In class, I provided two different derivations of the effective potential, $V_{\rm eff}(\Phi)$. The first method was based on employing the Green functions of the symmetric theory evaluated at zero external momentum. Setting $\Phi(x) = \overline{\Phi}$ to be a constant field, we obtained

$$V_{\rm eff}(\overline{\Phi}) = -\sum_{n-1}^{\infty} \frac{1}{n!}\Gamma^{(n)}(0,0,\ldots,0)\,\underbrace{\overline{\Phi}\,\overline{\Phi}\cdots\overline{\Phi}}_{n}\,. \tag{1}$$

A second derivation was based on the Green functions for the shifted theory which is obtained by shifting the scalar field, $\phi \to \phi + \Phi$. In this case, we obtained the formula,

$$\left.\frac{dV_{\rm eff}(\phi)}{d\phi}\right|_{\phi=\Phi} = -\Gamma^{(1)}_{\Phi}(0)\,, \tag{2}$$

where $i\Gamma^{(1)}_{\Phi}(0)$ is the sum of one-point Green functions (tadpoles) of the shifted theory with zero external momentum. One can then integrate eq. (2) to determine $V_{\rm eff}(\Phi)$ up to an overall integration constant which can be chosen such that $V_{\rm eff}(\Phi = 0) = 0$. In these notes, I shall provide a third derivation that makes use of the functional integral representation of the generating functional of the Green functions of the quantum field theory. This method of analysis was first introduced in Ref. [1] by Roman Jackiw.

For simplicity, we consider the field theory of a real scalar field. with Lagrangian density,

$$\mathscr{L} = \partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda^4}{4!}\phi^4\,. \tag{3}$$

Consider the generating functional,

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}\int d^4x\big[\mathscr{L}\big(\phi(x), \partial_\mu\phi(x)\big) + J(x)\phi(x)\big]\right\}\,, \tag{4}$$

where we have inserted the factor of Planck's constant, $\hbar$ explicitly (rather than setting it equal to one) for later convenience. The prefactor $\mathcal{N}$ is chosen such that $Z[0] = 1$.

Our strategy will be to perform the so-called loop expansion, which will be an expansion in $\hbar$. The semiclassical approximation is equivalent to evaluating the functional integral above in the limit of $\hbar \to 0$.[1] We shall follow closely the derivation provided in Ref. [4].

[1]Technically, we should really rotate from Minkowski space to Euclidean space before considering the semiclassical approximation. In this case, the evaluation of $Z[J]$ would be performed using the method of steepest descent (a well known technique in asymptotic analysis). The idea is to expand the integrand about the field ϕ where the integrand is stationary. In the limit where $\hbar^{-1}$ is large, the contributions to the functional integral from field configurations far away from the stationary point are suppressed. Although this argument is valid only if the stationary point represents a real minimum, the method can also generate a correct asymptotic series in the case of a saddle point. Nevertheless, I will stick to Minkowski space. Although this puts us on shakier ground mathematically, the results that we will end up deriving will be correct and can be justified by performing the appropriate Wick rotation.

We begin by writing,

$$\phi(x) = \phi_0(x) + \tilde{\phi}(x)\,, \tag{5}$$

where $\phi_0(x)$ corresponds to a stationary action, where the action is equal to the integral of $\mathscr{L} + J(x)\phi(x)$. This means that $\phi_0(x)$ satisfies the field equations of the Lagranigan density in the presence of the source term. That is, $\phi_0(x)$ satisfies

$$(\Box + m^2)\phi_0(x) + \tfrac{1}{6}\lambda\phi_0^3(x) = J(x)\,, \tag{6}$$

where $\Box \equiv \partial_\mu\partial^\mu$. If we plug eq. (5) into the expression for $\mathscr{L} + J(x)\phi(x)$, we obtain,

$$\begin{aligned}\mathscr{L}\big(\phi,\partial_\mu\phi\big) + J\phi = \mathscr{L}\big(\phi_0,\partial_\mu\phi_0\big) + J\phi_0 + \big[\partial_\mu\phi_0\partial^\mu - m^2\phi_0 - \tfrac{1}{6}\lambda\phi_0^3 + J\big]\tilde{\phi} \\ +\tfrac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \tfrac{1}{2}\left(m^2 + \tfrac{1}{2}\lambda\phi_0^2\right)\tilde{\phi}^2 - \tfrac{1}{6}\lambda\phi_0\tilde{\phi}^3 - \tfrac{1}{24}\lambda\tilde{\phi}^4\,,\end{aligned} \tag{7}$$

where we have isolated above the terms linear in $\tilde{\phi}$ in the first line of eq. (7). However, using eq. (6), it follows that

$$\big[\partial_\mu\phi_0\partial^\mu - m^2\phi_0 - \tfrac{1}{6}\lambda\phi_0^3 + J(x)\big]\tilde{\phi}(x) = \partial_\mu(\tilde{\phi}\partial^\mu\phi_0)\,, \tag{8}$$

which is a total divergence and thus does not contribute to the action. Thus, we can discard the linear term, which leaves

$$\mathscr{L}\big(\phi,\partial_\mu\phi\big) + J\phi = \mathscr{L}\big(\phi_0,\partial_\mu\phi_0\big) + J\phi_0 + \tfrac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \tfrac{1}{2}\left(m^2 + \tfrac{1}{2}\lambda\phi_0^2\right)\tilde{\phi}^2 - \tfrac{1}{6}\lambda\phi_0\tilde{\phi}^3 - \tfrac{1}{24}\lambda\tilde{\phi}^4\,. \tag{9}$$

