Wick Expansion in the functional integration formalism

Consider a quantum field theory of a real scalar field, $\phi(x)$, governed by the Lagrangian density,

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I\,, \qquad \text{where } \mathscr{L}_0 = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2. \tag{1}$$

and $\mathscr{L}_I$ contains the scalar self-interactions. For simplicity, we shall assume that $\mathscr{L}_I$ has a polynomial form in the field ϕ and does not contain any derivatives such as $\partial_\mu\phi$.

The generating functional for the Green functions of the scalar field theory is,

$$Z[J] = N\exp\left\{i\int d^4x\big[\mathscr{L} + J(x)\phi(x)\big]\right\}\,, \tag{2}$$

where N is a normalization constant chosen such that $Z[0] = 1$. If $\mathscr{L}_I = 0$, then the corresponding generating functional can be determined exactly, and is given by

$$\begin{aligned} Z_0[J] &= N\exp\left\{i\int d^4x\,\big[\mathscr{L}_0 + J(x)\phi(x)\big]\right\} \\ &= \exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\,d^4x\,d^4y\right)\,. \end{aligned} \tag{3}$$

where $\Delta(x)_F \equiv -(\Box + m^2 - i\epsilon)^{-1}$.

If $\mathscr{L}_I \neq 0$, then we can develop a perturbation series for $Z[J]$ by noting that

$$Z[J] = N\exp\left\{i\int d^4x\,\mathscr{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right\}\exp\left\{i\int d^4x\,\big[\mathscr{L}_0 + J(x)\phi(x)\big]\right\}\,, \tag{4}$$

since the effect of the operator $\mathscr{L}_I\,(-i\delta/\delta J(x))$ is to simply replace $-i\delta/\delta J(x)$ with $\phi(x)$ thereby recovering the result of eq. (2). Hence, it follows that

$$Z[J] = N\exp\left\{i\int d^4x\,\mathscr{L}_I\left(-i\frac{\delta}{\delta J}\right)\right\}\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\,d^4x\,d^4y\right)\,. \tag{5}$$

Expanding out the first exponential above yields,

$$\begin{aligned} Z[J] = N&\left[1 + \sum_{k=1}^{\infty}\frac{i^k}{k!}\int d^4x_1\cdots d^4x_k\,\mathscr{L}_I\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\mathscr{L}_I\left(-i\frac{\delta}{\delta J(x_k)}\right)\right] \\ &\times\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\,d^4x\,d^4y\right)\,. \end{aligned} \tag{6}$$

Eq. (6) is the Wick Expansion in the functional integration formalism and provides the perturbative expansion for the generating functional for the Green functions. From this expansion, one can obtain the perturbative expansion for the n-point Green functions of the theory,

$$G^{(n)}(x_1, x_2, \ldots, x_n) = i^{-n}\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}. \tag{7}$$

along with its Feynman diagrammatic representation.

There is some advantage to employing a modified version of the Wick Expansion that can be obtained by utilizing a functional identify first used by Sidney Coleman [1], which has since been dubbed the Coleman lemma [2, 3],

$$\boxed{F\left(-i\frac{\delta}{\delta J}\right)G[J]=G\left(-i\frac{\delta}{\delta\phi}\right)\left\{F[\phi]\exp\left(i\int J(x)\phi(x)\,d^4x\right)\right\}\Bigg|_{\phi=0}.} \tag{8}$$

Coleman proves this lemma by considering the analogous lemma applied to functions of n variables. Let $F(\vec{\boldsymbol{x}})\equiv F(x_1,x_2,\ldots,x_n)$ and $G(\vec{\boldsymbol{x}})\equiv G(x_1,x_2,\ldots,x_n)$. Then,

$$F\left(-i\frac{\partial}{\partial x}\right)G(\vec{\boldsymbol{x}})=G\left(-i\frac{\partial}{\partial y}\right)\left[F(\vec{\boldsymbol{y}})e^{i\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{y}}}\right]\Bigg|_{\vec{\boldsymbol{y}}=0}. \tag{9}$$

Coleman argues that it is sufficient to prove eq. (9) by taking $F(\vec{\boldsymbol{x}})=e^{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{x}}}$ and $G(\vec{\boldsymbol{x}})=e^{\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{x}}}$, since one can generalize to arbitrary functions by employing a Fourier transform. The computation in this case is straightforward,

$$\begin{aligned}F\left(-i\frac{\partial}{\partial x}\right)G(\vec{\boldsymbol{x}})&=\exp\bigl(-i\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{\nabla}}_x\bigr)e^{\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{x}}}=\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\bigl(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{\nabla}}_x\bigr)^k e^{\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{x}}}\\&=\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\bigl(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}\bigr)^k e^{\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{x}}}=e^{\vec{\boldsymbol{b}}\cdot(\vec{\boldsymbol{x}}-i\vec{\boldsymbol{a}})}.\end{aligned} \tag{10}$$

