DUE: THURSDAY OCTOBER 27, 2016

1. Consider the 4×4 matrices $S^{\mu\nu}$, defined by

$$(S^{\mu\nu})^\alpha{}_\beta = i(g^{\alpha\mu} g^\nu_\beta - g^{\alpha\nu} g^\mu_\beta)\,, \tag{1}$$

where the indices α and β label the rows and columns of the matrices $S^{\mu\nu}$. Noting that $S^{\mu\nu} = -S^{\nu\mu}$, it follows that there are six distinct matrices among the $S^{\mu\nu}$. We can identify these six matrices as follows,

$$S^i = \tfrac{1}{2}\epsilon^{ijk} S^{jk}\,, \qquad\qquad K^i \equiv S^{0i}\,. \tag{2}$$

where the indices i, j and k are space indices (1, 2 or 3).

(a) Write out the explicit matrix forms of the S^i and K^i. Then, evaluate the commutators, $\bigl[S^i\,,\,S^j\bigr]$, $\bigl[S^i\,,\,K^j\bigr]$ and $\bigl[K^i\,,\,K^j\bigr]$. Show that the results of these three commutation relations can be summarized by the following equation,

$$\bigl[S^{\mu\nu}\,,\,S^{\rho\sigma}\bigr] = i\bigl(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\bigr)\,. \tag{3}$$

(b) Show that the combinations,

$$S^i_+ \equiv \tfrac{1}{2}(S^i + iK^i) \quad \text{and} \quad S^i_- \equiv \tfrac{1}{2}(S^i - iK^i)\,, \tag{4}$$

commute with one another and separately satisfy the commutation relations of the generators of the rotation group.

(c) Consider a proper orthochronous Lorentz transformation that is a pure boost in the direction $\hat{\boldsymbol{\eta}}$. Define $\vec{\boldsymbol{\eta}} \equiv \eta\hat{\boldsymbol{\eta}}$, where η is the magnitude of the vector $\vec{\boldsymbol{\eta}}$. Then, verify the following result for the 4×4 Lorentz transformation matrix Λ,

$$\Lambda \equiv \exp[-i\vec{\boldsymbol{\eta}} \boldsymbol{\cdot} \vec{\boldsymbol{K}}] = I - i\hat{\boldsymbol{\eta}} \boldsymbol{\cdot} \vec{\boldsymbol{K}} \sinh\eta + (\hat{\boldsymbol{\eta}} \boldsymbol{\cdot} \vec{\boldsymbol{K}})^2[1 - \cosh\eta]\,, \tag{5}$$

where I is the 4×4 identity matrix, and $\vec{\boldsymbol{K}} = (K^1\,,\,K^2\,,\,K^3)$ are the 4×4 boost matrices defined in eq. (2).

HINT: The exponential is defined by its Taylor series. Show that $(\hat{\boldsymbol{\eta}}\boldsymbol{\cdot}\vec{\boldsymbol{K}})^3 = -\hat{\boldsymbol{\eta}}\boldsymbol{\cdot}\vec{\boldsymbol{K}}$.

(d) Consider the rest frame of a particle of mass m. Using the the Lorentz transformation matrix Λ given by eq. (5) to boost into a frame where the particle four-momentum is given by $(E\,;\vec{\boldsymbol{p}})$, show that $E = m\cosh\eta$ and $\vec{\boldsymbol{p}} = \hat{\boldsymbol{\eta}}\, m\sinh\eta$. Finally, verify that $\vec{\boldsymbol{\eta}} = \hat{\boldsymbol{\beta}}\tanh^{-1}\beta$, where $\vec{\boldsymbol{\beta}}$ is the velocity of the boosted frame (in units of $c = 1$), with $\beta \equiv |\vec{\boldsymbol{\beta}}|$ and $\hat{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{\beta}}/\beta$.

(e) Using eq. (5) and the results of part (d), show that the Lorentz transformation, $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$, can be written in the following form,

$$
\begin{aligned}
x'_0 &= \gamma(x_0 + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\,, \\
\vec{\boldsymbol{x}}\,' &= \vec{\boldsymbol{x}} + \left(\frac{\gamma-1}{\beta^2}\right)(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}x_0\,,
\end{aligned}
$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$.