Plugging eq. (9) back into eq. (4), we can now change the integration measure, $\mathcal{D}\phi = \mathcal{D}\tilde{\phi}$, since $\phi_0(x)$ is now fixed by the field equations given in eq. (6). It is then convenience to rescale the field $\tilde{\phi}$ by redefining

$$\tilde{\phi} = \hbar^{1/2}\phi\,. \tag{10}$$

Yes, I know I should use a different symbol instead of reusing the symbol ϕ in eq. (10). But since we no longer need to deal with eq. (5), I hope you can tolerate this redefinition of the symbol ϕ. Plugging eq. (10) back into eq. (9) and inserting the resulting expression back into eq. (4), we end up with

$$\begin{aligned}Z[J] = \mathcal{N}'\exp\left\{\frac{i}{\hbar}\int d^4x[\mathscr{L}(\phi_0,\partial_\mu\phi_0) + J\phi_0]\right\} \\ \times\int\mathcal{D}\phi\,\exp\left\{i\int d^4x\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}(m^2 + \tfrac{1}{2}\lambda\phi_0^2)\phi^2 - \tfrac{1}{6}\hbar^{1/2}\phi_0\phi^3 + \tfrac{1}{24}\hbar\lambda\phi^4\right]\right\},\end{aligned} \tag{11}$$

where $\mathcal{N}$ is a new constant that is fixed by the condition $Z[0] = 0$. Eq. (11) is the starting point for the loop expansion. We can now expand in powers of $\hbar$ and develop a set of Feynman rules in a very natural way.

One remarkable feature of eq. (11) is that one can derive the one-loop effective action by simply setting $\hbar = 0$ in terms that appear on the second line of eq. (11). (Those terms come into play for the first time when deriving the effective action at two loops.) The functional integral can now be explicitly evaluated since the integrand is a gaussian. In particular,

$$\int \mathcal{D}\phi\, \exp\left\{ i\int d^4x \left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}(m^2+\tfrac{1}{2}\lambda\phi_0^2)\phi^2\right]\right\} = \mathcal{N}'(\det A)^{-1/2}\,, \tag{12}$$

where $\mathcal{N}'$ is yet another constant and A is an hermitian differential operator,

$$A(x,x';\phi_0) = \left(\Box_x + m^2 + \tfrac{1}{2}\lambda\phi_0\right)\delta^4(x-x')\,. \tag{13}$$

It is convenient to employ the well-known identity,

$$(\det A)^{-1/2} = \exp\left(-\tfrac{1}{2}\,\mathrm{Tr}\ln A\right)\,. \tag{14}$$

Inserting these results back into eq. (11) yields,

$$Z[J] = \mathcal{N}\exp\left\{\frac{i}{\hbar}\int d^4x[\mathscr{L}(\phi_0,\partial_\mu\phi_0)+J\phi_0]\right\}\exp\left(-\tfrac{1}{2}\,\mathrm{Tr}\ln A\right)\,, \tag{15}$$

after adjusting the constant $\mathcal{N}$ again to ensure that $Z[0]=1$. Note that the determinant and trace that appear in the above expressions are the functional determinant and trace. However, if ϕ has internal degrees of freedom (e.g. suppressed color and/or flavor indices), then A will also be a matrix with color and/or flavor indices, in which case the determinant and trace operations also act in the usual way on the matrix indices.

Introducing the generating functional $W[J]$ such that

$$Z[J] = e^{iW[J]/\hbar}\,, \tag{16}$$

it follows from eq. (15) that

$$W[J] = \int d^4x\,[\mathscr{L}(\phi_0,\partial_\mu\phi_0)+J\phi_0] + \tfrac{1}{2}i\hbar\,\mathrm{Tr}\ln\left(\frac{A(x,x';\phi_0)}{A(x,x';0)}\right)\,. \tag{17}$$

which has the correct normalization $W[0]=0$, since the unique solution to the differential equation given by eq. (6) when $J=0$ is $\phi_0=0$, under the assumption that both $J(x)$ and $\phi_0(x)$ vanish at infinity.