and

$$\begin{aligned}G\left(-i\frac{\partial}{\partial y}\right)\left[F(\vec{\boldsymbol{y}})e^{i\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{y}}}\right]\Bigg|_{\vec{\boldsymbol{y}}=0}&=\exp\bigl(-i\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{\nabla}}_y\bigr)\left[e^{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{y}}}e^{i\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{y}}}\right]\Bigg|_{\vec{\boldsymbol{y}}=0}\\&=e^{\vec{\boldsymbol{b}}\cdot(\vec{\boldsymbol{x}}-i\vec{\boldsymbol{a}})}e^{i(\vec{\boldsymbol{x}}-i\vec{\boldsymbol{a}})\cdot\vec{\boldsymbol{y}}}\Bigg|_{\vec{\boldsymbol{y}}=0}=e^{\vec{\boldsymbol{b}}\cdot(\vec{\boldsymbol{x}}-i\vec{\boldsymbol{a}})}.\end{aligned} \tag{11}$$

Generalizing eq. (9) to functions, we let $x\to J$ and $y\to\phi$. Partial derivatives become functional derivatives, n-dimensional vectors become functions and sums become integrals. We shall now apply Coleman's lemma by taking

$$F[\phi]=\exp\left\{i\int d^4x\,\mathscr{L}_I\right\},\qquad G[J]=\exp\left\{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\,d^4x\,d^4y\right\}, \tag{12}$$

in eq. (8). In light of eq. (5), it then follows that

$$\boxed{Z[J]=N\exp\left\{\frac{i}{2}\int\frac{\delta}{\delta\phi(x)}\Delta_F(x-y)\frac{\delta}{\delta\phi(y)}\,d^4x\,d^4y\right\}\left[\exp\left(i\int\left[\mathscr{L}_I+J(x)\phi(x)\right]d^4x\right)\right]\Bigg|_{\phi=0}.} \tag{13}$$

Employing eq. (7), we immediately obtain,

$$\begin{aligned}G^{(n)}(x_1,\ldots,x_n)=N\exp&\left\{\frac{i}{2}\int d^4x\,d^4y\,\Delta_F(x-y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}\right\}\\&\times\phi(x_1)\cdots\phi(x_n)\exp\left\{i\int d^4x\bigl[\mathscr{L}_I+J(x)\phi(x)\bigr]\right\}\Bigg|_{\phi=J=0},\end{aligned} \tag{14}$$

which simplifies to

$$G^{(n)}(x_1,\ldots,x_n) = N\exp\left\{\frac{i}{2}\int d^4x\,d^4y\,\Delta_F(x-y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}\right\} \times\phi(x_1)\cdots\phi(x_n)\exp\left\{i\int d^4x\,\mathscr{L}_I(x)\right\}\bigg|_{\phi=0}. \tag{15}$$

Expanding out the second exponential yields

$$G^{(n)}(x_1,\ldots,x_n) = N\exp\left\{\frac{i}{2}\int d^4x\,d^4y\,\Delta_F(x-y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}\right\} \times\phi(x_1)\cdots\phi(x_n)\left(1+\sum_{k=1}^{\infty}\frac{i^k}{k!}\int d^4x_1\cdots d^4x_k\,\mathscr{L}_I(x_1)\cdots\mathscr{L}_I(x_k)\right)\bigg|_{\phi=0}. \tag{16}$$

This provides the alternative version of the Wick expansion in the functional integration formalism. One advantage of eq. (16) is that it provides a direct formula for the perturbative expansion of the n-point Green function.

The normalization factor N in eq. (15) is fixed by the condition that $Z[J=0]=1$. However, this would require one to perform another perturbative expansion in order to evaluate N. An alternative strategy is to set $N=1$, in which case $Z[J=0]\neq 1$. Since $Z[J]={}_J\langle\Omega|\Omega\rangle_J$ (where $|\Omega\rangle_{J=0}$ is the vacuum state), this would mean that the vacuum state is not properly normalized. In this case, a perturbative expansion yields a series of vacuum-to-vacuum Feynman diagrams.[1] However, this observation implies that if one performs the perturbative expansion shown in eq. (16) having set $N=1$, and simply deletes diagrams appearing in the perturbation series that contain disconnected vacuum-to-vacuum subdiagrams, then this is equivalent to restoring the proper normalization factor of N. Henceforth, we shall perform all computations of eq. (16) in this fashion.

