In these notes, I will demonstrate the relationship between the Ward-Takahashi identity of quantum electrodynamics (QED) and current conservation.

We begin by rewriting the Lagrangian density of QED in a form that is slightly different from its usual form by integrating by parts in the expression for the QED action. As a result, the QED Lagrangian density, with covariant gauge fixing, can be rewritten as,[1]

$$\mathscr{L} = \tfrac{1}{2}A_\mu \left[\Box\, g^{\mu\nu} - \left(1 - \frac{1}{a}\right)\partial^\mu\partial^\nu\right] A_\nu + \bar{\psi}(i \not{\partial} + e \not{A} - m)\psi\,, \tag{1}$$

which differs from the usual form of the QED Lagrangian density by a total derivative (which can be neglected as it does not contribute to the field equations). In momentum space, the operator inside the square brackets of eq. (1) is given by

$$iD^{-1}_{\mu\nu}(k) \equiv -g_{\mu\nu}k^2 + \left(1 - \frac{1}{a}\right)k^\mu k^\nu\,, \tag{2}$$

where $D^{-1}_{\mu\nu}(k)$ is the inverse of the momentum space tree-level propagator of the photon of four-momentum $k$, and $a$ is the gauge fixing parameter.

As a consequence of the gauge symmetry of QED, one can derive the conserved Noether current,

$$j_\mu(x) = -e\,\bar{\psi}\gamma_\mu\psi\,, \tag{3}$$

where normal ordering is implicit (but not explicitly indicated). The current is conserved, $\partial^\mu j_\mu = 0$ as a consequence of the fields equations. Note that $\mathscr{L}_{\rm int} = -j_\mu A^\mu$, as expected. Moreover, the photon field satisfies the field equations,

$$\Box\, A_\mu - \left(1 - \frac{1}{a}\right)\partial_\mu(\partial^\nu A_\nu) = j_\mu\,. \tag{4}$$

The fermions fields of QED obey canonical equal-time (anti-)commutation relations,

$$\left\{\psi_\alpha(\vec{\boldsymbol{x}},t)\,,\,\psi^\dagger_\beta(\vec{\boldsymbol{y}},t)\right\} = \delta_{\alpha\beta}\,\delta^3(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}})\,, \tag{5}$$

$$\left\{\psi_\alpha(\vec{\boldsymbol{x}},t)\,,\,\psi_\beta(\vec{\boldsymbol{y}},t)\right\} = \left\{\psi^\dagger_\alpha(\vec{\boldsymbol{x}},t)\,,\,\psi^\dagger_\beta(\vec{\boldsymbol{y}},t)\right\} = 0\,, \tag{6}$$

$$\left[\psi_\alpha(\vec{\boldsymbol{x}},t)\,,\,A_\mu(\vec{\boldsymbol{y}},t)\right] = \left[\psi^\dagger_\alpha(\vec{\boldsymbol{x}},t)\,,\,A_\mu(\vec{\boldsymbol{y}},t)\right] = 0\,. \tag{7}$$

Using these relations, one can derive the following equal-time commutation relations

$$\left[j_0(\vec{\boldsymbol{x}},t)\,,\,\psi(\vec{\boldsymbol{y}},t)\right] = \quad e\psi(x)\delta^3(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}})\,, \tag{8}$$

$$\left[j_0(\vec{\boldsymbol{x}},t)\,,\,\bar{\psi}(\vec{\boldsymbol{y}},t)\right] = -e\bar{\psi}(x)\delta^3(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}})\,, \tag{9}$$

$$\left[j_0(\vec{\boldsymbol{x}},t)\,,\,A_\mu(\vec{\boldsymbol{y}},t)\right] = \quad 0\,. \tag{10}$$

The time-independent charge operator is $Q = \int j_0(\vec{\boldsymbol{x}},t)\,d^3x$. Thus eqs. (8) and (9) yield,

$$\left[Q\,,\,\psi(y)\right] = e\psi(y)\,, \qquad\qquad \left[Q\,,\,\bar{\psi}(y)\right] = -e\bar{\psi}(y)\,. \tag{11}$$

That is, a positron of charge $e$ is created by the field operator $\psi$ (and annihilated by $\bar{\psi}$), whereas an electron of charge $-e$ is created by the field operator $\bar{\psi}$ (and annihilated by $\psi$).