2. Consider the field theory of a complex scalar field $\Phi(x)$ governed by the following Lagrangian density,

$$\mathscr{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi\,.$$

We can treat Φ and Φ^* as independent dynamical field variables.

(a) The conjugate momenta to $\Phi(x)$ and $\Phi^*(x)$ are denoted by $\Pi(x)$ and $\Pi^*(x)$, respectively, Obtain explicit expressions for the conjugate momenta and determine the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x \big(\Pi^*\,\Pi + \boldsymbol{\nabla}\Phi^* \cdot \boldsymbol{\nabla}\Phi + m^2\Phi^*\Phi\big)\,.$$

Compute the Heisenberg equation for $\Phi(x)$ and show that it is the Klein-Gordon equation.

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

HINT: In contrast to the case of a real scalar field, when expanding the complex field $\Phi(x)$ in terms of creation and annihilation operators, these operators are no longer related by hermitian conjugation. Thus, you should employ different symbols for the creation and annihilation operators. Traditionally, one uses a and $b^\dagger$, respectively.

(c) The Lagrangian density is invariant under a phase transformation, $\Phi \to e^{i\alpha}\Phi$. Evaluate the conserved Noether current and identity the corresponding conserved charge. Show that one can rewrite the conserved charge operator in the form,

$$Q = \tfrac{1}{2} i \int d^3x\,(\Phi^*\,\Pi^* - \Pi\,\Phi)\,. \tag{6}$$

Evaluate Q in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) Consider the case of two complex scalar fields with the same mass. Label the fields as $\Phi_a(x)$, where $a = 1$, 2. Show that there are now four conserved charges, one given by a generalization of eq. (6), and the other three given by

$$Q^i = \tfrac{1}{2} i \int d^3x\,\big(\Phi^*_a(\sigma^i)_{ab}\Pi^*_b - \Pi_a(\sigma^i)_{ab}\Phi_b\big)\,,$$

where the σ^i are the Pauli sigma matrices, and there is an implicit sum over the repeated indices a and b. Show that these three charges have the commutation relations of the generators of the rotation group.

3. (a) Let H be the second-quantized Hamiltonian of a free real scalar field. Prove the identity:

$$e^{-\beta H}\, a^\dagger_{\vec{\boldsymbol{p}}} = a^\dagger_{\vec{\boldsymbol{p}}}\, e^{-\beta(H+E_{\vec{\boldsymbol{p}}})}\,, \tag{7}$$

where $E_{\vec{\boldsymbol{p}}} \equiv (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$ and β is a real number.

(b) According to the rules of quantum mechanics, the expectation value of an operator $\mathcal{O}$, in a mixed state described by a density matrix ρ, is given by

$$\langle \mathcal{O} \rangle = \frac{\mathrm{Tr}\ (\rho\, \mathcal{O})}{\mathrm{Tr}\ \rho}\,.$$

For a thermal state with temperature $T = \beta^{-1}$, the density matrix is given by $\rho = e^{-\beta H}$, and therefore the thermal expectation values are defined as:

$$\langle \mathcal{O} \rangle_\beta = \frac{\mathrm{Tr}\ (\mathcal{O}\, e^{-\beta H})}{\mathrm{Tr}\ (e^{-\beta H})}\,,$$

where the trace is taken over the Fock space. Using the results of part (a), show that

$$\left\langle a^\dagger_{\vec{\boldsymbol{p}}}\, a_{\vec{\boldsymbol{q}}} \right\rangle_\beta = \frac{\delta_{\vec{\boldsymbol{p}}\vec{\boldsymbol{q}}}}{e^{\beta E_{\vec{\boldsymbol{p}}}} - 1}\,.$$

But, $\langle N_{\vec{\boldsymbol{p}}} \rangle_\beta \equiv \left\langle a^\dagger_{\vec{\boldsymbol{p}}}\, a_{\vec{\boldsymbol{p}}} \right\rangle_\beta$ is simply the thermal expectation value of the number operator. Thus, the above result shows that the $a^\dagger_{\vec{\boldsymbol{p}}}$ create quanta that obey the Bose-Einstein distribution.

