1. BRST invariance of QED

The Lagrangian density of QED is given by:

$$\mathscr{L}_{\rm QED} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/)\psi - m\overline{\psi}\psi - \frac{1}{2a}(\partial_\mu A^\mu)^2\,, \tag{1}$$

where the electron has charge $-e$ (in a convention where $e > 0$). The presence of the gauge fixing term, which is required in order to be able to define a photon propagator, naively spoils the gauge invariance of QED. To see why, consider the infinitesimal gauge transformation,

$$\delta A_\mu(x) = \partial_\mu\Lambda(x)\,, \tag{2}$$

$$\delta\psi(x) = ie\Lambda(x)\psi(x)\,, \tag{3}$$

$$\delta\overline{\psi}(x) = -ie\Lambda(x)\overline{\psi}(x)\,, \tag{4}$$

where $\Lambda(x)$ is an arbitrary real function of x that vanishes (sufficiently fast) as $|\vec{\boldsymbol{x}}| \to \infty$.

Under the infinitesimal gauge transformations given in eqs. (2)–(4),

$$\delta F_{\mu\nu} = \delta(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Lambda = 0\,,$$

since the partial derivatives commute under the assumption that $\Lambda(x)$ is a smooth function. Likewise, since $\Lambda(x)$ is a real function, we have

$$\begin{aligned}
\delta(\overline{\psi}\psi) &= \delta\overline{\psi}\psi + \overline{\psi}\delta\psi = -ie\Lambda\overline{\psi}\psi + ie\Lambda\overline{\psi}\psi = 0\,,\\
\delta(\overline{\psi}\partial\!\!\!/\psi) &= -ie\Lambda\overline{\psi}\partial\!\!\!/\psi + ie(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi + ie\Lambda\overline{\psi}\partial\!\!\!/\psi = ie(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi\,,\\
\delta(\overline{\psi}A\!\!\!/\psi) &= -ie\Lambda\overline{\psi}A\!\!\!/\psi + (\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi + ie\Lambda\overline{\psi}A\!\!\!/\psi = (\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi\,.
\end{aligned}$$

Hence, it follows that

$$\delta\bigl(\overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/)\psi\bigr) = -e(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi + e(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi = 0\,,$$

as expected. Finally, working to first order in the field variations,[1]

$$\delta(\partial_\mu A^\mu)^2 = 2(\partial_\mu A^\mu)\delta(\partial_\mu A^\mu) = 2(\partial_\mu A^\mu)\partial_\mu(\delta A^\mu) = 2(\Box\Lambda)\partial_\mu A^\mu\,.$$

Thus, the variation of the QED Lagrangian given in eq. (1) is

$$\delta\mathscr{L}_{\rm QED} = -\frac{1}{a}\,(\Box\Lambda)(\partial_\mu A^\mu)\,, \tag{5}$$

which is non-vanishing.

[1] Alternatively, we can write

$$\delta(\partial_\mu A^\mu)^2 = \partial_\mu(A^\mu + \partial^\mu\Lambda)\partial_\nu(A^\nu + \partial^\nu\Lambda) - (\partial_\mu A^\mu)^2 = 2(\Box\Lambda)\partial_\mu A^\mu$$

after dropping terms that are quadratic in Λ.

Consider the modified Lagrangian,[2]

$$\mathscr{L} = \mathscr{L}_{\rm QED} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi\,, \tag{6}$$

where $\phi(x)$ is a free (commuting) scalar field. We now define a so-called generalized gauge transformation, whose infinitesimal form is given by,

$$\delta A_\mu(x) = \epsilon\,\partial_\mu\phi(x)\,, \tag{7}$$
$$\delta\psi(x) = ie\epsilon\,\phi(x)\psi(x)\,, \tag{8}$$
$$\delta\overline{\psi}(x) = -ie\epsilon\,\overline{\psi}(x)\phi(x)\,, \tag{9}$$
$$\delta\phi(x) = -\frac{\epsilon}{a}\,\partial_\mu A^\mu\,, \tag{10}$$

where ϵ is an infinitesimal parameter. This transformation is called a Becchi-Rouet-Stora-Tyutin (BRST) transformation.

We now show that the modified action,

$$S[A_\mu,\psi,\overline{\psi},\phi] \equiv \int d^4x\,\left[\mathscr{L}_{\rm QED} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi\right]\,, \tag{11}$$

is BRST-invariant. Note that eq. (5) is still applicable with $\Lambda(x)$ replaced with $\epsilon\,\phi(x)$. It then follows from eq. (5) that

$$\delta\mathscr{L}_{\rm QED} = -\frac{\epsilon}{a}\,(\Box\phi)(\partial_\mu A^\mu)\,.$$

Using eq. (9) and working to first order in the field variations,

$$\delta(\partial_\mu\phi\partial^\mu\phi) = \partial_\mu(\delta\phi)\partial^\mu\phi + \partial_\mu\phi\partial^\mu(\delta\phi) = -\frac{2\epsilon}{a}(\partial_\mu\partial_\nu A^\nu)(\partial^\mu\phi)\,.$$

Hence,

$$\delta\mathscr{L} = -\frac{\epsilon}{a}\big[(\Box\phi)(\partial_\mu A^\mu) + (\partial_\mu\partial_\nu A^\nu)(\partial^\mu\phi)\big] = -\frac{\epsilon}{a}\,\partial^\mu\big[(\partial_\mu\phi)(\partial_\nu A^\nu)\big]\,,$$

which we recognize as a total divergence. Hence, the variation of the action,

$$\delta S = \int d^4x\,\delta\mathscr{L} = 0\,,$$

under the usual assumption that the fields at infinity vanish so that the surface terms vanish. That is, the modified action defined in eq. (6) is invariant under the BRST transformations.