Finally, we introduce the effective action,

$$\Gamma[\Phi] = W[J] - \int d^4x J(x)\Phi(x)\,, \tag{18}$$

where the classical field is defined by,

$$\Phi(x) = \frac{\delta W[J]}{\delta J(x)} = \phi_0(x) + \mathcal{O}(\hbar)\,, \tag{19}$$

in light of eq. (17). In particular, if we perform an $\hbar$ expansion,

$$\Gamma[\Phi] = \Gamma_0[\Phi] + \hbar\Gamma_1[\Phi] + \mathcal{O}(\hbar^2)\,, \tag{20}$$

then it follows from eqs. (17)–(19) that

$$\Gamma_0[\Phi] = \int d^4x\, \mathscr{L}\big(\phi_0, \partial_\mu\phi_0\big) = \int d^4x\big[\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \tfrac{1}{2}m^2\Phi^2 - \tfrac{1}{4!}\lambda\Phi^4\big]\,, \tag{21}$$

which we recognize as the classical action. Likewise, we can obtain an explicit expression for $\Gamma_1[\Phi]$ defined in eq. (20),

$$\hbar\Gamma_1[\Phi] = S[\phi_0] - S[\Phi] + \tfrac{1}{2}i\hbar\,\mathrm{Tr}\ln\left(\frac{A(x,x';\phi_0)}{A(x,x';0)}\right)\,, \tag{22}$$

where

$$S[\Phi] \equiv \int d^4x\,[\mathscr{L}(\Phi,\partial_\mu\Phi) + J\Phi]\,, \qquad\qquad S[\phi_0] \equiv \int d^4x\,[\mathscr{L}(\phi_0,\partial_\mu\phi_0) + J\phi_0]\,. \tag{23}$$

Noting that ϕ_0 is a stationary point of S, when we expand $S[\Phi]$ about ϕ_0, the linear term is absent. This implies that

$$S[\Phi] - S[\phi_0] \simeq \frac{1}{2}\int d^4x\,\left(\frac{\delta^2 S}{\delta\Phi^2}\right)_{\Phi=\phi_0}\big[\Phi(x) - \phi_0(x)\big]^2 \sim \mathcal{O}(\hbar^2)\,, \tag{24}$$

in light of eq. (19). Thus, this term actually belongs in the $\mathcal{O}(\hbar^2)$ term in eq. (20) and should be deleted from eq. (22). We end up with the exact expression,

$$\Gamma_1[\Phi] = \tfrac{1}{2}i\,\mathrm{Tr}\ln\left(\frac{A(x,x';\phi_0)}{A(x,x';0)}\right)\,. \tag{25}$$

Note that it is consistent within the $\hbar$ expansion to put $\phi_0 = \Phi$ on the right hand side of eq. (25) in light of eq. (19). Hence,

$$\boxed{\Gamma_1[\Phi] = \tfrac{1}{2}i\,\mathrm{Tr}\ln\left(\frac{A(x,x';\Phi)}{A(x,x';0)}\right)\,.} \tag{26}$$

To obtain the effective potential from eq. (26), we consider the case of a constant classical field, $\Phi(x) = \overline{\Phi}$. In this case,

$$\Gamma[\overline{\Phi}] = -\int d^4x\, V_{\rm eff}(\overline{\Phi})\,. \tag{27}$$

It immediately follows from eq. (21) that

$$V_{\rm eff}(\overline{\Phi}) = \tfrac{1}{2}m^2\overline{\Phi}^2 + \tfrac{1}{4!}\lambda\overline{\Phi}^4 + \mathcal{O}(\hbar)\,. \tag{28}$$

This equation provides the functional form of the effective potential, so we can simply omit the bars and write,

$$V_{\rm eff}(\Phi) = \tfrac{1}{2}m^2\Phi^2 + \tfrac{1}{4!}\lambda\Phi^4 + \mathcal{O}(\hbar)\,. \tag{29}$$

The one-loop effective potential is of $\mathcal{O}(\hbar)$. This is easily obtained from eq. (26). In light of eq. (13),

$$A(x,x';\Phi) = \big(\Box_x + m^2 + \tfrac{1}{2}\lambda\Phi\big)\delta^4(x-x') = \int \frac{d^4k}{(2\pi)^4}\big(\Box_x + m^2 + \tfrac{1}{2}\lambda\Phi\big)e^{ik(x-x')}$$
$$= \int \frac{d^4k}{(2\pi)^4}\big(-k^2 + m^2 + \tfrac{1}{2}\lambda\Phi\big)e^{ik(x-x')}\,. \tag{30}$$

It follows that

$$\operatorname{Tr}\ln A(x,x';\Phi) = \int d^4x \int \frac{d^4k}{(2\pi)^4}\ln\big(-k^2 + m^2 + \tfrac{1}{2}\lambda\Phi\big)\,. \tag{31}$$

Note that the trace of the operator $\ln A(x,x';\Phi)$ sums over the diagonal elements, i.e., by setting $x = x'$ and summing over x. Since x is a continuous variable, this means that the sum is in fact an integral over x as indicated above. If Φ has internal degrees of freedom, then the operator $\ln A(x,x';\Phi)$ would also be a finite dimensional matrix, and the trace would instruct you to sum over its diagonal elements. For simplicity, we assumed in eq. (3) that there no internal degrees of freedom associated with the scalar field, in which case no additional summation is required.