[1] By convention, the coupling $e$ is positive and the electric charge of the electron is $-e$.

We now consider the following Green function,

$$\langle\Omega|\, T\, j_\mu(x)\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)A_{\mu_1}(z_1)\cdots A_{\mu_p}(z_p)\,|\Omega\rangle\ , \tag{12}$$

where $|\Omega\rangle$ is the vacuum state. We would like to compute the partial derivative of this Green function with respect to $x$. In order to carry out this computation, we will need to pass the derivative past the time ordered product symbol $T$. Recall that for two operators $A$ and $B$,

$$T\big[A(x)B(y)\big] = \Theta(x_0-y_0)A(x)B(y) \pm \Theta(y_0-x_0)B(y)A(x)\,, \tag{13}$$

where the sign is chosen on the basis of the statistics of the operators $A$ and $B$. For example, given a bosonic operator $A_\mu(x)$,

$$\begin{aligned}\partial_x^\mu \langle\Omega|\, T\, A_\mu(x)B(y)\,|\Omega\rangle &= \langle\Omega|\, \partial_x^\mu\big\{\Theta(x_0-y_0)A_\mu(x)B(y)+\Theta(y_0-x_0)B(y)A_\mu(x)\big\}\,|\Omega\rangle \\ &= \langle\Omega|\, T\, \partial^\mu A_\mu(x)B(y)\,|\Omega\rangle + \delta(x_0-y_0)\,\langle\Omega|\,\big[A_0(x)\,,\,B(y)\big]\,|\Omega\rangle\ ,\end{aligned} \tag{14}$$

where $\partial_x^\mu \equiv \partial/\partial x_\mu$. The last term above is an equal time commutator, which can be evaluated by employing the canonical equal-time commutation relations. Generalizing eq. (14), it follows that

$$\begin{aligned}&\partial_x^\mu \langle\Omega|\, T\, j_\mu(x)\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)A_{\mu_1}(z_1)\cdots A_{\mu_p}(z_p)\,|\Omega\rangle \\ &\quad= \langle\Omega|\, T\, \partial^\mu j_\mu(x)\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)A_{\mu_1}(z_1)\cdots A_{\mu_p}(z_p)\,|\Omega\rangle \\ &\qquad+\sum_{i=1}^n \langle\Omega|\, T\, \psi(x_1)\cdots\psi(x_{i-1})\big[j_0(x),\psi(x_i)\big]\delta(x_0-x_{i0})\psi(x_{i+1})\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n) \\ &\qquad\qquad\times A_{\mu_1}(z_1)\cdots A_{\mu_p}(z_p)\,|\Omega\rangle \\ &\qquad+\sum_{i=1}^n \langle\Omega|\, T\, \psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_{i-1})\big[j_0(x),\bar{\psi}(y_i)\big]\delta(y_0-y_{0i})\bar{\psi}(y_{i+1})\cdots\bar{\psi}(y_n) \\ &\qquad\qquad\times A_{\mu_1}(z_1)\cdots A_{\mu_p}(z_p)\,|\Omega\rangle \\ &\qquad+\sum_{i=1}^p \psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)A_{\mu_1}(z_1)\cdots A_{\mu_{i-1}}(z_{i-1})\big[j_0(x),A_{\mu_i}(z_i)\big]\delta(z_0-z_{i0}) \\ &\qquad\qquad\times A_{\mu_{i+1}}(z_{i+1})\cdots A_{\mu_p}(z_p)\,.\end{aligned} \tag{15}$$