[2]In QED, we can choose $\phi(x)$ to be either a commuting or an anticommuting scalar field (if $\phi(x)$ is anticommuting, then ϵ must be an anticommuting Grassmann infinitesimal constant). When we generalize this construction to a nonabelian gauge theory, we will need to employ an anticommuting scalar in the modified Lagrangian.

2. Using BRST invariance to derive Ward identities

One can employ the BRST invariance of the modified QED action to derive Ward identities. As a first step, consider any string of electron and/or photon field operators, denoted symbolically by $\mathcal{O}$. Then, we shall derive the following relations among Green functions:

$$\langle\Omega|T(\phi(x)\mathcal{O})|\Omega\rangle = 0\,, \tag{12}$$

$$\langle\Omega|T(\phi(x)\phi(y)\mathcal{O})|\Omega\rangle = \langle\Omega|T\big(\phi(x)\phi(y)\big)|\Omega\rangle\langle\Omega|T(\mathcal{O})|\Omega\rangle\,. \tag{13}$$

In particular, let

$$\mathcal{O}(A_\mu,\overline{\psi},\psi) = \psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_n)A_{\mu_1}(z_1)\cdots A_{\mu_n}(z_n)\,.$$

The generating functional is given by

$$\begin{aligned}
Z[J_\mu,\overline{\zeta},\zeta,J] &= \mathcal{N}\int \mathcal{D}A^\mu\,\mathcal{D}\phi\,\exp\left[i\int d^4x\,(\mathscr{L}+J_\mu A^\mu + J\phi + \overline{\zeta}\psi + \overline{\psi}\zeta)\right] \\
&= \mathcal{N}\int \mathcal{D}A^\mu\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\exp\left[i\int d^4x\,(\mathscr{L}_{\rm QED}+J_\mu A^\mu) + \overline{\zeta}\psi + \overline{\psi}\zeta\right] \\
&\qquad\times\int\mathcal{D}\phi\,\exp\left\{i\int d^4x\,\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + J\phi\right]\right\}\,,
\end{aligned}$$

where $\mathcal{N}$ is chosen such that $Z[0,0]=1$, and $\mathscr{L}$ was given in eq. (6). Employing the explicit form for $\mathscr{L}$, we can carry out explicitly the path integral over the massless free field $\phi(x)$,

$$\begin{aligned}
Z[J_\mu,\overline{\zeta},\zeta,J] &= \mathcal{N}'\int \mathcal{D}A^\mu\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\exp\left[i\int d^4x\,(\mathscr{L}_{\rm QED}+J_\mu A^\mu) + \overline{\zeta}\psi + \overline{\psi}\zeta\right] \\
&\qquad\times\exp\left\{\tfrac{1}{2}i\int d^4x\,d^4y\,J(x)(\Box - i\epsilon)^{-1}J(y)\right\}\,,
\end{aligned} \tag{14}$$

where $\mathcal{N}'$ is chosen such that $Z[0,0]=1$. It follows that

$$\begin{aligned}
\langle\Omega|T(\phi(x)\mathcal{O})|\Omega\rangle &= \mathcal{N}\int \mathcal{D}A^\mu\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\mathcal{O}(A_\mu,\overline{\psi},\psi)\exp\left[i\int d^4x\,(\mathscr{L}_{\rm QED}+J_\mu A^\mu)\right] \\
&\qquad\times\int\mathcal{D}\phi\,\phi(x)\exp\left\{i\int d^4x\,\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + J\phi\right]\right\} = 0\,,
\end{aligned}$$

where we have noted that the path integral over $\phi(x)$ vanishes since the integrand is an odd function of $\phi(x)$. One can reach the same conclusion by taking a functional derivative of eq. (14) with respect to $J(x)$ and then setting $J=0$. Likewise,

$$
\begin{aligned}
\langle\Omega|T(\phi(x)\phi(y)\mathcal{O})|\Omega\rangle &= \mathcal{N}\int \mathcal{D}A^\mu\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\mathcal{O}(A_\mu,\overline{\psi},\psi)\exp\left[i\int d^4x\,(\mathscr{L}_{\rm QED}+J_\mu A^\mu)\right] \\
&\qquad\qquad \times\int\mathcal{D}\phi\,\phi(x)\phi(y)\exp\left\{i\int d^4x\,\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+J\phi\right]\right\} \\
&= \langle\Omega|T(\mathcal{O})|\Omega\rangle\langle\Omega|T\big(\phi(x)\phi(y)\big)|\Omega\rangle\,,
\end{aligned}
$$

since the path integral over A_μ, $\overline{\psi}$ and ψ and the path integral over ϕ factorize. That is, $Z[J_\mu,\overline{\zeta},\zeta,J]$ can be written as a product of the QED generating functional and the generating functional for a free massless scalar field. Indeed, by taking functional derivatives with respect to the sources and then setting those sources to zero, we see that the corresponding Green functions also factorize.