Therefore eq. (26) yields,

$$\tfrac{1}{2}i\operatorname{Tr}\ln\left(\frac{A(x,x';\Phi)}{A(x,x';0)}\right) = \tfrac{1}{2}i\int d^4x\int\frac{d^4k}{(2\pi)^4}\ln\left(1 - \frac{\frac{1}{2}\lambda\Phi^2}{k^2-m^2}\right)\,. \tag{32}$$

Finally, choosing $\Phi(x) = \overline{\Phi}$ and employing eq. (28) we end up with

$$\boxed{V_{\rm eff}(\Phi) = \tfrac{1}{2}m^2\Phi^2 + \tfrac{1}{4!}\lambda\Phi^4 - \tfrac{1}{2}i\hbar\int\frac{d^4k}{(2\pi)^4}\ln\left(1 - \frac{\frac{1}{2}\lambda\Phi^2}{k^2-m^2+i\varepsilon}\right) + \mathcal{O}(\hbar^2)\,,} \tag{33}$$

after dropping the bars as we did in obtaining eq. (29), and restoring the usual factor of $i\varepsilon$. Indeed, eq. (33) is the same formula obtained for the effective potential either via eq. (1) or via eq. (2). Note that the effective potential is defined such that the vacuum energy is zero. Of course, in dimensional regularization, we would replace $d^4k/(2\pi)^4$ with $d^nk/(2\pi)^n$.

Let us denote the one loop effective potential by $V^{(1)}_{\rm eff}$. Then, the above results are equivalent to statement that [1],

$$V^{(1)}_{\rm eff}(\overline{\Phi}) = -\tfrac{1}{2}i\hbar\int\frac{d^4k}{(2\pi)^4}\ln\left(\frac{i\mathcal{D}^{-1}(\overline{\Phi};k)}{i\mathcal{D}^{-1}(0;k)}\right)\,, \tag{34}$$

where $\mathcal{D}^{-1}(\Phi;k)$ is the inverse tree-level propagator obtained from a free field Lagrangian that consists of the terms quadratic in the fields obtained by shifting the scalar field of the original Lagrangian by $\phi \to \phi + \overline{\Phi}$, where $\overline{\Phi}$ is a constant field. To check this assertion, we can perform the indicated shift to obtain,

$$\tfrac{1}{2}\int d^4x\Big\{-\phi(x)\Box_x\phi(x) - (m^2 + \tfrac{1}{2}\lambda\overline{\Phi}^2)\phi^2(x)\Big\} \equiv \tfrac{1}{2}\int d^4x\,d^4y\,\phi(x)i\mathcal{D}^{-1}(\overline{\Phi};x,y)\phi(y)\,, \tag{35}$$

after an integration by parts, where

$$i\mathcal{D}^{-1}(\overline{\Phi};x,y)\phi_b(y) = \big[-\Box_x - (m^2 + \tfrac{1}{2}\lambda\overline{\Phi}^2)\big]\delta^4(x-y)\,. \tag{36}$$

In momentum space,

$$i\mathcal{D}^{-1}(\overline{\Phi};k) = \int d^4x\, e^{ikx}\, i\mathcal{D}^{-1}(\overline{\Phi};x,0) = k^2 - (m^2 + \tfrac{1}{2}\lambda\overline{\Phi}^2)\,. \tag{37}$$

Plugging this result back into eq. (34) reproduces eq. (33), as advertised.

The derivation eq. (33) or its equivalent form given in eq. (34) is easily extended to the case in which the scalar field possesses internal degrees of freedom. In this case, the scalar fields that appear previously possess a suppressed index, in which case one must take these indices into account when evaluating the determinant and trace. For example, the generalization of eq. (34) is as follows in the case of multiple scalar fields,

$$V_{\rm eff}^{(1)}(\overline{\Phi}_a) = -\tfrac{1}{2}i\hbar\int\frac{d^4k}{(2\pi)^4}\big[\ln\det i\mathcal{D}^{-1}_{ab}(\overline{\Phi};k) - \ln\det i\mathcal{D}^{-1}_{ab}(0;k)\big]\,, \tag{38}$$

where the determinant [whose origin is the multi-field generalization of eq. (12)] acts on the matrix inverse propagators $\mathcal{D}^{-1}$.