Employing $\partial^\mu j_\mu = 0$ and the equal-time commutation relations given in eqs. (8)–(10), we end up with,

$$\boxed{\begin{aligned}&\partial_x^\mu \langle\Omega|\, T\, j_\mu(x)\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)A_{\mu_1}(z_1)\cdots A_{\mu_p}(z_p)\,|\Omega\rangle \\ &\quad= e\,\langle\Omega|\, T\, \psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)A_{\mu_1}(z_1)\cdots A_{\mu_p}(z_p)\,|\Omega\rangle \sum_{i=1}^n\big[\delta^4(x-x_i)-\delta^4(x-y_i)\big].\end{aligned}} \tag{16}$$

Eq. (16) is another form of the Ward-Takahashi identity, whose derivation relied on the conservation of the current, $\partial^\mu j_\mu = 0$.

In order to understand the connection between eq. (16) and the Ward-Takahashi identity obtained in class, we must relate the Green function on the left hand side of eq. (16) to the ordinary Green functions given by the time ordered product of photon and fermion fields. We can accomplish this by considering

$$\left[\Box\, g^{\mu\nu} - \left(1 - \frac{1}{a}\right)\partial_x^\mu \partial_x^\nu\right] \langle\Omega|\, T\, A_\nu(x)\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\,|\Omega\rangle$$
$$= \langle\Omega|\, T\, j_\mu(x)\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\,|\Omega\rangle\;, \tag{17}$$

after employing eq. (4). Note that we can bring the differential operator in the first line of eq. (17) through the time ordered product symbol $T$, since additional terms arising that are proportional to equal time commutation relations vanish in light of eq. (7). We now Fourier transform the above equation by operating on both sides of eq. (17) with

$$\int\cdots\int e^{i[p_1x_1+\ldots+p_nx_n+kx-(p_1'y_1+\ldots+p_n'y_n)]}d^4x\, d^4x_1\cdots d^4x_n\, d^4y_1\cdots d^4y_n\,. \tag{18}$$

Note the signs in the exponent correspond to an incoming photon of four-momentum $k$, incoming fermions of four-momenta $p_i$, and outgoing fermions of four-momenta $p_i'$. The differential operator can then be moved over to operate on the exponential factor in eq. (17) by two successive integration by parts (dropping all surface terms by assuming that the fields die off at infinity), thereby producing a factor of $iD^{-1}_{\mu\nu}(k)$ [cf. eq. (2)]. We thus end up with,

$$iD^{-1}_{\mu\nu}(k)\int\cdots\int d^4x\, d^4x_1\cdots d^4x_n\, d^4y_1\cdots d^4y_n\, e^{i[p_1x_1+\ldots+p_nx_n+kx-(p_1'y_1+\ldots+p_n'y_n)]}$$
$$\times\langle\Omega|\, T\, A^\nu(x)\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\,|\Omega\rangle$$
$$= \int\cdots\int d^4x\, d^4x_1\cdots d^4x_n\, d^4y_1\cdots d^4y_n\, e^{i[p_1x_1+\ldots+p_nx_n+kx-(p_1'y_1+\ldots+p_n'y_n)]}$$
$$\times\langle\Omega|\, T\, j_\mu(x)\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\,|\Omega\rangle\;, \tag{19}$$

where $D^{-1}_{\mu\nu}$ is given in eq. (2).

We now apply eq. (19) to the three-point connected Green function of the photon field and two fermion fields,[2]

$$\mathcal{V}_\mu(x, x_1, y_1) \equiv \langle\Omega|\, A_\mu(x)\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle\;. \tag{20}$$

The corresponding momentum-space Green function is defined by

$$\mathcal{V}_\mu(p,p')(2\pi)^4\delta(p+k-p') = \int d^4x\, d^4x_1\, d^4y_1\, e^{i(px_1+kx-p'y_1)}\,\langle\Omega|\, T\, A_\mu(x)\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle$$
$$= -iD_{\mu\nu}(k)\int d^4x\, d^4x_1\, d^4y_1\, e^{i(px_1+kx-p'y_1)}\,\langle\Omega|\, T\, j^\nu(x)\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle\,, \tag{21}$$

after employing eq. (19).