Eqs. (12) and (13) also hold for the corresponding connected Green functions, since a similar proof can be presented in which the generating functional $Z=\exp(iW)$ is replaced by W. Henceforth, all Green functions that appear below are assumed to be connected Green functions,

In light of eq. (12),

$$
\langle\Omega|T(\psi(x)\overline{\psi}(y)\phi(z))|\Omega\rangle = 0\,.
$$

Using the BRST-invariance of the modified QED action, this Green function must remain zero under an (infinitesimal) BRST-transformation. That is,

$$
\delta\langle\Omega|T(\psi(x)\overline{\psi}(y)\phi(z))|\Omega\rangle = 0\,. \tag{15}
$$

Using the functional integral representation of the (connected) Green function, it immediately follows that the operation of δ obeys the product rule of the derivative. That is, eq. (15) yields,

$$
\langle\Omega|T(\delta\psi(x)\overline{\psi}(y)\phi(z))|\Omega\rangle+\langle\Omega|T(\psi(x)\delta\overline{\psi}(y)\phi(z))|\Omega\rangle+\langle\Omega|T(\psi(x)\overline{\psi}(y)\delta\phi(z))|\Omega\rangle = 0\,. \tag{16}
$$

Making use of eqs. (8)–(10), it follows that

$$
\begin{aligned}
&ie\big[\langle\Omega|T(\psi(x)\overline{\psi}(y)\phi(x)\phi(z))|\Omega\rangle-\langle\Omega|T(\psi(x)\overline{\psi}(y)\phi(y)\phi(z))|\Omega\rangle\big] \\
&\qquad -\frac{1}{a}\langle\Omega|T(\psi(x)\overline{\psi}(y)\partial_\mu A^\mu(z))|\Omega\rangle = 0\,.
\end{aligned} \tag{17}
$$

Note that

$$
\frac{\partial}{\partial z^\mu}\,T\big(\psi(x)\overline{\psi}(y)A^\mu(z)\big) = T\big(\psi(x)\overline{\psi}(y)\partial_\mu A^\mu(z)\big)\,, \tag{18}
$$

since the equal time commutators that arise when pushing the derivative through the T symbol vanish,

$$
\delta(z_0-x_0)\big[A_\mu(z),\overline{\psi}(y)\big] = \delta(z_0-x_0)\big[A_\mu(z),\psi(x)\big] = 0\,. \tag{19}
$$

Hence, it follows that

$$ie\big[\langle\Omega|T(\psi(x)\overline{\psi}(y)\phi(x)\phi(z))|\Omega\rangle - \langle\Omega|T(\psi(x)\overline{\psi}(y)\phi(y)\phi(z))|\Omega\rangle\big]$$
$$-\frac{1}{a}\frac{\partial}{\partial z^\mu}\langle\Omega|T(\psi(x)\overline{\psi}(y)A^\mu(z))|\Omega\rangle = 0\,. \tag{20}$$

We now introduce the full fermion propagator and vertex function (using the same notation as in the class lectures),

$$iS(p) = \int d^4x\, e^{ipx}\langle\Omega|\psi(x)\overline{\psi}(0)|\Omega\rangle\,, \tag{21}$$

$$\mathcal{V}_\mu(p,p+q) = \int d^4x\, d^4z\, e^{i(px+qz)}\langle\Omega|\psi(x)\overline{\psi}(0)A_\mu(z)|\Omega\rangle\,, \tag{22}$$

where translational invariance of the Green functions has been used to set the coordinate $y = 0$. Using eq. (13), it follows that

$$\int d^4x\, d^4z\, e^{i(px+qz)}\langle\Omega|T\psi(x)\overline{\psi}(0)\phi(x)\phi(z))|\Omega\rangle$$
$$= \int d^4x\, d^4z\, e^{i(px+qz)}\langle\Omega|T\psi(x)\overline{\psi}(0)|\Omega\rangle\langle\Omega|T\phi(x)\phi(z)|\Omega\rangle\,. \tag{23}$$

Since $\phi(x)$ is a free massless scalar field, its full propagator corresponds to a free field propagator,

$$\langle\Omega|T\phi(x)\phi(z)|\Omega\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-z)}\frac{i}{k^2 - i\epsilon}\,. \tag{24}$$

Inserting this result into eq. (23) yields,

$$\int d^4x\, d^4z\, e^{i(px+qz)}\langle\Omega|T\psi(x)\overline{\psi}(0)\phi(x)\phi(z))|\Omega\rangle$$
$$= \int d^4x\, d^4z\, \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2 - i\epsilon}\, e^{-ik(x-z)}\, e^{i(px+qz)}\langle\Omega|T\psi(x)\overline{\psi}(0)|\Omega\rangle$$
$$= \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2 - i\epsilon}\, iS(p-k)\,\delta^4(q+k) = -\frac{1}{q^2 - i\epsilon}\, S(p+q)\,, \tag{25}$$

after recognizing the integral representation of the delta function. Likewise,

$$\int d^4x\, d^4z\, e^{i(px+qz)}\langle\Omega|T\psi(x)\overline{\psi}(0)\phi(0)\phi(z))|\Omega\rangle = -\frac{1}{q^2 - i\epsilon}\, S(p)\,. \tag{26}$$

