DUE: THURSDAY, JANUARY 21, 2016

1. Show that for complex scalar fields,

$$\int \mathcal{D}\Phi^* \, \mathcal{D}\Phi \exp\left\{ i \int d^4x \, d^4y \big[\Phi^*(x) M(x,y) \Phi(y)\big] + i \int d^4x \,\big[J^*(x)\Phi(x) + \Phi^*(x) J(x)\big] \right\}$$

$$= \mathcal{N} \, \frac{1}{\det M} \exp\left\{ -i \int d^4x \, d^4y \, J^*(x) M^{-1}(x,y) J(y) \right\} ,$$

for some infinite constant $\mathcal{N}$. This is problem 14.1 on p. 283 of Schwartz.

2. (a) Derive the result:

$$\int d^4z \, \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} = -\delta^4(x-y) ,$$

and interpret diagrammatically. Here, $W[J]$ is the generating functional for the connected Green functions and $\Gamma[\Phi]$ is the generating functional for the one particle irreducible (1PI) Green functions.

(b) By taking one further functional derivative, show that Γ generates the amputated connected three-point function.

3. Consider the quantum field theory of a real scalar field governed by the Lagrangian,

$$\mathscr{L} = \tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 .$$

(a) Evaluate the generating functional $Z[J]$ perturbatively, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$ as follows. First, show that $Z[J]$ can be written in the following form,

$$Z[J] = \mathcal{N} \left[1 - \frac{i\lambda}{4!} \int d^4y \left(\frac{1}{i} \frac{\delta}{\delta J(y)} \right)^4 + \mathcal{O}(\lambda^2) \right] \exp\left\{ -\frac{i}{2} \int d^4x_1 \, d^4x_2 \, J(x_1) \Delta_F(x_1 - x_2) J(x_2) \right\} , \tag{1}$$

where $\mathcal{N}$ is the J-independent constant. Then, carry out the functional derivatives with respect to J, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Using the result just obtained for $Z[J]$, obtain an expression for the generating functional for the connected Green functions, $W[J]$, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$.

(b) Using the result of part (a) for $W[J]$, compute the four-point connected Green function. Check that the same result is obtained by making use of Coleman's lemma derived in class to obtain the $\mathcal{O}(\lambda)$ contribution to $G^{(4)}(x_1, x_2, x_3, x_4)$. By taking the appropriate Fourier transform, verify that you obtain the momentum space Feynman rule for the four-point scalar interaction obtained in class.

(c) Evaluate the classical field $\phi_c(x)$ and the generating functional for the 1PI Green functions, $\Gamma[\phi_c]$, perturbatively, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Then, repeat part (b) for the four-point 1PI Green function.

4. Consider a scalar field theory defined by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\,\partial^\mu\phi(x)\partial_\mu\phi(x) - V(\phi(x))\,, \tag{2}$$

and the corresponding equation of motion,

$$\Box\,\phi(x) + V'(\phi) = 0\,,$$

where $\Box \equiv \partial^\mu\partial_\mu$ and $V' \equiv dV/d\phi$.

(a) Starting from eq. (14.122) on p. 276 of Schwartz, derive the equation of motion for the Green function $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$,

$$\Box_x\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = -\langle\Omega|T\{V'(\phi(x))\phi(y)\}|\Omega\rangle - i\delta^4(x-y)\,. \tag{3}$$

(b) Derive eq. (3) by the following technique. Start from the path integral definition of the generating functional,

$$Z[J] = \mathcal{N}\int\mathcal{D}\phi\,\exp\left\{i\int d^4x\,\big[\mathscr{L} + J(x)\phi(x)\big]\right\}\,, \tag{4}$$

where $\mathcal{N}$ is chosen such that $Z[0]=1$. Perform a change of variables in the path integral, $\phi(x)\to\phi(x)+\varepsilon(x)$, where $\varepsilon(x)$ is an arbitrary infinitesimal function of x. Noting that a change of variables[1] does not change the value of of $Z[J]$, show that to first order in $\varepsilon(x)$,

$$\int\mathcal{D}\phi\,\exp\left\{i\int d^4x\,\big[\mathscr{L} + J(x)\phi(x)\big]\right\}\int d^4x\,\varepsilon(x)\big[-\Box\Phi - V'(\phi) + J(x)\big] = 0\,. \tag{5}$$

Since $\varepsilon(x)$ is arbitrary, we may choose $\varepsilon(x) = \epsilon\,\delta^4(x-y)$, where ϵ is an infinitesimal constant. With this choice for $\varepsilon(x)$, show that by taking the functional derivative of the eq. (5) with respect to $J(x)$ and then setting $J=0$, one can derive eq. (3).

HINT: What is the Jacobian corresponding to the change of variables, $\phi(x)\to\phi(x)+\varepsilon(x)$?

[1] Just as in the case of ordinary functional integration, a change of integration variables does not change the value of the functional integral.