[2] In QED, all three-point Green functions are connected since all one-point functions vanish (and disconnected vacuum graphs do not appear once the generating functional is normalized such that $Z[0]=1$).

One can relate $\mathcal{V}_\mu(p,p')$ to the momentum space three-point 1PI Green function of the photon field and two fermion fields by following the derivation given in part (b) of problem 1 on Problem Set 1. The analogue of eq. (18) of the solutions to Problem Set 1 is

$$\begin{aligned}\mathcal{V}_\mu(x,x_1,y_1) &\equiv \langle\Omega|\, T\, A_\mu(x)\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle \\ &= i\int d^4z_1\, d^4z_2\, d^4z_3\, \mathscr{D}_{\mu\nu}(x,z_1)G_c^{(2)}(y_1,z_3)\Gamma^\nu(z_1,z_2,z_3)G_c^{(2)}(x_1,z_2)\,, \end{aligned}\tag{22}$$

where $\mathscr{D}_{\mu\nu}(x,z_1)$ is the exact two-point photon Green function and $G_c^{(2)}(x_1,z_2)$ and $G_c^{(2)}(y_1,z_3)$ are exact two-point fermion Green functions (all of which are connected Green functions in light of footnote 2), and $\Gamma^\nu(z_1,z_2,z_3)\equiv\Gamma^{(3)\,\nu}(z_1,z_2,z_3)$ is the 1PI three-point (amputated) Green function of the photon field and two fermion fields. The order of the terms appearing in the second line eq. (22) is dictated by the suppressed spinor indices (and corresponds to the natural order expected for the multiplication of matrices).

We can convert the above equation to momentum space by multiplying both sides of eq. (22) by

$$\int d^4x\, d^4x_1\, d^4y_1\, e^{i(px_1+kx-p'y_1)}\,. \tag{23}$$

In addition, we can write

$$\mathscr{D}_{\mu\nu}(x,z_1) = \int \frac{d^4p_2}{(2\pi)^4}\, e^{-ip_2(x-z_1)}\mathscr{D}_{\mu\nu}(p_2)\,, \tag{24}$$

$$G_c^{(2)}(x_1,z_2) = \int \frac{d^4p_1}{(2\pi)^4}\, e^{-ip_1(x_1-z_2)}G_c^{(2)}(p_1)\,, \tag{25}$$

$$G_c^{(2)}(y_1,z_3) = \int \frac{d^4p_3}{(2\pi)^4}\, e^{-ip_3(y_1-z_3)}G_c^{(2)}(p_3)\,, \tag{26}$$

$$\Gamma^\nu(z_1,z_2,z_3) = \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\, e^{-i(q_1z_1+q_2z_2-q_3z_3)}(2\pi)^4\delta^4(q_1+q_2-q_3)\Gamma^\nu(q_1,q_3)\,. \tag{27}$$

Plugging these results back into eq. (22), we see that the integrals over $x_1, x_2, x_3, z_1, z_2$ and $z_3$ produce six delta functions which then allow us to immediately perform the remaining integrals over $p_1$, $p_2$, $p_3$, $q_1$, $q_2$ and $q_3$. The end result is,

$$\mathcal{V}_\mu(p,p') = i\mathscr{D}_{\mu\nu}(k)G_c^{(2)}(p')\Gamma^\nu(p,p')G_c^{(2)}(p)\,. \tag{28}$$