Finally, one integration by parts yields,

$$\int d^4x\, d^4z\, e^{i(px+qz)}\frac{\partial}{\partial z^\mu}\langle\Omega|\psi(x)\overline{\psi}(0)A_\mu(z)|\Omega\rangle = -iq_\mu\mathcal{V}^\mu(p,p+q)\,. \tag{27}$$

Thus, we can act with

$$\int d^4x\, d^4z\, e^{i(px+qz)}$$

on eq. (20) and make use of eqs. (25)–(27) to obtain,

$$iq_\mu \mathcal{V}^\mu = \frac{iea}{q^2 - i\epsilon}\big[S(p+q) - S(p)\big]\,. \tag{28}$$

We can manipulate eq. (28) into a more familiar form by introducing the 1PI three-point Green function Γ_μ via,

$$\mathcal{V}_\mu(x_1, x_2, x_3) = i\int d^4y_1\, d^4y_2\, d^4y_3\, G_c^{(2)}(x_1, y_1) G_c^{(2)}(x_2, y_1) G_c^{(2)}(x_3, y_3)\Gamma_\mu(y_1, y_2, y_3)\,, \tag{29}$$

and its momentum space equivalent,

$$\mathcal{V}_\mu(p, p') = i\mathscr{D}_{\mu\nu}(q)\, iS(p')\, \Gamma^\mu(p, p')\, iS(p)\,, \tag{30}$$

where $p' \equiv p + q$, $G_c^{(2)}(p) \equiv iS(p)$, and $\mathscr{D}_{\mu\nu}(q)$ is the full photon propagator. Multiplying eq. (30) by q^μ and employing the identity proved in class (which states that the longitudinal part of the full photon propagator is equal to that of the tree-level photon propagator),

$$q^\mu \mathscr{D}_{\mu\nu}(q) = -\frac{iaq_\nu}{q^2 - i\epsilon}\,, \tag{31}$$

it follows from eqs. (28), (30) and (31) that

$$\frac{ie}{q^2 - i\epsilon}\big[S(p+q) - S(p)\big] = -\frac{iq_\nu}{q^2 - i\epsilon}\, S(p+q)\, \Gamma_\nu(p, p')\, S(p)\,. \tag{32}$$

Finally, multiplying the above equation by $S^{-1}(p+q)$ on the left and $S^{-1}(p)$ on the right yields the famous QED Ward-Takahashi identity relating the inverse propagator and the 1PI vertex function,

$$q^\mu \Gamma_\mu(p, p+q) = e\big[S^{-1}(p+q) - S^{-1}(p)\big]\,. \tag{33}$$

3. BRST invariance in nonabelian gauge theory

Consider the Lagrangian density for a non-abelian Yang-Mills gauge theory, with gauge field A_μ^a and gauge field strength tensor $F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c$,

$$\mathscr{L}_{\rm YM} = -\tfrac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}\,. \tag{34}$$

Eq. (34) is invariant under the gauge transformation,

$$\delta A_\mu^a(x) = \epsilon D_\mu^{ab}\omega_b(x)\,, \tag{35}$$

where ϵ is an infinitesimal constant, g is the gauge coupling constant, and $\omega_b(x)$ is an arbitrary function of x. The covariant derivative acting on a field in the adjoint representation is $D_\mu^{ab} = \delta^{ab}\partial_\mu + ig(T^c)^{ab}A_\mu^c$, where the generators of the Lie group in the adjoint representation are given by $(T^c)^{ab} = -if^{cab}$. That is,

$$D_\mu^{ab} = \delta^{ab}\partial_\mu + gf^{abc}A_\mu^c\,. \tag{36}$$

Note that generators of the gauge group (in any representation) satisfy the commutation relations,

$$[T^a\,,\,T^b] = if^{abc}T^c\,, \tag{37}$$

and have been chosen such that the structure constants f^{abc} are completely antisymmetric under the interchange of any pair of indices.

As in QED, order to be able to define a propagator for the gauge field, one must add a gauge-fixing term,

$$\mathscr{L}_{\rm GF} = -\frac{1}{2a}(\partial^\mu A^a_\mu)^2\,. \tag{38}$$

However, this term spoils the gauge invariance of the theory. In particular, under an infinitesimal gauge transformation given in eq. (35), and working to first order in ϵ,

$$\delta(\partial^\mu A^a_\mu)^2 = 2(\partial^\mu A^a_\mu)\delta(\partial^\nu A^a_\nu) = 2(\partial^\mu A^a_\mu)(\partial^\nu \delta A^a_\nu) = \epsilon(\partial^\mu A^a_\mu)\big(\partial^\nu D^{ab}_\nu \omega_b\big)\,,$$

after noting that $\delta(\partial^\mu A^a_\mu) \equiv \partial^\mu(A^a_\mu + \delta A^a_\mu) - \partial^\mu A^a_\mu = \partial^\mu(\delta A^a_\mu)$. Hence,

$$\delta\mathscr{L}_{\rm GF} = -\frac{\epsilon}{a}(\partial^\mu A^a_\mu)(\partial^\nu D^{ab}_\nu \omega_b)\,, \tag{39}$$

which is non-vanishing. Of course, we discovered a similar result for QED in eq. (5).

