We first consider a massless spin-1 particle moving in the z-direction with four-momentum $k^\mu = E(1\,;\,0\,,\,0\,,\,1)$. The textbook expressions for the helicity ± 1 polarization vectors of a massless spin-1 boson are given by [1–4]:

$$\varepsilon^\mu(\hat{\boldsymbol{z}}, \pm 1) = \frac{1}{\sqrt{2}}\,(0\,;\,\mp 1\,,\,-i\,,\,0)\,. \tag{1}$$

Note that the $\varepsilon^\mu(\hat{\boldsymbol{z}}, \lambda)$ are normalized eigenvectors of the spin-1 operator $\vec{\boldsymbol{\mathcal{S}}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}$,

$$(\vec{\boldsymbol{\mathcal{S}}}\boldsymbol{\cdot}\hat{\boldsymbol{z}})^\mu{}_\nu\,\varepsilon^\nu(\hat{\boldsymbol{z}}, \lambda) = \lambda\,\varepsilon^\mu(\hat{\boldsymbol{z}}, \lambda)\,, \qquad \text{for } \lambda = \pm 1, \tag{2}$$

where $\mathcal{S}^i \equiv \frac{1}{2}\epsilon^{ijk}\mathcal{S}_{jk}$ (with $i, j, k = 1, 2, 3$ and $\epsilon^{123} = +1$), and the matrix elements of the 4×4 matrices $\mathcal{S}_{jk}$ are given by[1]

$$(\mathcal{S}_{\rho\sigma})^\mu{}_\nu = i(g_\rho{}^\mu\, g_{\sigma\nu} - g_\sigma{}^\mu\, g_{\rho\nu})\,. \tag{3}$$

To accommodate photons traveling along the $\hat{\boldsymbol{k}}$-direction, one can transform $\varepsilon^\mu(\hat{\boldsymbol{z}}, \lambda)$ to $\varepsilon^\mu(\hat{\boldsymbol{k}}, \lambda)$ by employing a three-dimensional rotation $\mathcal{R}$ such that $\hat{\boldsymbol{k}} = \mathcal{R}\,\hat{\boldsymbol{z}}$. Explicitly, the rotation operator can be parameterized in terms of three Euler angles (e.g., see Refs. [5,6]):

$$\mathcal{R}(\phi\,,\,\theta\,,\,\gamma) \equiv R(\hat{\boldsymbol{z}}\,,\,\phi)\,R(\hat{\boldsymbol{y}}\,,\,\theta)\,R(\hat{\boldsymbol{z}}\,,\,\gamma)\,, \tag{4}$$

The Euler angles can be chosen to lie in the range $0 \leq \theta \leq \pi$ and $0 \leq \phi\,,\,\gamma < 2\pi$. Here, $R(\hat{\boldsymbol{n}}\,,\,\theta)$ is a 3×3 orthogonal matrix that represents a rotation by an angle θ about a fixed axis $\hat{\boldsymbol{n}}$,

$$R^{ij}(\hat{\boldsymbol{n}}\,,\,\theta) = \exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}) = n^i n^j + (\delta^{ij} - n^i n^j)\cos\theta - \epsilon^{ijk} n^k \sin\theta\,, \tag{5}$$

where the $\vec{\boldsymbol{S}} = (S^1\,,\,S^2\,,\,S^3)$ are three 3×3 matrices whose matrix elements are given by $(S^i)^{jk} = -i\epsilon^{ijk}$ [i.e., the lower right hand 3×3 block of the matrices $\mathcal{S}^i$ defined above eq. (3)].

Thus, the polarization vector for a massless spin-1 boson of energy E moving in the direction $\hat{\boldsymbol{k}} = (\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)$ is obtained as follows:

$$\varepsilon^\mu(\hat{\boldsymbol{k}}, \lambda) = \Lambda^\mu{}_\nu(\phi\,,\theta\,,\,\gamma)\,\varepsilon^\nu(\hat{\boldsymbol{z}}, \lambda)\,, \tag{6}$$

[1] Recall that the most general proper orthochronous Lorentz transformation (which is continuously connected to the identity), corresponding to a rotation by an angle θ about an axis $\hat{\boldsymbol{n}}$ [$\vec{\boldsymbol{\theta}} \equiv \theta\hat{\boldsymbol{n}}$] and a boost vector $\vec{\boldsymbol{\zeta}} \equiv \hat{\boldsymbol{v}}\tanh^{-1}\beta$ [where $\hat{\boldsymbol{v}} \equiv \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}|$ and $\beta \equiv |\vec{\boldsymbol{v}}|$], is a 4×4 matrix given by:

$$\Lambda = \exp\left(-\tfrac{1}{2}i\,\theta^{\rho\sigma}\mathcal{S}_{\rho\sigma}\right) = \exp\left(-i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\mathcal{S}}} - i\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\mathcal{K}}}\right)\,,$$