It is traditional to change the notation slightly by defining

$$iS(p)\equiv G_c^{(2)}(p)\,. \tag{29}$$

Then, in light of eqs. (21) and (28), it follows that

$$\begin{aligned}&(2\pi)^4\delta^4(p+k-p')\mathscr{D}_{\mu\nu}(k)S(p')\Gamma^\nu(p,p')S(p) \\ &\qquad = D_{\mu\nu}(k)\int d^4x\, d^4x_1\, d^4y_1\, e^{i(px_1+kx-p'y_1)}\,\langle\Omega|\, T\, j^\nu(x)\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle\,. \end{aligned}\tag{30}$$

We now multiply both sides of eq. (30) by $k^\mu$. In class, we proved that

$$k^\mu \mathscr{D}_{\mu\nu}(k) = k^\mu D_{\mu\nu}(k) = -\frac{iak_\nu}{k^2}\,, \tag{31}$$

since the multiplication by $k^\mu$ annihilates the transverse piece of the photon two-point function, whereas the longitudinal piece is not renormalized. Moreover, by writing

$$k_\nu\, e^{i(px_1+kx-p'y_1)} = -i\partial^x_\nu e^{i(px_1+kx-p'y_1)}\,, \tag{32}$$

we can then perform an integration by parts to move the derivative over to act on the term $\langle\Omega|\,T\,j^\nu(x)\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle$ in the second line of eq. (30). Then, it follows that

$$\begin{aligned}(2\pi)^4\delta^4(p+k-p')&S(p')k_\nu\Gamma^\nu(p,p')S(p)\\ &= i\int d^4x\, d^4x_1\, d^4y_1\, e^{i(px_1+kx-p'y_1)}\partial^x_\nu\,\langle\Omega|\,T\,j^\nu(x)\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle\,. \end{aligned}\tag{33}$$

Using the Ward-Takahashi identity [cf. eq. (16)], it follows that

$$\begin{aligned}\partial^x_\nu\,\langle\Omega|\,T\,j^\nu(x)\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle &= e\,\langle\Omega|\,T\,\psi(x_1)\bar{\psi}(y_1)\,|\Omega\rangle\,\big[\delta^4(x-x_1)-\delta^4(x-y_1)\big]\\ &= e\,G_c^{(2)}(x_1,y_1)\big[\delta^4(x-x_1)-\delta^4(x-y_1)\big]\,.\end{aligned}\tag{34}$$

Hence, eq. (33) yields,

$$\begin{aligned}(2\pi)^4\delta^4(p+k-p')S(p')k_\nu\Gamma^\nu(p,p')S(p) &= ie\int d^4x_1\, d^4x_2 G_c^{(2)}(x_1,y_1)\big[e^{i[(p+k)x_1-p'y_1]}-e^{i[px_1+(k-p')y_1]}\big]\\ &= ie(2\pi)^4\delta^4(p+k-p')\big[G_c^{(2)}(p')-G_c^{(2)}(p)\big]\,,\end{aligned}\tag{35}$$

after recognizing the definition of the momentum space fermion two-point Green function,

$$G_c^{(2)}(p)(2\pi)^4\delta^4(p+p') = \int d^4x\, d^4y\, e^{i(px+p'y)}G_c^{(2)}(x,y)\,. \tag{36}$$

After putting $iS(p)\equiv G_c^{(2)}(p)$ on the right hand side of eq. (35), we end up with,

$$S(p+k)k_\nu\Gamma^\nu(p,p')S(p) = e\big[S(p)-S(p+k)\big]\,, \tag{37}$$

where we have used the momentum conserving delta function to set $p'=p+k$. Finally, multiplying the above equation on the left by $S^{-1}(p+k)$ and on the right by $S^{-1}(p)$, we obtain our final result,

$$\boxed{k_\nu\Gamma^\nu(p,p') = e\big[S^{-1}(p+k)-S^{-1}(p)\big]\,,\qquad \text{where } p'=p+k} \tag{38}$$

which we recognize as the Ward identity of QED that relates the vertex function to the inverse fermion propagators.