Suppose we attempt to define a generalized gauge symmetry by adding a new field $\eta_a(x)$ that transforms under the adjoint representation of the gauge group, along with a new term to the Lagrangian:

$$\mathscr{L}_{\rm G} = -\eta_a(\partial^\mu D^{ab}_\mu \omega_b)\,, \tag{40}$$

and by postulating the transformation law:

$$\delta\eta_a(x) = -\frac{\epsilon}{a}(\partial_\mu A^\mu_a)\,. \tag{41}$$

Using eq. (36), we can rewrite eq. (40) as

$$\mathscr{L}_{\rm G} = -\eta_a\,\partial^\mu\big(\partial_\mu\omega_a + gf^{abc}\omega_b A^c_\mu\big)\,. \tag{42}$$

Next, we apply an infinitesimal gauge transformation to eq. (42). Working to first order in ϵ,

$$\begin{aligned}\delta\mathscr{L}_{\rm G} &= -(\delta\eta_a)(\partial^\mu D^{ab}_\mu\omega_b) - \eta_a(\partial^\mu \delta D^{ab}_\mu \omega_b) = -(\delta\eta_a)(\partial^\mu D^{ab}_\mu\omega_b) - gf^{abc}\eta_a\partial^\mu\big(\omega_b\delta A^c_\mu\big)\\ &= \frac{\epsilon}{a}(\partial_\nu A^\nu_a)(\partial^\mu D^{ab}_\mu\omega_b) - \epsilon\, gf^{abc}\eta_a\partial^\mu\big(\omega_b D^{cd}_\mu\omega_d\big)\,.\end{aligned} \tag{43}$$

Hence,

$$\delta(\mathscr{L}_{\rm GF} + \mathscr{L}_{\rm G}) = -\epsilon\, gf^{abc}\eta_a\partial^\mu\big(\omega_b D^{cd}_\mu\omega_d\big) \neq 0\,.$$

In contrast to QED (where $f^{abc} = 0$), we have failed to restore the generalized gauge symmetry. However, all is not lost. Let us save the day by promoting the function $\omega_a(x)$ to a field and postulating the transformation law:

$$\delta\omega_a(x) = \tfrac{1}{2}\epsilon g f^{abc}\omega_b\omega_c\,, \tag{44}$$

where summation over repeated indices is implied. Note that since the f^{abc} are totally antisymmetric under interchange of a, b and c, the only way to have $\delta\omega_a(x) \neq 0$ is to require that $\omega_a(x)$ is an *anticommuting* field that transforms under the adjoint representation of the gauge group. This immediately implies that $\eta_a(x)$ is an anticommuting field and ϵ is an anticommuting infinitesimal constant.[3]

In light of eq. (44), eq. (43) is modified as follows:

$$\begin{aligned}
\delta\mathscr{L}_{\rm G} &= -(\delta\eta_a)(\partial^\mu D_\mu^{ab}\omega_b) - \eta_a[\partial^\mu(\delta D_\mu^{ab})\omega_b] - \eta_a(\partial^\mu D_\mu^{ab}\delta\omega_b) \\
&= \frac{\epsilon}{a}(\partial_\nu A_a^\nu)(\partial^\mu D_\mu^{ab}\omega_b) - g f^{abc}\eta_a\partial^\mu\big(\omega_b\epsilon D_\mu^{cd}\omega_d\big) - \tfrac{1}{2}g f^{bcd}\eta_a\,\partial^\mu\epsilon D_\mu^{ab}(\omega_c\omega_d) \\
&= \frac{\epsilon}{a}(\partial_\nu A_a^\nu)(\partial^\mu D_\mu^{ab}\omega_b) - \epsilon g\eta_a\partial^\mu\left\{f^{abc}\big(\omega_b D_\mu^{cd}\omega_d\big) - \tfrac{1}{2}f^{bcd}D_\mu^{ab}(\omega_c\omega_d)\right\}, \qquad (45)
\end{aligned}$$

where we have made use of the anticommuting properties of ϵ, η_a and ω_a.