where $\theta^i \equiv \frac{1}{2}\epsilon^{ijk}\theta_{jk}$, $\zeta^i \equiv \theta^{i0} = -\theta^{0i}$, $\mathcal{S}^i \equiv \frac{1}{2}\epsilon^{ijk}\mathcal{S}_{jk}$, $\mathcal{K}^i \equiv \mathcal{S}^{0i} = -\mathcal{S}^{i0}$ and the $(\mathcal{S}_{\rho\sigma})^\mu{}_\nu$ are given by eq. (3). The $\mathcal{S}_{jk}$ correspond to the generators of rotation and thus provide the relevant matrix representations for the spin-1 operators.

where

$$\Lambda^0{}_0 = 1\,, \qquad \Lambda^i{}_0 = \Lambda^0{}_i = 0\,, \quad \text{and} \quad \Lambda^i{}_j = \mathcal{R}^{ij}(\phi\,,\theta\,,\gamma)\,, \tag{7}$$

and $\mathcal{R}(\phi\,,\theta\,,\gamma)$ is the rotation matrix introduced in eq. (4). Actually, the angle γ can be chosen arbitrarily, since the desired rotation is accomplished by employing the angles θ and ϕ. In the literature, one typically finds conventions where $\gamma = -\phi$ [1,2,7] or $\gamma = 0$ [3]. Ultimately, the dependence of the polarization vectors on the angle γ yields an unimportant overall phase factor. A simple computation yields:

$$\varepsilon^\mu(\hat{\boldsymbol{k}}, \pm 1) = \frac{1}{\sqrt{2}}\, e^{\mp i\gamma}\,(0\,;\, \mp\cos\theta\cos\phi + i\sin\phi\,,\, \mp\cos\theta\sin\phi - i\cos\phi\,,\, \pm\sin\theta)\ . \tag{8}$$

Note that $\varepsilon^\mu(\hat{\boldsymbol{k}}, \pm 1)$ depends only on the direction of $\vec{\boldsymbol{k}}$ and not on its magnitude $E = |\vec{\boldsymbol{k}}|$. One can easily check that the $\varepsilon^\mu(\hat{\boldsymbol{k}}, \pm 1)$ are normalized eigenstates of $\vec{\boldsymbol{\mathcal{S}}}\boldsymbol{\cdot}\hat{\boldsymbol{k}}$ with corresponding eigenvalues ± 1. The positive and negative helicity massless spin-1 polarization vectors satisfy:

$$k\!\cdot\!\epsilon(\hat{\boldsymbol{k}}\,,\,\lambda) = 0\,, \qquad\qquad \epsilon(k\,,\,\lambda)\!\cdot\!\epsilon(\hat{\boldsymbol{k}}\,,\,\lambda')^* = -\delta_{\lambda\lambda'}\,. \tag{9}$$

Consider again the case of a massless spin-1 particle moving in the z-direction with four-momentum

$$k^\mu = E(1\,;\,0\,,\,0\,,\,1)\,. \tag{10}$$

The positive and negative helicity polarization vectors are given in eq. (1). We now introduce a fixed timelike four-vector,

$$n^\mu = (1\,;\,0\,,\,0\,,\,0)\,. \tag{11}$$

To construct the polarization sum over the physical (positive and negative helicity) polarization states, it is convenient to introduce two additional (unphysical) polarization vectors,

$$\varepsilon^\mu(\hat{\boldsymbol{z}}, 0) = n^\mu = (1\,;\,0\,,\,0\,,\,0)\ , \tag{12}$$

$$\varepsilon^\mu(\hat{\boldsymbol{z}}, 3) = k^\mu/E - n^\mu = (0\,;\,0\,,\,0\,,\,1)\ , \tag{13}$$

where k^μ is given by eq. (10). It then follows that[2]

$$\epsilon(k\,,\,\lambda)\!\cdot\!\epsilon(k\,,\,\lambda')^* = \eta_\lambda \delta_{\lambda\lambda'}\,, \tag{14}$$

where

$$\eta_\lambda = \begin{cases} +1\,, & \text{for } \lambda = 0, \\ -1\,, & \text{for } \lambda = \pm 1 \text{ and } 3. \end{cases} \tag{15}$$

[2]Alternatively, one can choose the physical polarization states to be linear combinations of the positive and negative helicity states given in eq. (1) with

$$\varepsilon^\mu(\hat{\boldsymbol{z}}, 1) = (0\,;\,1\,,\,0\,,\,0)\ , \qquad \varepsilon^\mu(\hat{\boldsymbol{z}}, 2) = (0\,;\,0\,,\,1\,,\,0)\ .$$

in which case, eq. (14) can be written as

$$\epsilon(k\,,\,\lambda)\!\cdot\!\epsilon(k\,,\,\lambda')^* = g_{\lambda\lambda'}\,,$$

where $g_{\lambda\lambda'} = \text{diag}(1\,,\,-1\,,\,-1\,,\,-1)$ is the Minkowski metric tensor [8].

