The running mass, $\overline{m}(s)$ is the solution to the differential equation,[1]

$$s\frac{d\overline{m}(s)}{ds} = \overline{m}(s)\big[\gamma_m\big(\overline{g}_s(s)\big) - 1\big]\,, \qquad \text{where } \overline{m}(s=1) = m. \tag{1}$$

In eq. (1), $\overline{g}(s)$ is the running coupling, γ_m is the mass anomalous dimension, and m is the renormalized mass parameter. In this note, we assume that the renormalized parameters of QCD are defined in the $\overline{\rm MS}$ renormalization scheme. The parameter s is dimensionless and can be identified as $s = Q/\mu$, where Q is a physical momentum scale[2] and μ is an arbitrary mass parameter introduced in the dimensional regularization procedure to ensure that the renormalized coupling g is dimensionless.

It is convenient to set $s = e^t$, in which case eq. (1) takes the form,

$$\frac{d\overline{m}(t)}{dt} = \overline{m}(t)\big[\gamma_m\big(\overline{g}_s(t)\big) - 1\big]\,, \qquad \text{where } \overline{m}(t=0) = m. \tag{2}$$

The solution to this differential equation is given by

$$\overline{m}(t) = \overline{m}(0)\, e^{-t} \exp\left\{\int_0^t \gamma_m\big(\overline{g}_s(t')\big)dt'\right\}\,. \tag{3}$$

For $s = Q/\mu$, then

$$t = \tfrac{1}{2}\ln\left(\frac{Q^2}{\mu^2}\right)\,. \tag{4}$$

Then, one can rewrite eq. (3) as

$$\overline{m}(Q^2) = \overline{m}(\mu^2)\left(\frac{\mu^2}{Q^2}\right)^{1/2} \exp\left\{\int_0^{\frac{1}{2}\ln(Q^2/\mu^2)} \gamma_m\big(\overline{g}_s(t')\big)\, dt'\right\}\,. \tag{5}$$

The one-loop running coupling constant of QCD is given by

$$\overline{g}^2(Q^2) = \frac{16\pi^2}{b_0\ln(Q^2/\Lambda^2)}\,, \tag{6}$$

where the one-loop QCD β-function is $\beta(g_s) = -b_0 g_s^3/(16\pi^2)$ with

$$b_0 = \frac{11N}{3} - \frac{2}{3}n_f\,, \tag{7}$$

[1]My notation follows that of Pierre Ramond, *Field Theory: A Modern Primer*, Second Edition (Westview Press, Boulder, CO, 1990). In most of the physics literature, γ_m is defined with the opposite sign.

[2]In practical applications, one often defines $Q \equiv (q^2)^{1/2}$ for a timelike four-momentum q and $Q \equiv (-q^2)^{1/2}$ for a spacelike four-momentum q.

in the case of an SU(N) color group and n_f flavors of quarks. The parameter Λ is a physical parameter of QCD,

$$\Lambda^2 \equiv \mu^2 \exp\left(\frac{-16\pi^2}{b_0\,\overline{g}_s^2(\mu)}\right) . \tag{8}$$

One can formally prove that $d\Lambda/d\mu = 0$ since the explicit dependence on μ (in the one-loop approximation) is exactly cancelled out by the implicit dependence on μ in $\overline{g}_s^2(\mu^2)$.

The one-loop mass anomalous dimension will be denoted by

$$\gamma_m(\overline{g}_s) = \frac{\overline{g}_s^2}{16\pi^2}\,\gamma_m^{(0)} . \tag{9}$$

As in eq. (4), we define $t' = \frac{1}{2}\ln(Q'^2/\Lambda^2)$. Then,

$$\tfrac{1}{2}\ln\left(\frac{Q'^2}{\Lambda^2}\right) = \tfrac{1}{2}\ln\left(\frac{Q'^2}{\mu^2}\right) + \tfrac{1}{2}\ln\left(\frac{\mu^2}{\Lambda^2}\right) = t' + \tfrac{1}{2}\ln\left(\frac{\mu^2}{\Lambda^2}\right) , \tag{10}$$

and it follows that

$$\overline{m}(Q^2) = \overline{m}(\mu^2)\left(\frac{\mu^2}{Q^2}\right)^{1/2} \exp\left\{\frac{\gamma_m^{(0)}}{2b_0}\int_0^{\frac{1}{2}\ln(Q^2/\mu^2)} \frac{dt'}{t' + \frac{1}{2}\ln(\mu^2/\Lambda^2)}\right\} . \tag{11}$$