As an illustration of eq. (38), consider the Lagrangian density,

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2\,. \tag{39}$$

Performing the shifts, $\phi_1\to\phi_1+\overline{\Phi}_1$ and $\phi_2\to\phi_2+\overline{\Phi}_2$ and retaining only the terms that are quadratic in the fields, the corresponding action is given by,

$$\begin{aligned}\tfrac{1}{2}\int d^4x\,\Big\{&-\phi_1(x)\Box_x\phi_1(x) - \phi_2(x)\Box_x\phi_2(x) - \lambda\big[(3\overline{\Phi}_1^2+\overline{\Phi}_2^2)\phi_1^2(x) + (\overline{\Phi}_1^2+3\overline{\Phi}_2^2)\phi_2^2(x)\\ &+4\overline{\Phi}_1\overline{\Phi}_2\phi_1(x)\phi_2(x)\big]\Big\} \equiv \tfrac{1}{2}\sum_{a,b}\int d^4x\,d^4y\,\phi_a(x)i\mathcal{D}^{-1}_{ab}(\overline{\Phi};x,y)\phi_b(y)\,,\end{aligned} \tag{40}$$

after an integration by parts, where

$$i\mathcal{D}^{-1}_{ab}(\overline{\Phi};x,y) = \Big\{-\delta_{ab}\Box_x - \lambda\big[\delta_{ab}(\overline{\Phi}_1^2+\overline{\Phi}_2^2) + 2\overline{\Phi}_a\,\overline{\Phi}_b\big]\Big\}\delta^4(x-y)\,. \tag{41}$$

In momentum space,

$$i\mathcal{D}^{-1}_{ab}(\overline{\Phi};k) = \int d^4x\,e^{ikx}\,i\mathcal{D}^{-1}_{ab}(\overline{\Phi};x,0) = k^2\delta_{ab} - \lambda\big[\delta_{ab}(\overline{\Phi}_1^2+\overline{\Phi}_2^2) + 2\overline{\Phi}_a\,\overline{\Phi}_b\big]\,. \tag{42}$$

A straightforward computation yields,

$$\det i\mathcal{D}^{-1}_{ab}(\overline{\Phi};k) = \det\begin{pmatrix} k^2-\lambda(3\overline{\Phi}_1^2+\overline{\Phi}_2^2) & -2\lambda\overline{\Phi}_1\,\overline{\Phi}_2 \\ -2\lambda\overline{\Phi}_1\,\overline{\Phi}_2 & k^2-\lambda(\overline{\Phi}_1^2+3\overline{\Phi}_2^2)\end{pmatrix} = (k^2-3\lambda\overline{\Phi}^2)(k^2-\lambda\overline{\Phi}^2)\,, \tag{43}$$

where $\overline{\Phi}^2\equiv\overline{\Phi}_1^2+\overline{\Phi}_2^2$. Hence, eqs. (38) and (43) yields,

$$V_{\rm eff}(\Phi_1,\Phi_2) = -\tfrac{1}{2}i\hbar\int\frac{d^4k}{(2\pi)^4}\left\{\ln\left(1-\frac{3\lambda(\Phi_1^2+\Phi_2^2)}{k^2-i\varepsilon}\right)+\ln\left(1-\frac{\lambda(\Phi_1^2+\Phi_2^2)}{k^2-i\varepsilon}\right)\right\}, \qquad (44)$$

which reproduces the result obtained in class based on eq. (2).

Due to the SO(2) global symmetry of eq. (39), we could have simplified the above computation by using the SO(2) symmetry to "rotate" the scalar shift entirely into one of the two scalar fields, say $\phi_1 \to \phi_1 + \overline{\Phi}$ and $\phi_2 \to \phi_2$, where $\overline{\Phi}$ is defined below eq. (43). In this case, the matrix $i\mathcal{D}^{-1}$ is diagonal, and we easily obtain eq. (44).

Indeed, since the general form for the inverse propagator is

$$i\mathcal{D}^{-1}_{ab} = k^2\delta_{ab} - M^2(\overline{\Phi})_{ab}\,, \qquad (45)$$

it follows that by diagonalizing the $\overline{\Phi}$-dependent squared mass matrix, M^2_{ab},

$$S^{-1}M^2S = \mathrm{diag}(m_1^2, m_2^2, \ldots)\,, \qquad (46)$$

one obtains,

$$\ln\det i\mathcal{D}^{-1} = \mathrm{Tr}\ln i\mathcal{D}^{-1} = \mathrm{Tr}\big[S^{-1}\ln i\mathcal{D}^{-1}S\big] = \mathrm{Tr}\ln\big[S^{-1}i\mathcal{D}^{-1}S\big] = \sum_i \ln(k^2-m_i^2)\,, \qquad (47)$$

which is consistent with the result exhibited in eq. (44).

