*This is a three hour "open book" exam. During the exam, you may consult with the textbook (Schwartz or another textbook of your choosing), any of the class handouts (including solutions to the five problem sets), and your personal handwritten notes. However, you may not not collaborate with anyone else during the exam. The exam consists of three problems, with ten parts (worth ten points each) in total. Ten bonus points are available if you correctly answer part (d) of problem 1.*

*Please note that you do not have to derive all results from scratch. However, if you use a particular result, you should cite its source (e.g. an equation in Schwartz or an equation in a Solution Set or a class handout.*

1. The helicity spinor $u(p,\lambda)$ satisfies $\overline{u}(p,\lambda)u(p,\lambda) = 2m$. In parts (a) and (b), you may assume that $m \neq 0$.

(a) Evaluate $\overline{u}(p,\lambda)\,\gamma_5\,u(p,\lambda)$ by any method of your choosing.

*HINT:* Although one can simply employ explicit representations for the helicity spinor and the gamma matrices, one can often save time by employing representation-independent techniques. For example, under the assumption that $m \neq 0$, use the Dirac equation to show that

$$\overline{u}(p,\lambda)\,\gamma_5\,u(p,\lambda) = \frac{1}{2m}\overline{u}(p,\lambda)\,(\not{p}\gamma_5 + \gamma_5\not{p})\,u(p,\lambda)\,.$$

Then simplify the expression $\not{p}\gamma_5 + \gamma_5\not{p}$ to obtain the final result.

(b) Evaluate $\overline{u}(p,\lambda)\,\gamma^\mu\,u(p,\lambda)$.

*HINT:* Consider again the hint given for part (a), suitably modified. Express your result in terms of the momentum four-vector $p^\mu$.

(c) Evaluate $\overline{u}(p,\lambda)\,\gamma^\mu\gamma_5\,u(p,\lambda)$, assuming the the fermion mass $m = 0$.

(d) [*EXTRA CREDIT*] Evaluate $\overline{u}(p,\lambda)\,\gamma^\mu\gamma_5\,u(p,\lambda)$, assuming that the fermion mass, $m \neq 0$. Express your result in terms of the spin four-vector $s^\mu$. How would you use this result in the case of $m = 0$ to reproduce the answer obtained in part (c)?

2. Quantum electrodynamics (QED) is a theory of spin-1/2 electrons and positrons interacting with photons. Suppose we add to QED a real massive pseudoscalar field $\Phi$ that couples to electrons and positrons via

$$\mathscr{L}_{\rm int} = -ig\overline{\Psi}(x)\gamma_5\Psi(x)\Phi(x) \tag{1}$$

(a) Which, if any, of the discrete symmetries P, C and T are violated if the interaction in eq. (1) is added to the QED Lagrangian? You may assume that the only scalar self-interaction term allowed is $\Phi^4(x)$, although this has no bearing on the rest of the problem.

(b) Draw all distinct Feynman diagrams that contribute to the second-order scattering process $\gamma e^- \to e^- \Phi$.

(c) What is the Feynman rule for the $\Phi\, e^+ e^-$ vertex? Write down the invariant matrix element corresponding to the Feynman diagrams of part (b). Do not assume that the mass of the electron is zero.

(d) Your answer in part (c) has the form $\mathcal{M} = \mathcal{M}_\mu \epsilon^\mu(k, \lambda)$, where $k$ is the photon four-momentum and $\lambda$ is the photon helicity. Check your answer by verifying that $k^\mu \mathcal{M}_\mu = 0$.

*HINT:* Use four-momentum conservation and the Dirac equation, and express intermediate results in terms of the kinematic invariants $s$, $t$ and $u$ whenever possible.

3. The $Z$ boson is a massive spin 1 particle (of mass $m_Z$). Suppose there exists a light spin-0 particle $\phi$ of mass $m_\phi$. Then one possible decay mode of the $Z$ boson is $Z \to \phi\gamma$.[1] Denote the four-momenta of the $Z$ boson, the spin-0 particle $\phi$ and the photon ($\gamma$) by $p$, $q$ and $k$, respectively. An explicit computation of the decay matrix element yields the following result for the invariant matrix element for $Z \to \phi\gamma$,

$$\mathcal{M}_{s\lambda} = A(g^{\mu\nu} p{\cdot}k - k^\mu p^\nu)\epsilon_\mu(p, s)\epsilon_\nu(k, \lambda)^*\,,$$

where $\epsilon_\mu(p, s)$ is the polarization vector of the $Z$ boson and $\epsilon_\mu(k, \lambda)$ is the polarization vector of the photon. The $Z$ boson and photon helicities are denoted by $s$ and $\lambda$, respectively. The coefficient $A$, which has units of inverse mass and is a function of $m_\phi$, $m_Z$ and couplings that govern the decay, will not be specified further.

(a) Square the invariant matrix element, sum over the final state photon helicities and average over the initial state $Z$ boson helicities. At the end of your calculation, any dot product of four-vectors should be re-expressed in terms of $m_Z$ and $m_\phi$.

*HINT:* In evaluating the average over the $Z$ boson polarizations, be sure to average over the three possible polarizations. When evaluating the sum over the photon polarizations, you are allowed to make the substitution,

$$\sum_{\lambda=\pm 1} \epsilon_\beta(k, \lambda)\epsilon_\nu(k, \lambda)^* \quad \longrightarrow \quad -g_{\beta\nu}\,, \tag{2}$$

which will significantly reduce the algebra involved in completing this calculation.

(b) Explain why it is permissible to make the substitution shown in eq. (2), even though the formula for the photon polarization sum given in part (c) of problem 4 on Problem Set 4 is a much more complicated expression.

(c) Using the result obtained in part (a), derive an expression for the decay width, $\Gamma(Z \to \phi\gamma)$. Your final result should depend on $m_Z$, $m_\phi$ and $A$.

[1]Prior to the discovery of the Higgs boson, one of the Higgs boson ($H$) search techniques employed at the LEP collider in the 1990s was a search for $Z \to H\gamma$. No evidence was ever found. We now know that the Higgs boson is heavier than the $Z$ boson, so this decay is not permitted.