The integral is elementary and we end up with,

$$\begin{aligned}\overline{m}(Q^2) &= \overline{m}(\mu^2)\left(\frac{\mu^2}{Q^2}\right)^{1/2} \exp\left\{\frac{\gamma_m^{(0)}}{2b_0}\left[\ln\left(\tfrac{1}{2}\ln\frac{Q^2}{\Lambda^2}\right) - \ln\left(\tfrac{1}{2}\ln\frac{\mu^2}{\Lambda^2}\right)\right]\right\} \\ &= \overline{m}(\mu^2)\left(\frac{\mu^2}{Q^2}\right)^{1/2}\left[\frac{\frac{1}{2}\ln\left(\frac{\mu^2}{\Lambda^2}\right)}{\frac{1}{2}\ln\left(\frac{Q^2}{\Lambda^2}\right)}\right]^{-\gamma_m^{(0)}/(2b_0)} .\end{aligned} \tag{12}$$

In the $\overline{\rm MS}$ renormalization scheme in the one-loop approximation (with an SU(N) color group),

$$\gamma_m^{(0)} = -6\left(\frac{N^2-1}{2N}\right) , \tag{13}$$

which along with b_0 is gauge independent. One can define a renormalization group invariant mass parameter $\hat{m}$ as follows,

$$\hat{m} \equiv \overline{m}(\mu^2)\left[\tfrac{1}{2}\ln\left(\frac{\mu^2}{\Lambda^2}\right)\right]^{-\gamma_m^{(0)}/(2b_0)} . \tag{14}$$

We leave it as an exercise to the reader to show that

$$\frac{d\hat{m}}{d\mu} = 0 , \tag{15}$$

by showing that the explicit μ dependent is exactly canceled out (in the one-loop approximation) by the implicit μ dependence of $\overline{m}(\mu^2)$. It then follows that the running mass of QCD in the one-loop approximation is given by,

$$\boxed{\overline{m}(Q^2) = \left(\frac{\mu^2}{Q^2}\right)^{1/2} \frac{\hat{m}}{\left[\frac{1}{2}\ln(Q^2/\Lambda^2)\right]^{-\gamma_m^{(0)}/(2b_0)}}\,.} \tag{16}$$

In the literature, one often defines dimensionless quantities, $x \equiv m/\mu$ and $\overline{x}(s) = \overline{m}(s)/\mu$, in which case eq. (16) is rewritten as,

$$\overline{x}(Q^2) = \left(\frac{1}{Q^2}\right)^{1/2} \frac{\hat{m}}{\left[\frac{1}{2}\ln(Q^2/\Lambda^2)\right]^{-\gamma_m^{(0)}/(2b_0)}}\,. \tag{17}$$

In this convention, a slightly different running mass is introduced,

$$\widetilde{m}(Q^2) \equiv (Q^2)^{1/2}\,\overline{x}(Q^2) = \frac{\hat{m}}{\left[\frac{1}{2}\ln(Q^2/\Lambda^2)\right]^{-\gamma_m^{(0)}/(2b_0)}}\,. \tag{18}$$

Note that $\overline{m}(\mu) = \widetilde{m}(\mu) = m$. One advantage of employing $\widetilde{m}(Q^2)$ is that it is independent of μ.

We end these notes with a few remarks. First, note that $\overline{m}(Q^2) \to 0$ as $Q^2 \to \infty$. Recall that the solution to the renormalization group equation for the N-point 1PI Green function of QCD is given by,

$$\Gamma^{(N)}(sp_1, sp_2, \ldots, sp_N; g_s, m, a, \mu) = s^d \exp\left\{-\int_1^s \frac{ds'}{s'}\gamma_\Gamma\left(\overline{g}_s(s'), \overline{a}(s')\right)\right\} \times \Gamma^{(N)}(p_1, p_2, \ldots, p_N; \overline{g}_s(s), \overline{m}(s), \overline{a}(s), \mu)\,, \tag{19}$$

where $d = 4 - n_B - 3n_F$ (for an N-point 1PI Green function with n_B external gluons and $2n_F$ external fermions), $\gamma_\Gamma \equiv n_B\gamma_3 + 2n_F\gamma_2$ with $\gamma_i = \frac{1}{2}d\ln Z_i/d\ln\mu$ (for $i = 2, 3$), and $\overline{a}(s)$ is the running gauge parameter. In the asymptotic regime corresponding to $s \to \infty$, we see that both $\overline{m}(s) \to 0$ and $\overline{g}(s) \to 0$ [the latter assumes that $b_0 < 0$; i.e., the number of fermions is not so large as to spoil asymptotic freedom of the QCD β-function]. This means that for large values of s, the perturbative expansion of $\Gamma^{(N)}$ becomes more reliable and mass effects can be ignored to a very good approximation.

It is noteworthy that if one takes ratios of running quark masses at some fixed value of Q^2, then

$$\frac{\overline{m}_i(Q^2)}{\overline{m}_j(Q^2)} = \frac{\hat{m}_i}{\hat{m}_j}\,, \tag{20}$$

independently of the value of Q^2 and of μ^2, where the indices i and j label the quark flavor. This means that these ratios are renormalization group invariant quantities. They can also be related to ratios of on-shell masses defined at the pole of the quark propagator. Nevertheless, relating either of these quantities to physical observables is subtle since, strictly speaking, quarks are confined and thus cannot be observed as free particles.