Focusing on the term inside the braces in eq. (45),

$$\begin{aligned}
f^{abc}\omega_b D_\mu^{cd}\omega_d - \tfrac{1}{2}f^{bcd}D_\mu^{ab}(\omega_c\omega_d) &= f^{abc}\omega_b(\partial_\mu\omega_c + g f^{cde}\omega_d A_\mu^e) - \tfrac{1}{2}f^{bcd}(\delta^{ab}\partial_\mu + g f^{abe}A_\mu^e)(\omega_c\omega_d) \\
&= f^{abc}\omega_b\partial_\mu\omega_c - \tfrac{1}{2}f^{acd}\big[\omega_c\partial_\mu\omega_d + (\partial_\mu\omega_c)\omega_d\big] + gA_\mu^e\big(f^{abc}f^{cde}\omega_b\omega_d - \tfrac{1}{2}f^{bcd}f^{abe}\omega_c\omega_d\big)\,.
\end{aligned}$$

After an appropriate relabeling of indicies,

$$f^{abc}\omega_b\partial_\mu\omega_c - \tfrac{1}{2}f^{acd}\big[\omega_c\partial_\mu\omega_d + (\partial_\mu\omega_c)\omega_d\big] = f^{abc}\big[\omega_b\partial_\mu\omega_c - \tfrac{1}{2}\omega_b\partial_\mu\omega_c - \tfrac{1}{2}(\partial_\mu\omega_b)\omega_c\big] = 0\,,$$

after using the anticommuting properties of ω_b and ω_c and antisymmetry properties of the f^{abc}. Likewise, using the same properties and appropriate relabeling of indices,

$$\begin{aligned}
f^{abc}f^{cde}\omega_b\omega_d - \tfrac{1}{2}f^{bcd}f^{abe}\omega_c\omega_d &= \omega_c\omega_d(f^{bac}f^{bde} + \tfrac{1}{2}f^{bcd}f^{bae}) \\
&= \tfrac{1}{2}\omega_c\omega_d(f^{bac}f^{bde} + f^{bad}f^{bec} + f^{bae}f^{bcd}) = 0\,, \qquad (46)
\end{aligned}$$

where the last step is a consequence of the Jacobi identity (that is always satisfied by the structure constants of a Lie algebra). Hence, the expression inside the braces in eq. (45) vanishes, and we are left with

$$\delta\mathscr{L}_{\rm G} = \frac{\epsilon}{a}(\partial_\nu A_a^\nu)(\partial^\mu D_\mu^{ab}\omega_b)\,.$$

Combining with eq. (39) yields

$$\delta(\mathscr{L}_{\rm GF} + \mathscr{L}_{\rm G}) = 0\,.$$

In conclusion, we have shown that the modified Lagrangian density,

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2a}(\partial^\mu A_\mu^a)^2 - \eta_a(\partial^\mu D_\mu^{ab}\omega_b)\,, \qquad (47)$$

is invariant under the following infinitesimal generalized gauge transformations, collectively

[3] In some books, ϵ is taken to be a commuting infinitesimal constant, in which case one would have to view δ as an anticommuting operator. In this convention, e.g., $\delta(\omega_a\omega_b) = (\delta\omega_a)\omega_b - \omega_a\delta\omega_b$. In these notes, we will not adopt this convention, in which case δ satisfies the usual product rule for derivatives when acting on a product of anticommuting fields, e.g., $\delta(\omega_a\omega_b) = (\delta\omega_a)\omega_b + \omega_a\delta\omega_b$. Instead, the anticommuting nature of ϵ will generate an extra minus sign when passing through an anticommuting field.

known as the BRST transformations,

$$\delta A^a_\mu(x) = \epsilon D^{ab}_\mu \omega_b(x)\,, \tag{48}$$

$$\delta\eta_a(x) = -\frac{\epsilon}{a}\big(\partial_\mu A^\mu_a(x)\big)\,, \tag{49}$$

$$\delta\omega_a(x) = \tfrac{1}{2}\epsilon g f^{abc}\omega_b(x)\omega_c(x)\,. \tag{50}$$

We recognize $\mathcal{L}_G$ [cf. eqs. (40) and (42)] as the Faddeev-Popov Lagrangian,[4] which is usually derived via the path integral formalism of gauge field theory. Note that the Faddeev-Popov fields, η_a and ω_a, are sometimes called η^*_a and η_a, respectively (just to keep you on your toes).

If we add fermions to the theory, one can easily extend the BRST transformation law. The fermion Lagrangian is given by

$$\mathscr{L}_F = i\overline{\psi}_i\gamma_\mu D^\mu_{ij}\psi_j\,, \tag{51}$$

with $D^\mu_{ij} = \delta_{ij}\partial^\mu + igT^a_{ij}A^\mu_a$ (where we employ the generators T^a appropriate to the representation of the fermions under the gauge group). Under a generalized gauge transformation,

$$\delta\psi(x) = -i\epsilon\, gT^a\omega_a(x)\psi(x)\,, \tag{52}$$

$$\delta\overline{\psi}(x) = i\epsilon\, gT^a\omega_a(x)\overline{\psi}(x)\,. \tag{53}$$

This is to be expected, as eq. (51) is invariant under ordinary gauge transformations.

One critical property of the BRST transformation operator δ is that it is nilpotent. That is, applying δ twice to any on-shell field produces zero. In particular, we define δ^2 to mean the application of δ with anticommuting parameter ϵ_1 followed by δ with anticommuting parameter ϵ_2. Let us see what happens when we apply δ^2 to the fields A^a_μ, ψ, $\overline{\psi}$, ω_a and η_a.

