state spins, we are free to choose the z axis arbitrarily. Thus, the integration over Ω is trivial and yields 4π. Hence the decay rate of the unstable particle is given by

$$\Gamma = \frac{p_{\rm CM}}{8\pi M^2} \sum_f |\mathcal{M}_{fi}|^2 \,, \tag{14}$$

where the sum over f is a sum over final state spins (and any other internal degrees of freedom if present). It should be noted that the an invariant matrix element for the decay of a particle into a two particle final state has a mass dimension equal to 1. Thus, Γ has mass dimension 1 and $\tau = \Gamma^{-1}$ has mass dimension -1, which in units of $\hbar = c = 1$ corresponds to a quantity with the dimensions of time. (More precisely, $\tau \equiv \hbar/\Gamma$.)

If the unstable particle has a non-zero spin J, then one should also average over the initial spins in the computation of decay rate,

$$\Gamma = \frac{p_{\rm CM}}{8\pi M^2} \frac{1}{2J+1} \sum_{i,f} |\mathcal{M}_{fi}|^2 \,.$$

That is, to perform the average over initial spins, we sum over the initial spins and divide by the number of possible spin states. Finally, one can rewrite this result using eq. (8) with $\sqrt{s} = M$ to obtain

$$\boxed{\Gamma = \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{16\pi M^3} \frac{1}{2J+1} \sum_{i,f} |\mathcal{M}_{fi}|^2} \tag{15}$$

Note that in deriving eq. (14), we integrated over the full 4π steradians. In the case where the two decay products are identical particles, the integration over 4π steradians is double counting, since one cannot distinguish whether particle 1 or particle 2 has been emitted at a solid angle $\Omega = (\theta, \phi)$ with respect to a fixed z-axis. To avoid double counting, we simply integrate over 2π steradians. That is, the width computed in eq. (14) must be reduced by a factor of 2. We can write this more explicitly as

$$\boxed{\Gamma = \left(\frac{1}{1+\delta_{12}}\right) \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{16\pi M^3} \frac{1}{2J+1} \sum_{i,f} |\mathcal{M}_{fi}|^2} \tag{16}$$

where $\delta_{12} = 1$ of the two particles in the final state are identical; otherwise, $\delta_{12} = 0$.

4. Cross section for the scattering of two particles into a two-particle final state

Consider the two-particle scattering process, $a + b \to 1 + 2$, with corresponding four momentum vectors p_a, p_b, p_1 and p_2. It is convenient to introduced the Lorentz invariant Mandelstam variables,

$$s = (p_a + p_b)^2 = (p_1 + p_2)^2 \,, \tag{17}$$

$$t = (p_a - p_1)^2 = (p_2 - p_b)^2 \,, \tag{18}$$

$$u = (p_a - p_2)^2 = (p_1 - p_b)^2 \,, \tag{19}$$

where we have used the conservation of four-momentum, $p_a + p_b = p_1 + p_2$. A straightforward computation reveals that s, t and u are not independent variables. Using the mass-shell conditions, $p_a^2 = m_a^2$, $p_b^2 = m_b^2$, $p_1^2 = m_1^2$ and $p_2^2 = m_b^2$, it follows that

$$s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2 \,.$$

One is free to evaluate s, t and u in any reference frame. It is convenient to work out the Mandelstam variables in the center-of- momentum frame where $\vec{\boldsymbol{p}}_b = -\vec{\boldsymbol{p}}_a$ and $\vec{\boldsymbol{p}}_2 = -\vec{\boldsymbol{p}}_1$. The four-momenta of the scattering process are then given by,

$$p_a = (E_a\,;\,\vec{\boldsymbol{p}}_a)\,, \qquad p_b = (E_b\,;\,-\vec{\boldsymbol{p}}_a)\,, \qquad p_1 = (E_1\,;\,\vec{\boldsymbol{p}}_1)\,, \qquad p_2 = (E_2\,;\,-\vec{\boldsymbol{p}}_1)\,, \tag{20}$$

It is convenient to introduce the notation for the magnitudes of the initial and final state center-of-momentum three-momentum,

$$p_i \equiv |\vec{\boldsymbol{p}}_a| = |\vec{\boldsymbol{p}}_b|\,, \qquad p_f \equiv |\vec{\boldsymbol{p}}_1| = |\vec{\boldsymbol{p}}_2|\,.$$

Consequently, the mass-shell conditions yield

$$E_a = (p_i^2 + m_a^2)^{1/2}\,, \qquad E_b = (p_i^2 + m_b^2)^{1/2}\,, \qquad E_1 = (p_f^2 + m_1^2)^{1/2}\,, \qquad E_2 = (p_f^2 + m_2^2)^{1/2}\,.$$

It immediately follows from eq. (20) that

$$s = (E_a + E_b)^2\,.$$

That is, $\sqrt{s}$ is the total relativistic energy available in the center-of-momentum frame available for the collision process. Next, we consider

$$\begin{aligned} t = (p_a - p_1)^2 &= m_a^2 + m_1^2 - 2p_a{\cdot}p_1\,, \\ &= m_a^2 + m_1^2 - 2E_aE_1 + 2\vec{\boldsymbol{p}}_a{\cdot}\vec{\boldsymbol{p}}_1\,, \\ &= m_a^2 + m_1^2 - 2E_aE_1 + 2p_ip_f\cos\theta\,, \end{aligned} \tag{21}$$

where the center-of-momentum scattering angle θ is the polar angle between particles a and 1.