That is, the four polarization vectors, $\varepsilon^\mu(\hat{\boldsymbol{z}},0)$, $\varepsilon^\mu(\hat{\boldsymbol{z}},\pm1)$, and $\varepsilon^\mu(\hat{\boldsymbol{z}},3)$ constitute an orthonormal basis for four-vectors in Minkowski space. Consequently, they must obey the following completeness relation,

$$\varepsilon_\mu(\hat{\boldsymbol{z}},0)\varepsilon_\nu(\hat{\boldsymbol{z}},0)^* - \sum_{\lambda=\pm1,3}\varepsilon_\mu(\hat{\boldsymbol{z}},\lambda)\varepsilon_\nu(\hat{\boldsymbol{z}},\lambda)^* = g_{\mu\nu}\,. \tag{16}$$

We can therefore isolate the sum over the physical polarization states,

$$\sum_{\lambda=\pm1}\varepsilon_\mu(\hat{\boldsymbol{z}},\lambda)\varepsilon_\nu(\hat{\boldsymbol{z}},\lambda)^* = -g_{\mu\nu} + \varepsilon_\mu(\hat{\boldsymbol{z}},0)\varepsilon_\nu(\hat{\boldsymbol{z}},0)^* - \varepsilon_\mu(\hat{\boldsymbol{z}},3)\varepsilon_\nu(\hat{\boldsymbol{z}},3)^*\,. \tag{17}$$

It is convenient to rewrite eq. (17) with the help of eqs. (12) and (13). Noting that $k\cdot n = E$ [cf. eqs. (10) and (11)], it follows that

$$\sum_{\lambda=\pm1}\varepsilon_\mu(\hat{\boldsymbol{k}},\lambda)\varepsilon_\nu(\hat{\boldsymbol{k}},\lambda)^* = -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot n)^2} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n}\,. \tag{18}$$

Although eq. (18) was derived for the case of $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$, it is straightforward to check that eq. (18) is also valid for the case where $\hat{\boldsymbol{k}}$ points in an arbitrary direction.[3]

An alternative form for eq. (18) is obtained by introducing the four-vector

$$\overline{k}^\mu \equiv 2(k\cdot n)n^\mu - k^\mu\,. \tag{19}$$

More explicitly, if $k^\mu = (E\,;\,\vec{\boldsymbol{k}})$, where $E = |\vec{\boldsymbol{k}}|$ for a massless particle, then $k\cdot n = E$ and $\overline{k}^\mu = (E\,;\,-\vec{\boldsymbol{k}})$. Then, it follows that

$$\sum_{\lambda=\pm1}\varepsilon_\mu(\hat{\boldsymbol{k}},\lambda)\varepsilon_\nu(\hat{\boldsymbol{k}},\lambda)^* = -g_{\mu\nu} + \frac{k_\mu\overline{k}_\nu + k_\nu\overline{k}_\mu}{k\cdot\overline{k}}\,. \tag{20}$$

Indeed, using the fact that $k\cdot\overline{k} = 2(k\cdot n)^2$ [since $k^2 = 0$ for a massless particle], one can easily verify that eqs. (18) and (20) are equivalent.

It should be appreciated that the sum over physical massless spin-1 polarization states is not Lorentz covariant, since n^μ is fixed and does not transform under a Lorentz transformation.[4] That is, the sum over physical massless spin-1 polarization states depends on the frame of reference of the spin-1 particle. Nevertheless, in scattering or decay processes involving massless spin-1 particles, the dependence on the four-vector n^μ (or equivalently, the four-vector $\overline{k}^\mu$) must drop out of any expression for a physical (i.e., measurable) observable.

[3] After raising the indices μ and ν in eq. (18), one simply multiplies both sides of the resulting equation by $\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu$, where the matrix elements of Λ are specified in eq. (7). In particular, let us denote the four-momentum defined in eq. (10) by $k_z^\mu = E(1\,;\,0\,,\,0\,,\,1)$. Then it follows that $k^\alpha = \Lambda^\alpha{}_\mu k_z^\mu = E(1\,;\,\hat{\boldsymbol{k}})$. In addition, in light of eq. (11), we have $\Lambda^\alpha{}_\mu n^\mu = n^\alpha$ and $k\cdot n = E$ independently of the direction of $\hat{\boldsymbol{k}}$.