The following example is instructive. We consider a quantum field theory of a real scalar field, with a self-interaction, where

$$\mathscr{L}_I(x) = -\frac{\lambda}{4!}\phi^4(x)\,. \tag{17}$$

In this example, we shall compute the connected four-point Green function, $G_c^{(4)}(x_1,x_2,x_3,x_4)$ in perturbation theory at $\mathcal{O}(\lambda)$. This can be obtained by computing the full four-point Green function, $G^{(4)}(x_1,x_2,x_3,x_4)$ based on eq. (16) with $N=1$, and then explicitly dropping all terms that correspond diagrammatically to disconnected Feynman diagrams.[2]

[1] To make matters worse, each of these diagrams is divergent. However this divergence is not physical, and we can ignore this fact in what follows.

[2] As discussed above, by taking $N=1$ in eq. (16) rather than using the proper normalization factor, we will obtain extra terms corresponding to diagrams that contain disconnected vacuum-to-vacuum subdiagrams. However, such terms are of no interest in the computation of the connected Green function.

Using eq. (6) with $n = 4$, and keeping terms up to and including terms of $\mathcal{O}(\lambda)$,

$$G^{(4)}(x_1, x_2, \ldots, x_n) = G_0^{(4)}(x_1, x_2, \ldots, x_n)$$
$$+ \exp\left\{\tfrac{1}{2}i \int d^4y\, d^4z\, \Delta_F(y-z) \frac{\delta}{\delta\phi(y)} \frac{\delta}{\delta\phi(z)}\right\} \left[\phi(x_1)\phi(x_2)\cdots\phi(x_4)\, i \int d^4x\, \mathscr{L}_I\right]\Bigg|_{\phi=0}, \tag{18}$$

where $G_0^{(4)}(x_1, x_2, \ldots, x_n)$ is the four-point Green function of the free scalar field theory. It is straightforward to verify that setting $\mathscr{L}_I = 0$ in eq. (16) yields the well-known free scalar field theory result,

$$G_0^{(4)}(x_1, x_2, x_3, x_4) = \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4)$$
$$+\Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)\,, \tag{19}$$

which consists entirely of disconnected pieces, and thus does not contribute to the connected four-point Green function $G_c^{(4)}(x_1, x_2, x_3, x_4)$. Using eq. (17), it is convenient to write

$$\int d^4x\, \mathscr{L}_I(x) = -\frac{\lambda}{4!} \int d^4w_1\, d^4w_2\, d^4w_3\, d^4w_4 \phi(w_1)\phi(w_2)\phi(w_3)\phi(w_4)$$
$$\times \delta^4(w_1 - w_2)\delta^4(w_1 - w_3)\delta^4(w_1 - w_4)\,. \tag{20}$$

Then, the $\mathcal{O}(\lambda)$ term of $G^{(4)}(x_1, x_2, \ldots, x_n)$, denoted below by $G_1^{(4)}(x_1, x_2, \ldots, x_n)$, arises entirely from the fourth term of the expansion of the exponential in eq. (18), which involves eight functional derivatives,

$$G_1^{(4)}(x_1, x_2, x_3, x_4) = -\frac{i\lambda}{4!} \frac{1}{4!} \left(\frac{i}{2}\right)^4 \int d^4w_1 \cdots d^4w_4\, d^4y_1 \cdots d^4y_4\, d^4z_1 \cdots d^4z_4$$
$$\times \delta^4(w_1 - w_2)\delta^4(w_1 - w_3)\delta^4(w_1 - w_4)$$
$$\times \Delta_F(y_1 - z_1)\Delta_F(y_2 - z_2)\Delta_F(y_3 - z_3)\Delta_F(y_4 - z_4)$$
$$\times \frac{\delta}{\delta\phi(y_1)} \cdots \frac{\delta}{\delta\phi(y_4)} \frac{\delta}{\delta\phi(z_1)} \cdots \frac{\delta}{\delta\phi(z_4)} \Big\{\phi(x_1)\phi(x_2)$$
$$\times \phi(x_3)\phi(x_4)\phi(w_1)\phi(w_2)\phi(w_3)\phi(w_4)\Big\}. \tag{21}$$

Note that the expression obtained after evaluating the eight functional derivatives contain no factors of the scalar field. Indeed, when we set $\phi = 0$ in eq. (18), the $\mathcal{O}(\lambda)$ term exhibited in eq. (21) is the only term that survives. Evaluating the eight functional derivatives in eq. (21) leads to a sum of products of eight delta-functions,