It is not too difficult to extend the analysis presented in these notes to theories that involve fermions and gauge bosons. One can easily include the functional integration over fermion fields and gauge fields, if present. The analysis employed above quickly reveals that the one-loop effective action simply requires one to perform the scalar field shifts mentioned above and retain only the terms of the Lagrangian that are quadratic in the fields that are present in the theory. The corresponding functional integration then can be carried out directly and results in an inverse determinant [as in eq. (12)] when integrating over scalar and/or vector boson fields, or a determinant when integrating over anticommuting fermion fields or anticommuting Faddeev-Popov ghost fields.[2] Thus, the generalization of eq. (38) is

$$V^{(1)}_{\rm eff}(\overline{\Phi}_a) = -\tfrac{1}{2}i\hbar C\int\frac{d^4k}{(2\pi)^4}(-1)^A\big[\ln\det i\mathcal{D}^{-1}_{ab}(\overline{\Phi};k) - \ln\det i\mathcal{D}^{-1}_{ab}(0;k)\big]\,, \qquad (48)$$

where $(-1)^A = +1$ $[-1]$ for commuting [anticommuting] fields and $C = 1$ $[C = 2]$ for real (complex) fields. The inverse propagators are computed in the usual way by shifting the scalar fields, $\phi_a \to \phi_a + \overline{\Phi}_a$. Note that one does not shift the fermion, vector boson or ghost fields. Nevertheless, due to the interactions of these fields with the scalars, the result of shifting the scalar fields will yield additional quadratic terms involving fermion fields, gauge boson fields and Faddeev-Popov ghost fields.

[2] More precisely, the functional integration over real commuting fields yields the square root of an inverse determinant [as in eq. (12)], whereas integration over complex commuting fields yields an inverse determinant.

To give one example, consider the Lagrangian density given in eq. (39). If we now gauge the SO(2) symmetry by adding a gauge field A_μ and replacing the derivatives with covariant derivatives, we then arrive at the abelian Higgs model with a massless charged scalar field,

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(\partial_\mu\phi_1 - eA_\mu\phi_2)^2 + \tfrac{1}{2}(\partial_\mu\phi_2 + eA_\mu\phi_1)^2 - \tfrac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\,, \quad (49)$$

where we have included the Lorenz gauge fixing term. Note that the Faddeev-Popov ghosts [not shown explicitly in eq. (49)] are free fields and thus have no impact on the considerations below.

After shifting the scalar fields, the terms quadratic in the gauge field are:

$$\tfrac{1}{2}\int d^4x\, A^\mu(x)\left[(\Box_x + e^2\overline{\Phi}^2)g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]A^\nu(x) \equiv \tfrac{1}{2}\int d^4x\, d^4y\, A^\mu(x)\mathcal{D}^{-1}_{\mu\nu}(\overline{\Phi};x,y)A^\nu(y)\,, \quad (50)$$

after an integration by parts, where $\overline{\Phi}^2 \equiv \overline{\Phi}_1^2 + \overline{\Phi}_2^2$ and

$$i\mathcal{D}^{-1}_{\mu\nu}(\overline{\Phi};x,y) = \left\{g_{\mu\nu}(\Box_x + e^2\overline{\Phi}^2) - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right\}\delta^4(x-y)\,. \quad (51)$$

In momentum space,

$$i\mathcal{D}^{-1}_{\mu\nu}(\overline{\Phi};k) = \int d^4x\, e^{ikx}\, i\mathcal{D}^{-1}_{\mu\nu}(\overline{\Phi};x,0) = (-k^2 + e^2\overline{\Phi}^2)g_{\mu\nu} + \left(1-\frac{1}{\xi}\right)k_\mu k_\nu\,. \quad (52)$$

With a little help from Mathematica, it follows that

$$\det i\mathcal{D}^{-1} \equiv \det(i\mathcal{D}^{-1})^\mu{}_\nu = -(k^2 - e^2\overline{\Phi}^2)^3\left(\frac{1}{\xi}k^2 - e^2\overline{\Phi}^2\right)\,. \quad (53)$$

Unfortunately, shifting the scalar fields, $\phi_i \to \phi_i + \overline{\Phi}_i$ in eq. (49) also generates a mixing between the gauge field and the derivative of the scalar fields. Thus, in order to compute the one-loop effective potential, we need to evaluate the determinant of a more complicated inverse propagator that includes the effects of this mixing. Starting from eq. (49), one can easily identify the mixing of the scalar and vector fields,

$$\begin{aligned} eA^\mu\big[(\phi_1+\overline{\Phi}_1)\partial_\mu\phi_2 - (\phi_2+\overline{\Phi}_2)\partial_\mu\phi_1\big] &= eA^\mu\big[\overline{\Phi}_1\partial_\mu\phi_2 - \overline{\Phi}_2\partial_\mu\phi_1\big] + \text{cubic and quartic terms} \\ &= \tfrac{1}{2}e\overline{\Phi}_1(A^\mu\partial_\mu\phi_2 - \phi_2\partial_\mu A^\mu) - \tfrac{1}{2}e\overline{\Phi}_2(A^\mu\partial_\mu\phi_1 - \phi_1\partial_\mu A^\mu) \\ &\quad + \text{total derivative terms} + \text{cubic and quartic terms}. \end{aligned} \quad (54)$$