HINT: It is sufficient to use the cyclicity of the trace and the commutation relations of the creation and annihilation operators in this derivation.

(c) Suppose that the commutation relations of the creation and annihilation operators are replaced by anticommuation relations. Show that eq. (7) still holds. How would the results of part (b) change?

4. Eq. (1) provides a four-dimensional matrix representation of the commutation relations of the generators of the Lorentz group [cf. eq. (3)]. However, other matrix representations of the Lorentz group exist. Indeed, In light of part (b) of problem 1, it follows that the finite dimensional irreducible representations of the Lorentz group can be specified by a pair of non-negative integers or half-integers, corresponding to pairs of representations of the rotation group.

(a) Write out explicitly the infinitesimal transformation laws of the two-component fields transforming according to the $(\frac{1}{2}\,,\,0)$ and $(0\,,\,\frac{1}{2})$ representations of the Lorentz group.

HINT: The spin-$\frac{1}{2}$ representation of the generators of the rotation group are $\frac{1}{2}\vec{\boldsymbol{\sigma}}$ where $\vec{\boldsymbol{\sigma}} = (\sigma^1\,,\,\sigma^2\,,\,\sigma^3)$ are the Pauli sigma matrices. In contrast, the spin-0 representation of the generators is trivial (i.e. all generators are zero). As an example, for the $(\frac{1}{2}\,,\,0)$ representation, it then follows that $S^i_+ = \frac{1}{2}\sigma^i$ and $S^i_- = 0$. Using eq. (4), one can then obtain the corresponding two-dimensional matrix representations for S^i and K^i.

(b) Using the same notation as in part (c) of problem 1, prove the identity

$$\exp\left(\tfrac{1}{2}\vec{\boldsymbol{\eta}}\cdot\vec{\boldsymbol{\sigma}}\right) = \cosh(\eta/2) + \hat{\boldsymbol{\eta}}\cdot\vec{\boldsymbol{\sigma}}\,\sinh(\eta/2)\,.$$

(c) It is convenient to define $\sigma^\mu \equiv (1\,;\,\vec{\boldsymbol{\sigma}})$ and $\overline{\sigma}^\mu \equiv (1\,;\,-\vec{\boldsymbol{\sigma}})$. Prove the following two identities:

$$\sqrt{p\cdot\sigma} \equiv \frac{E+m-\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{\sqrt{2(E+m)}}\,,$$

$$\sqrt{p\cdot\overline{\sigma}} \equiv \frac{E+m+\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{\sqrt{2(E+m)}}\,,$$

where $p^\mu \equiv (E\,;\,\vec{\boldsymbol{p}})$ and $m = (E^2 - |\vec{\boldsymbol{p}}|^2)^{1/2}$. The matrix square root of $p\cdot\sigma$ [or $p\cdot\overline{\sigma}$] as defined here is the unique hermitian matrix with non-negative eigenvalues whose square is equal to $p\cdot\sigma$ [or $p\cdot\overline{\sigma}$].

(d) Using the results of parts (b) and (c), and recalling the expressions for E and $\vec{\boldsymbol{p}}$ obtained in part (d) of problem 1, show that for a pure boost, the Lorentz transformations for the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations, respectively, are given by:

$$\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = \begin{cases} \exp\left(-\dfrac{1}{2}\vec{\boldsymbol{\eta}}\cdot\vec{\boldsymbol{\sigma}}\right) = \sqrt{\dfrac{p\cdot\sigma}{m}}\,, & \text{for } (\tfrac{1}{2},0)\,, \\[2ex] \exp\left(\dfrac{1}{2}\vec{\boldsymbol{\eta}}\cdot\vec{\boldsymbol{\sigma}}\right) = \sqrt{\dfrac{p\cdot\overline{\sigma}}{m}}\,, & \text{for } (0,\tfrac{1}{2})\,, \end{cases}$$

where $\omega_{ij} = 0$ and $\eta^i = \omega^{i0} = -\omega^{0i}$.