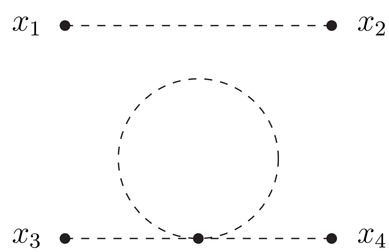
$$G_1^{(4)}(x_1, x_2, x_3, x_4) = -\frac{i\lambda}{4!} \frac{1}{4!} \left(\frac{i}{2}\right)^4 \int d^4w_1 \cdots d^4w_4\, d^4y_1 \cdots d^4y_4\, d^4z_1 \cdots d^4z_4$$
$$\times \delta^4(w_1 - w_2)\delta^4(w_1 - w_3)\delta^4(w_1 - w_4)$$
$$\times \Delta_F(y_1 - z_1)\Delta_F(y_2 - z_2)\Delta_F(y_3 - z_3)\Delta_F(y_4 - z_4)$$
$$\times \left[\delta^4(x_1 - y_1) \cdots \delta^4(x_4 - y_4)\, \delta^4(w_1 - z_1) \cdots \delta^4(w_4 - z_4) + \ldots\right], \tag{22}$$

where we do not explicitly exhibit all possible 8! permutations of $\{x_1, x_2, x_3, x_4, w_1, w_2, w_3, w_4\}$ in the sum of products of delta-functions.

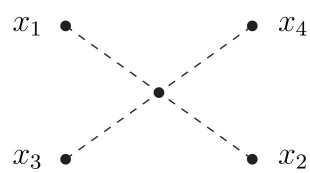
Integrating over $y_1, \ldots, y_4$ and $z_1, \ldots z_4$ in eq. (22) using the eight delta functions yields,

$$G_1^{(4)}(x_1,x_2,x_3,x_4) = -\frac{i\lambda}{4!}\frac{1}{4!}\left(\frac{i}{2}\right)^4 \int d^4w_1 \cdots d^4w_4\, \delta^4(w_1-w_2)\delta^4(w_1-w_3)\delta^4(w_1-w_4)$$
$$\times\big[\Delta_F(x_1-w_1)\Delta_F(x_2-w_2)\Delta_F(x_3-w_3)\Delta_F(x_4-w_4)+\ldots\big]. \quad (23)$$

where the terms not shown above are obtained by allowing for all possible permutations of $\{x_1, x_2, x_3, x_4, w_1, w_2, w_3, w_4\}$. The possible contributions to $G_1^{(4)}(x_1,x_2,x_3,x_4)$ fall into two classes. In one class of terms, at least one propagator factor of the form $\Delta_F(w_i - w_j)$ will appear in the product of four propagator factors for some $i, j = 1, 2, 3, 4$. When the integration over w_2, w_3 and w_4 is carried out, a factor of $\Delta_F(0)$ will appear due to the presence of the delta functions in eq. (23). Such a term corresponds to a disconnected piece of the 4-point Green function. Diagrammatically, a term of this type is represented by, e.g.,[3]



In the second class of terms, all propagator factors are of the form $\Delta_F(x_i - w_j)$. When the integration over w_2, w_3 and w_4 is carried out, the resulting contributions to $G_1^{(4)}(x_1,x_2,x_3,x_4)$ are diagrammatically represented by the connected diagram,



It is straightforward to count the number of terms of this type. We add all possible permutations of the product of propagator factors in eq. (23), consisting of the 4! permutations of $\{x_1, x_2, x_3, x_4\}$, the 4! permutations of $\{w_1, w_2, w_3, w_4\}$, and an additional $2^4 = 16$ terms (for each term previously identified) that are obtained by interchanging $x_i \leftrightarrow w_i$ for $i = 1, 2, 3$ and 4. Since $\Delta_F(x_i - w_i) = \Delta_F(w_i - x_i)$, it follows that there are $2^4 \times (4!)^2$ identical terms after integrating over w_2, w_3 and w_4. Setting $x \equiv w_1$, we end up with

$$G_c^{(4)}(x_1,x_2,x_3,x_4) = -i\lambda \int d^4x\, \Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(x_3-x)\Delta_F(x_4-x)\,. \quad (24)$$

[3]Contributions arising from terms with two propagator factors of the form $\Delta_F(w_i - w_j)$, when integrated over w_2, w_3 and w_4, yield two factors of $\Delta_F(0)$. These contributions are represented by disconnected diagrams that include two-loop vacuum bubbles containing an interaction vertex, which can be discarded as discussed previously.

References

[1] Sidney Coleman, *Aspects of Symmetry*, Selected Erice lectures (Cambridge University Press, Cambridge, UK, 1985). See pp. 152–153.

[2] Jan Ambjørn and Jens Lyng Petersen, *Quantum Field Theory lecture notes* (January, 1998), https://www.nbi.ku.dk/bibliotek/noter-og-undervisningsmateriale-i-fysik/quantum-field-theory-by-jan-ambjoern-and-jens-lyng-petersen/AandP_Quantum_Field_Theory.pdf. See also: Timo Weigand, *Quantum Field Theory* I + II, http://www.thphys.uni-heidelberg.de/ weigand/QFT2-14/SkriptQFT2.pdf.

[3] Horaţiu Năstase, *Introduction to Quantum Field Theory* (Cambridge University Press, Cambridge, UK, 2020).