The relevant terms in the action that are quadratic in the fields is then given by,

$$\begin{aligned} \tfrac{1}{2}\int d^4x\, \Big\{&-\phi_1(x)\Box_x\phi_1(x) - \phi_2(x)\Box_x\phi_2(x) - \lambda\big[(3\overline{\Phi}_1^2 + \overline{\Phi}_2^2)\phi_1^2(x) + (\overline{\Phi}_1^2 + 3\overline{\Phi}_2^2)\phi_2^2(x) \\ &+ 4\overline{\Phi}_1\overline{\Phi}_2\phi_1(x)\phi_2(x)\big]\Big\} + eA^\mu\big(\overline{\Phi}_1\partial_\mu\phi_2 - \overline{\Phi}_2\partial_\mu\phi_1\big) - e\big(\overline{\Phi}_1\phi_2 - \overline{\Phi}_2\phi_1\big)\partial_\mu A^\mu \\ &+ A^\mu(x)\left[\big(\Box_x + e^2(\overline{\Phi}_1^2 + \overline{\Phi}_2^2)\big)g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]A^\nu(x)\,. \end{aligned} \quad (55)$$

Eq. (55) can be rewritten in matrix form,

$$\tfrac{1}{2}\int d^4x\, d^4y\, F^{\mathsf{T}}(x) i\mathcal{D}^{-1}(\overline{\Phi}_1,\overline{\Phi}_2;x,y)F(y)\,, \tag{56}$$

where $F^{\mathsf{T}}(x)$ is a six component row vector of fields and $F(x)$ is a six component column vector of fields,

$$F^{\mathsf{T}}(x) = \begin{pmatrix}\phi_1(x) & \phi_2(x) & A^\mu(x)\end{pmatrix}\,, \qquad\qquad F(y) = \begin{pmatrix}\phi_1(y)\\ \phi_2(y)\\ A^\nu(y)\end{pmatrix}\,, \tag{57}$$

and the 6×6 matrix $i\mathcal{D}^{-1}$ is given by,

$$i\mathcal{D}^{-1}(\overline{\Phi}_1,\overline{\Phi}_2;x,y) = \delta^4(x-y)\times$$
$$\begin{pmatrix} -\Box_x - \lambda(3\overline{\Phi}_1^2+\overline{\Phi}_2^2) & -2\lambda\overline{\Phi}_1\,\overline{\Phi}_2 & e\overline{\Phi}_2\partial_\nu \\ -2\lambda\overline{\Phi}_1\,\overline{\Phi}_2 & -\Box_x - \lambda(\overline{\Phi}_1^2+3\overline{\Phi}_2^2) & -e\overline{\Phi}_1\partial_\nu \\ -e\overline{\Phi}_2\partial_\mu & e\overline{\Phi}_1\partial_\mu & \left(\Box_x + e^2(\overline{\Phi}_1^2+\overline{\Phi}_2^2)\right)g_{\mu\nu} - \left(1-\tfrac{1}{\xi}\right)\partial_\mu\partial_\nu \end{pmatrix}. \tag{58}$$

which is an hermitian matrix differential operator. In momentum space, $i\mathcal{D}^{-1}$ is an hermitian matrix,

$$i\mathcal{D}^{-1}(\overline{\Phi}_1,\overline{\Phi}_2;k) = \int d^4x\, e^{ikx}\, i\mathcal{D}^{-1}_{\mu\nu}(\overline{\Phi}_1,\overline{\Phi}_2;x,0) =$$
$$\begin{pmatrix} k^2 - \lambda(3\overline{\Phi}_1^2+\overline{\Phi}_2^2) & -2\lambda\overline{\Phi}_1\,\overline{\Phi}_2 & -ie\overline{\Phi}_2 k_\nu \\ -2\lambda\overline{\Phi}_1\,\overline{\Phi}_2 & k^2 - \lambda(\overline{\Phi}_1^2+3\overline{\Phi}_2^2) & ie\overline{\Phi}_1 k_\nu \\ ie\overline{\Phi}_2 k_\mu & -ie\overline{\Phi}_1 k_\mu & \left(-k^2 + e^2(\overline{\Phi}_1^2+\overline{\Phi}_2^2)\right)g_{\mu\nu} + \left(1-\tfrac{1}{\xi}\right)k_\mu k_\nu \end{pmatrix}. \tag{59}$$