We first compute

$$\begin{aligned}
\delta^2 A^a_\mu(x) &= \epsilon_1\delta\big[\partial_\mu\omega_a + gf^{abc}\omega_b A^c_\mu\big] \\
&= \epsilon_1\epsilon_2\big\{\tfrac{1}{2}gf^{abc}\partial_\mu(\eta_b\omega_c) + gf^{abc}(D^{cd}_\mu\omega_d)\omega_b + \tfrac{1}{2}gf^{abc}f^{bde}\omega_d\omega_e A^c_\mu\big\} \\
&= \epsilon_1\epsilon_2\big\{\tfrac{1}{2}gf^{abc}\big[(\partial_\mu\omega_b)\omega_c + \omega_b(\partial_\mu\omega_c)\big] + gf^{abc}\big[\partial_\mu\omega_c + gf^{cde}A^d_\mu\omega_e\big]\omega_b \\
&\qquad\qquad + \tfrac{1}{2}g^2 f^{abc}f^{bde}\omega_d\omega_e A^c_\mu\big\} \\
&= \epsilon_1\epsilon_2\, g^2\omega_e\omega_b A^d_\mu\big(f^{cab}f^{cde} + \tfrac{1}{2}f^{cad}f^{ceb}\big) = 0\,,
\end{aligned} \tag{54}$$

after an appropriate relabeling of indices. The final steps are the same as in eq. (46), where after some manipulation the Jacobi identity is invoked. Next,

$$\begin{aligned}
\delta^2\psi(x) &= -i\epsilon_1 gT^a\big[\big(\delta\omega_a\big)\psi + \omega_a\delta\psi\big] \\
&= -i\epsilon_1\epsilon_2\, g^2\big[\tfrac{1}{2}T^a f^{abc}\omega_b\omega_c + iT^aT^b\omega_a\omega_b\big] \\
&= -i\epsilon_1\epsilon_2\, g^2\big[\tfrac{1}{2}T^c f^{cab}\omega_a\omega_b + \tfrac{1}{2}i(T^aT^b - T^bT^a)\omega_a\omega_b\big] \\
&= \tfrac{1}{2}\epsilon_1\epsilon_2\, g^2\omega_a\omega_b\big[T^aT^b - T^bT^a - if^{abc}T^c\big] = 0\,,
\end{aligned} \tag{55}$$

where we have made use of eq. (37) and the anticommuting properties of ϵ_2 and ω_a. Likewise a similar computation yields $\delta^2\overline{\psi}(x) = 0$.

[4] One typically integrates by parts in the Faddeev-Popov action to obtain the form, $\mathscr{L}_{\rm G} = \partial^\mu\eta_a D^{ab}_\mu\omega_b$.

A slightly more involved computation yields,

$$\begin{aligned}
\delta^2\omega_a(x) &= \tfrac{1}{2}\epsilon_1 g f^{abc}\delta(\omega_b\omega_c) \\
&= \tfrac{1}{2}\epsilon_1 g f^{abc}\big[(\delta\omega_b)\omega_c + \omega_b\delta\omega_c\big] \\
&= \tfrac{1}{4}\epsilon_1\epsilon_2 g^2 (f^{abc}f^{bde}\omega_d\omega_e\omega_c - f^{abc}f^{cde}\omega_b\omega_d\omega_e) \\
&= \tfrac{1}{4}\epsilon_1\epsilon_2 g^2 (f^{abc}f^{bde} - f^{acb}f^{bde})\omega_c\omega_d\omega_e \\
&= \tfrac{1}{2}\epsilon_1\epsilon_2 g^2 f^{abc}f^{bde}\omega_c\omega_d\omega_e \\
&= \tfrac{1}{6}\epsilon_1\epsilon_2 g^2 f^{abc}f^{bde}(\omega_c\omega_d\omega_e + \omega_e\omega_c\omega_d + \omega_d\omega_e\omega_c) \\
&= \tfrac{1}{6}\epsilon_1\epsilon_2 g^2 (f^{abc}f^{bde} + f^{abd}f^{bec} + f^{abe}f^{bcd})\omega_c\omega_d\omega_e = 0\,,
\end{aligned} \tag{56}$$

after appropriate relabeling of indices and using the anticommuting properties of ϵ_2 and ω_a and the antisymmetry properties of f^{abc}. The Jacobi identity was employed in the final step.

However, we do not obtain $\delta^2\eta_a(x) = 0$. An explicit computation yields

$$\delta^2\eta_a(x) = -\frac{1}{a}\epsilon_1\delta(\partial^\mu A^a_\mu) = -\frac{1}{a}\epsilon_1\epsilon_2\partial^\mu D^{ab}_\mu\omega_b \neq 0\,. \tag{57}$$

Nevertheless, in light of eq. (40), the Lagrange field equations yield,

$$\partial_\mu\frac{\partial\mathscr{L}_{\rm G}}{\partial(\partial_\mu\eta_a)} - \frac{\partial\mathscr{L}_{\rm G}}{\partial\eta_a} = \partial^\mu D^{ab}_\mu\omega_b = 0\,.$$

That is, if we apply the field equations for ω_b in eq. (57), we do obtain $\delta^2\eta_a = 0$. Thus, when we say that δ^2 acting on all on-shell fields is zero, we mean that one may have to invoke the field equations, which (by definition) are satisfied by an on-shell field.