[4] However, the polarization sum is covariant with respect to three-dimensional rotations, since n^μ is rotationally invariant. In contrast, n^μ (which by definition remains fixed under a Lorentz transformation) does not behave like a four-vector with respect to Lorentz boosts.

The results above should be contrasted with the case of a spin-1 particle of mass $m \neq 0$. The expressions given by eqs. (1) and (8) also apply in the case of a massive spin-1 particle. In addition, there exists an helicity $\lambda = 0$ polarization vector that depends on the magnitude of the momentum as well as its direction:

$$\varepsilon^\mu(|\vec{\boldsymbol{k}}|\hat{\boldsymbol{z}}\,,\,0) = (|\vec{\boldsymbol{k}}|/m\,;\,0\,,\,0\,,\,E/m)\,, \tag{21}$$

where $E = (|\vec{\boldsymbol{k}}|^2 + m^2)^{1/2}$. One can use eq. (6) to obtain the helicity zero polarization vector for a massive spin-1 particle moving in an arbitrary direction

$$\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,0) = \frac{1}{m}\Big(|\vec{\boldsymbol{k}}|\,;\,E\sin\theta\cos\phi\,,\,E\sin\theta\sin\phi\,,\,E\cos\theta\Big)\,. \tag{22}$$

Note that both the massless and massive spin-1 polarization vectors satisfy:[5]

$$\epsilon^\mu(\vec{\boldsymbol{k}}\,,\lambda)^* = (-1)^\lambda\epsilon^\mu(\vec{\boldsymbol{k}}\,,-\lambda)\,. \tag{23}$$

One can check that the $\epsilon^\mu(\vec{\boldsymbol{k}},\lambda)$ of a massive spin-one particle also satisfy,

$$k\cdot\epsilon(\vec{\boldsymbol{k}}\,,\,\lambda) = 0\,, \qquad\qquad \epsilon(\vec{\boldsymbol{k}}\,,\,\lambda)\cdot\epsilon(\vec{\boldsymbol{k}}\,,\,\lambda')^* = -\delta_{\lambda\lambda'}\,, \tag{24}$$

for helicity states $\lambda = -1, 0, +1$.

To construct the polarization sum for a massive spin-1 particle, we introduce a fourth unphysical polarization vector,

$$\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,S) = k^\mu/m\,, \tag{25}$$

where S stands for the unphysical "scalar" mode. Note that since $k^2 = m^2$ for a particle of mass m, it follows that $\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,S)\cdot\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,S)^* = 1$. In addition, $\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,S)\cdot\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,\lambda)^* = 0$ for $\lambda = -1, 0, +1$. Hence the four polarization vectors $\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,\lambda)$, $\lambda = S, -1, 0, +1$ form an orthonormal basis for four-vectors in Minkowski space. It then follows that the corresponding completeness relation,

$$\varepsilon_\mu(\vec{\boldsymbol{k}}\,,\,S)\varepsilon_\nu(\vec{\boldsymbol{k}}\,,\,S)^* - \sum_{\lambda=-1,0,+1}\varepsilon_\mu(\vec{\boldsymbol{k}}\,,\,\lambda)\varepsilon_\nu(\vec{\boldsymbol{k}}\,,\,\lambda)^* = g_{\mu\nu}\,, \tag{26}$$

must be satisfied. In light of eq. (25), we obtain the following expression for the sum over physical polarization states of a spin-1 particle of mass $m \neq 0$,

$$\sum_{\lambda=-1,0,+1}\varepsilon_\mu(\vec{\boldsymbol{k}}\,,\,\lambda)\,\varepsilon_\nu(\vec{\boldsymbol{k}}\,,\,\lambda)^* = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\,. \tag{27}$$

In contrast to the case of the massless spin-1 particle, the polarization sum for a massive spin-1 particle given in eq. (27) is Lorentz covariant.

There is no smooth limit as $m \to 0$ for the polarization states of a spin-1 particle. This is because the massive spin-1 particle exhibits three possible helicities, whereas the massless spin-1 particle exhibits two possible helicities. Further details on polarization vectors for both massless and massive spin-1 states can be found in Refs. [8, 10].

[5]Some authors introduce polarization vectors where the sign factor $(-1)^\lambda$ in eq. (23) is omitted. One motivation for eq. (23) is to maintain consistency with the Condon-Shortley phase conventions [9] for the eigenfunctions of the spin-1 angular momentum operators $\vec{\boldsymbol{S}}^{\boldsymbol{2}}$ and S_z. In particular, if we denote the unit three-vector in the radial direction by $\hat{\boldsymbol{r}}$, then the relation $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\varepsilon}}^\mu(\hat{\boldsymbol{z}},\pm 1) = (4\pi/3)^{1/2}Y_{1,\pm 1}(\theta,\phi)$ between the polarization three-vectors and the $\ell = 1$ spherical harmonics holds without any additional sign factors.

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