With a little help from Mathematica, we readily obtain,

$$\begin{aligned}\det i\mathcal{D}^{-1}(\overline{\Phi}_1,\overline{\Phi}_2;k) &= -\left(k^2 - 3\lambda(\overline{\Phi}_1^2+\overline{\Phi}_2^2)\right)\left[k^2 - e^2(\overline{\Phi}_1^2+\overline{\Phi}_2^2)\right]^3 \\ &\times\left\{\left(k^2-\lambda(\overline{\Phi}_1^2+\overline{\Phi}_2^2)\right)\left(\frac{k^2}{\xi} - e^2(\overline{\Phi}_1^2+\overline{\Phi}_2^2)\right) + e^2k^2(\overline{\Phi}_1^2+\overline{\Phi}_2^2)\right\}.\end{aligned} \tag{60}$$

Another technique for evaluating the determinant given in eq. (60) makes use of the well-known formula for the determinant of a partitioned matrix [2],

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det A \det(D - CA^{-1}B)\,, \tag{61}$$

in the special case of $C = B^\dagger$, under the assumption that $\det A \neq 0$. This is essentially the method employed by Ref. [1]. Hence, the one-loop contribution to the effective potential for the massless Abelian Higgs model is given by,

$$V^{(1)}(\overline{\Phi}_1, \overline{\Phi}_2) = -\tfrac{1}{2} i\hbar \int \frac{d^4k}{(2\pi)^4} \Biggl\{ \ln\left(\frac{k^2 - 3\lambda(\overline{\Phi}_1^2 + \overline{\Phi}_2^2)}{k^2}\right) + 3\ln\left(\frac{k^2 - e^2(\overline{\Phi}_1^2 + \overline{\Phi}_2^2)}{k^2}\right)$$
$$+ \ln\left[\left(\frac{k^2 - \lambda(\overline{\Phi}_1^2 + \overline{\Phi}_2^2)}{k^2}\right)\left(\frac{k^2 - \xi e^2(\overline{\Phi}_1^2 + \overline{\Phi}_2^2)}{k^2}\right) + \frac{\xi e^2}{k^2}(\overline{\Phi}_1^2 + \overline{\Phi}_2^2)\right]\Biggr\}\,, \tag{62}$$

in agreement with the result obtained in Ref. [1]. For notational convenience, the $i\varepsilon$ factors have been suppressed in eq. (62).

If we set $\xi = 0$ in eq. (62), we recover the well known expression for the one-loop contribution to the effective potential for the massless Abelian Higgs model in the Landau gauge [3]. Remarkably, this observation implies that in the Landau gauge, one can simply neglect the mixing of the scalar bosons and vector bosons in the computation of the one-loop effective potential.

One subtle point should be mentioned here. If dimensional regularization is used, then one must replace $d^4k/(2\pi)^4$ by $d^nk/(2\pi)^n$ to regulate the integrals, and then replace the bare parameters and fields with the corresponding renormalized parameters and fields. In particular, the determinant of the 4×4 matrix evaluated in eq. (53) must be replaced by the determinant of an $n \times n$ matrix. For example, it is not difficult to convince yourself that in $n = 4 - 2\epsilon$ dimensions, one eq. (53) should be replaced by

$$\det i\mathcal{D}^{-1} = -(k^2 - e^2\overline{\Phi}^2)^{3-2\epsilon}\left(\frac{1}{\xi}k^2 - e^2\overline{\Phi}^2\right)\,. \tag{63}$$

Similarly, in eq. (60), one would replace the power 3 with $3 - 2\epsilon$. Hence, it follows that eq. (62) should be modified by replacing $d^4k/(2\pi)^4$ by $d^nk/(2\pi)^n$ and by replacing the 3 with $3 - 2\epsilon$.

Eq. (62) is not the result obtained in eq. (13.234) of Ref. [4] in the case of $\xi \neq 0$ (even taking into account the different normalization used in defining λ and the inclusion of a tree-level scalar mass term). The result obtained by Ref. [4] simply included the separate contributions of the scalar and vector bosons while ignoring the mixing of the scalar and vector fields. The authors justified the latter by making use of the R_ξ gauge. However, the R_ξ gauge fixing term was employed only after performing the shift of the scalar fields, which seems contrary to the rules of the formalism developed in these notes. An attempt to correct this error was proposed in Refs. [5, 6] where a slightly modified $\overline{R}_\xi$ gauge fixing term was introduced. However, the calculations presented in these works made the assumption that the effective potential was a function of $\overline{\Phi}_1^2 + \overline{\Phi}_2^2$ alone (in which case they could perform their calculations by setting $\overline{\Phi}_2 = 0$). In light of the $\overline{R}_\xi$ gauge fixing term employed in these works, I believe that their assumption was incorrect, and thus the end result of their calculations is untrustworthy.

One can extend the functional derivation of the effective potential presented in these notes beyond the one-loop level. Indeed, such an analysis was presented already in Ref. [1].

However, many of the simplifications that we were able to make good use of in the one-loop analysis cannot be extended beyond one-loop. Nevertheless, the functional techniques still provide an excellent framework for computing the two-loop contributions (and beyond) to the effective potential. In this regard, Refs. [7,8] are especially useful additions to the recent literature.

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