Of course, it would be convenient if δ^2 would annihilate all fields independently of their equations of motion, in which case we would say that δ^2 applied to any off-shell field yields zero. In fact, this can be arranged by the following trick (which is also employed to great advantage in supersymmetric field theories). The idea is to introduce an appropriate auxiliary field, $B_a(x)$. By definition, its Lagrangian density does not contain any derivative, $\partial_\mu B_a(x)$, in which case one can trivially eliminate $B_a(x)$ via its field equation.

In particular, we shall introduce a modified gauge fixing term to replace eq. (38),

$$\mathscr{L}_{\rm GF} = B_a\partial_\mu A^\mu_a + \tfrac{1}{2}aB_aB_a\,, \tag{58}$$

which depends on a new auxiliary scalar field $B_a(x)$ that transforms under the adjoint representation of the gauge group. Applying the Lagrange field equations for $B_a(x)$,

$$\partial_\mu\frac{\partial\mathscr{L}_{\rm GF}}{\partial(\partial_\mu B_a)} - \frac{\partial\mathscr{L}_{\rm GF}}{\partial B_a} = -\partial_\mu A^a_\mu - aB_a = 0\,.$$

That is, $B_a = -a^{-1}\partial_\mu A^a_\mu$. Inserting this result back into eq. (58) yields,

$$\mathscr{L}_{\rm GF} = -\frac{1}{2a}(\partial_\mu A^\mu_a)^2\,, \tag{59}$$

which coincides with eq. (38). Thus, the gauge theory with gauge fixing term given by eq. (58) is completely equivalent to the gauge theory with gauge fixing term given by eq. (38).

However, one can now make use of this new field to modify the BRST transformation law of $\eta_a(x)$, while also specifying a transformation law for $B_a(x)$,

$$\delta\eta_a(x) = \epsilon\, B_a(x)\,, \tag{60}$$

$$\delta B_a(x) = 0\,. \tag{61}$$

Applying the transformation law given by eq. (61) to eq. (58), and using eq. (35),

$$\delta\mathscr{L}_{\rm GF} = (\delta B_a)\partial^\mu A^a_\mu + B_a\partial^\mu(\delta A^a_\mu) + aB_a\delta B_a = \epsilon B_a\partial^\mu D^{ab}_\mu\omega_b\,. \tag{62}$$

In light of eq. (60), we see that eq. (45) is now given by

$$\delta\mathscr{L}_{\rm G} = -(\delta\eta_a)(\partial^\mu D^{ab}_\mu\omega_b) = -\epsilon B_a\partial^\mu D^{ab}_\mu\omega_b\,.$$

since the terms inside the braces in eq. (45) cancel by virtue of eq. (46). Hence it again follows that

$$\delta(\mathscr{L}_{\rm GF} + \mathscr{L}_{\rm G}) = 0\,.$$

That is, the full Lagrangian with the modified gauge fixing term [eq. (58)] is still invariant with respect to the new version of the BRST transformation laws [which have been modified as specified in eqs. (60) and (61)].

Furthermore, the computations of $\delta^2 A^a_\mu(x)$, $\delta^2\psi(x)$, $\delta^2\overline{\psi}(x)$ and $\delta^2\omega_a(x)$ are unchanged from the ones presented above; each of these second variations yields zero. However, we now have

$$\delta^2\eta(x) = \epsilon_1\delta B(x) = 0\,, \tag{63}$$

$$\delta^2 B(x) = 0 \tag{64}$$

as a consequence of eqs. (60) and (61). Thus, $\delta^2 = 0$ when applied to *all* fields of the theory, independently of the field equations.

In summary, we have shown that the Lagrangian density,

$$\mathscr{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + B_a\partial_\mu A^\mu_a + \tfrac{1}{2}aB_aB_a - \eta_a(\partial^\mu D^{ab}_\mu\omega_b) + i\overline{\psi}_i\gamma_\mu D^\mu_{ij}\psi_j\,, \tag{65}$$

is invariant under the following infinitesimal BRST (generalized gauge) transformations,

$$\delta A^a_\mu(x) = \epsilon D^{ab}_\mu\omega_b(x)\,, \tag{66}$$

$$\delta\psi(x) = -i\epsilon\, gT^a\omega_a(x)\psi(x)\,, \tag{67}$$

$$\delta\overline{\psi}(x) = i\epsilon\, gT^a\omega_a(x)\overline{\psi}(x)\,, \tag{68}$$

$$\delta\omega_a(x) = \tfrac{1}{2}\epsilon g f^{abc}\omega_b(x)\omega_c(x) \tag{69}$$

$$\delta\eta_a(x) = \epsilon\, B_a(x)\,, \tag{70}$$

$$\delta B_a(x) = 0\,. \tag{71}$$

Moreover,

$$\delta^2 A^a_\mu(x) = \delta^2\psi(x) = \delta^2\overline{\psi}(x) = \delta^2\omega_a(x) = \delta^2 B(x) = 0\,, \tag{72}$$

independently of the field equations.