Using energy and three-momentum conservation along with the mass-shell conditions, it is straightforward to obtain the following expressions for the center-of-momentum frame energies and three-momenta,

$$\begin{aligned} E_a &= \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}\,, \qquad & E_b &= \frac{s + m_b^2 - m_a^2}{2\sqrt{s}}\,, \\ E_1 &= \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}\,, \qquad & E_2 &= \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}\,, \end{aligned} \tag{22}$$

and

$$p_i = \frac{1}{2\sqrt{s}}\lambda^{1/2}(s, m_a^2, m_b^2)\,, \qquad p_f = \frac{1}{2\sqrt{s}}\lambda^{1/2}(s, m_1^2, m_2^2)\,, \tag{23}$$

where $\lambda^{1/2}$ is the square root of the triangle function introduced in eq. (9). Finally, the cosine of the scattering angle in the center-of-momentum frame is given by

$$\cos\theta = \frac{s(t-u) + (m_a^2 - m_b^2)(m_1^2 - m_2^2)}{\lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(s, m_1^2, m_2^2)}\,.$$

Recall the formula obtained in class for the total cross section for the scattering of two particles into an n particle final state,

$$\sigma = \frac{1}{4\sqrt{(p_a{\cdot}p_b)^2 - m_a^2m_b^2}} \int d\,\mathrm{Lips}\,(2\pi)^4\delta^4\left(p_a + p_b - \sum_f p_f\right) \sum_f |\mathcal{M}_{fi}|^2\,, \tag{24}$$

where $\mathcal{M}_{fi}$ is the invariant scattering amplitude, and we are instructed to sum over all internal spin degrees of freedom. We shall evaluate eq. (24) in the case of $n = 2$. Furthermore, note that in the center-of-momentum frame,

$$(p_a{\cdot}p_b)^2 - m_a^2m_b^2 = \tfrac{1}{4}\big[(s - m_a - m_b)^2 - 4m_a^2m_b^2\big] = \tfrac{1}{4}\lambda(s, m_a^2, m_b^2)\,,$$

where we have used eqs. (9) and (17).

If we do not measure the initial state spins of the colliding particles a and b, then we should average over the initial state spin states. Denoting the initial spins by J_a and J_b, we obtain the spin-averaged cross section by summing the squared invariant matrix element over initial spin states and dividing by the total number of initial spin states, $(2J_a+1)(2J_b+1)$. Thus, in light of eq. (3) [where we identify $p = p_1 + p_2$],

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\lambda^{1/2}(s, m_a^2, m_b^2)} \frac{dR_2(s)}{d\Omega} \frac{1}{(2J_a+1)(2J_b+1)} \sum_{i,f} |\mathcal{M}_{fi}|^2 \,,$$

where $\Omega = (\theta, \phi)$ refers to the center-of-momentum frame polar and azimuthal angles of particle 1 with respect to the z-axis that is defined to lie in the direction of $\vec{\boldsymbol{p}}_a$. Using the explicit expression obtained for $R_2(s)$ in eq. (13),

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{\lambda^{1/2}(s, m_a^2, m_b^2)} \frac{1}{(2J_a+1)(2J_b+1)} \sum_{i,f} |\mathcal{M}_{fi}|^2} \tag{25}$$

The expression $d\sigma/d\Omega$ is not Lorentz invariant since it depends on angles defined in the center-of-momentum frame. It is possible to define a Lorentz invariant differential cross section by noting that in view of eq. (21), the variable t can be used in place of the center-of-momentum scattering angle. In particular, using eqs. (22) and (23) to rewrite the form of eq. (21),

$$t = m_a^2 + m_1^2 - \frac{(s - m_a^2 + m_b^2)(s - m_1^2 + m_2^2)}{2s} + \frac{1}{2s}\,\lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(s, m_1^2, m_2^2)\cos\theta \,.$$

It follows that

$$dt = \frac{1}{2s}\lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(s, m_1^2, m_2^2)\, d\cos\theta \,.$$

The azimuthal angle ϕ of particle 1 in the center-of-momentum frame corresponds to rotations around the z-axis. The integral over this angle is trivial. Hence,

$$d\Omega = 2\pi\, d\cos\theta = 4\pi s \lambda^{-1/2}(s, m_a^2, m_b^2)\lambda^{-1/2}(s, m_1^2, m_2^2)\, dt \,.$$

Inserting this result into eq. (25) yields

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{16\pi\lambda(s, m_a^2, m_b^2)} \frac{1}{(2J_a+1)(2J_b+1)} \sum_{i,f} |\mathcal{M}_{fi}|^2} \tag{26}$$

The above expression is manifestly Lorentz invariant. One can use eq. (26) to evaluate the differential cross section in any reference frame.

Finally, it is often convenient to make use of eq. (23) to rewrite eq. (26) as

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{64\pi p_i^2 s} \frac{1}{(2J_a+1)(2J_b+1)} \sum_{i,f} |\mathcal{M}_{fi}|^2} \tag{27}$$

Note that having derived eq. (15) for the decay rate for an unstable particle to decay into two final state particles and eq. (27) for the differential cross section for a $2 \to 2$ scattering process, the one remaining task is to evaluate the spin-averaged squared invariant amplitude,

$$\frac{1}{(2J_a+1)(2J_b+1)} \sum_{i,f} |\mathcal{M}_{fi}|^2 \,,$$

relevant for the corresponding process.

Finally, the total cross section can be obtained from eq. (25) by integrating over the solid angle. In light of the discussion following eq. (15) in the case of identical final state particles,

$$\sigma = \left(\frac{1}{1+\delta_{12}}\right) \int \frac{d\sigma}{d\Omega}\, d\Omega\,,$$

where $\delta_{12} = 1$ of the two particles in the final state are identical; otherwise, $\delta_{12} = 0$. No extra factor is needed in the case of identical initial state particles, since the issue of double counting does not arise in the latter case.

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