In the computation of the one-loop amplitude for the Higgs boson decay to two photons, the following integral arises [1–4],

$$F(z) = \lim_{\epsilon \to 0^+} F(z + i\epsilon) \equiv \lim_{\epsilon \to 0^+} \int_0^1 \frac{dx}{x} \ln\bigl[1 - zx(1-x) - i\epsilon\bigr] \,, \tag{1}$$

where z is a real parameter and ϵ is a real positive infinitesimal. The goal of this note is to provide an explicit computation of $F(z)$.

First, let us examine the range of the parameter z for which $\operatorname{Im} F(z) \neq 0$. Let us denote the argument of the logarithm in eq. (1) by the function,

$$f(x) \equiv zx^2 - zx + 1 \geq 0 \,. \tag{2}$$

Observe that $\operatorname{Im} F(z) = 0$ if $f(z) > 0$ for $0 \leq x \leq 1$. In particular, $\operatorname{Im} F(z) = 0$ if $z < 4$ since the maximal value of $x(1-x)$ is $\frac{1}{4}$ over the integration range. Note that $x = \frac{1}{2}$ is a minimum of $f(x)$ if $z > 0$ and $f(\frac{1}{2}) = 1 - \frac{1}{4}z$, which implies that the minimum value of $f(x)$ at $x = \frac{1}{2}$ is negative when $z > 4$. Since $f(0) = f(1) = 1$, it follows that $f(x) < 0$ for values of x such that $0 < x_- < x < x_+ < 1$, where $x_\pm$ are the roots of $f(x)$,

$$x_\pm = \tfrac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{z}}\,\right] . \tag{3}$$

Thus,

$$\operatorname{Im} F(z) = \Theta(z-4) \int_{x_-}^{x_+} \frac{dx}{x} \operatorname{Im} \ln\bigl[1 - zx(1-x) - i\epsilon\bigr] \,, \tag{4}$$

where we have explicitly included the step function to enforce the condition that $\operatorname{Im} F(z) = 0$ if $z \leq 4$. To evaluate the imaginary part of the logarithm, we employ the principal value of the complex-valued logarithm, with the branch cut taken along the negative real axis. In particular, for any nonzero real number y and real positive infinitesimal ϵ,

$$\ln(y - i\epsilon) = \ln|y| - i\pi\Theta(-y) \,. \tag{5}$$

It then follows that $\operatorname{Im} \ln(y - i\epsilon) = -\pi\Theta(-y)$. Employing this result in eq. (4),

$$\begin{aligned}\operatorname{Im} F(z) &= -\pi\Theta(z-4) \int_{x_-}^{x_+} \frac{dx}{x} = -\pi\,\Theta(z-4) \ln\left(\frac{x_+}{x_-}\right) = -\pi\,\Theta(z-4) \ln\left(\frac{1 + \sqrt{1 - \frac{4}{z}}}{1 - \sqrt{1 - \frac{4}{z}}}\right) \\ &= -2\pi\,\Theta(z-4) \ln\left(\frac{\sqrt{z}}{2} + \sqrt{\frac{z}{4} - 1}\,\right) \,, \end{aligned} \tag{6}$$

after using the explicit forms for $x_\pm$ given in eq. (3).

For $0 \leq z < 4$, $\mathrm{Im}\, F(z) = 0$, and we can simply drop the $-i\epsilon$ in eq. (1) and write,

$$F(z) \equiv \int_0^1 \frac{dx}{x} \ln\bigl[1 - zx(1-x)\bigr]\,, \qquad \text{for } 0 \leq z < 4. \tag{7}$$

To evaluate $F(z)$, it is convenient to define,

$$z = 4\sin^2\theta\,, \qquad \text{for } 0 < \theta \leq \tfrac{1}{2}\pi. \tag{8}$$

Next, we take the derivative of F with respect to θ,

$$\frac{dF}{d\theta} = \frac{d}{d\theta}\int_0^1 \frac{dx}{x} \ln\bigl[1 - 4x(1-x)\sin^2\theta\bigr] = -4\sin 2\theta \int_0^1 \frac{(1-x)dx}{1 - 4x(1-x)\sin^2\theta}\,. \tag{9}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1 - 4x(1-x)\sin^2\theta = 4\sin^2\theta(x - x_+)(x - x_-)\,, \qquad \text{where } x_\pm \equiv \pm\frac{ie^{\mp i\theta}}{2\sin\theta}. \tag{10}$$

Hence, it follows that

$$\frac{dF}{d\theta} = -\frac{\sin 2\theta}{\sin^2\theta}\int_0^1 \frac{(1-x)dx}{(x-x_+)(x-x_-)} = -\frac{2\cos\theta}{(x_+ - x_-)\sin\theta}\int_0^1 \left(\frac{1-x_+}{x-x_+} - \frac{1-x_-}{x-x_-}\right) dx\,. \tag{11}$$

Using eq. (10), it follows that

$$x_+ - x_- = \frac{i\cos\theta}{\sin\theta}\,, \qquad x_+ + x_- = 1\,. \tag{12}$$

Moreover,

$$\int_0^1 \frac{(1-x_-)dx}{x - x_-} = \int_0^1 \frac{x_+\,dx}{x - x_-} = \int_0^1 \frac{x_+\,dx}{x - 1 + x_+} = -\int_0^1 \frac{x_+\,dx}{x - x_+}\,, \tag{13}$$

after changing the integration variable $x \to 1 - x$ in the final step above. In light of these last two results, eq. (11) yields,

$$\frac{dF}{d\theta} = 2i\int_0^1 \frac{dx}{x - x_+} = 2i\ln\left(\frac{1-x_+}{-x_+}\right) = 2i\ln\left(\frac{x_-}{-x_+}\right) = 2i\ln\left(e^{2i\theta}\right)\,, \tag{14}$$

after using eq. (10) for $x_\pm$ to obtain the final result.

To complete our analysis, recall that the principal value of the complex logarithm is given by,

$$\ln z = \ln|z| + i\arg z\,, \tag{15}$$

where the principal value of the argument function is defined such that $-\pi < \arg z \leq \pi$. Since $0 < \theta \leq \frac{1}{2}\pi$ [cf. eq. (8)], it follows that $\ln(e^{2i\theta}) = 2i\theta$. Hence, eq. (14) yields,

$$\frac{dF}{d\theta} = -4\theta\,, \qquad \text{for } 0 < \theta \leq \tfrac{1}{2}\pi. \tag{16}$$

Setting $z = 0$ in eq. (7) and noting that $z = 0$ implies that $\theta = 0$, it follows that $F(\theta = 0) = 0$, which serves as an initial condition for eq. (16). Integrating eq. (16) subject to this initial condition yields,

$$F(\theta) = -2\theta^2\,, \qquad \text{for } 0 < \theta \leq \tfrac{1}{2}\pi. \tag{17}$$

From eq. (8), $\sin\theta = \frac{1}{2}\sqrt{z}$. Hence,

$$\theta = \arcsin\left(\tfrac{1}{2}\sqrt{z}\right)\,. \tag{18}$$

Plugging this result back into eq. (17) yields our final result,

$$F(z) = -2\big[\arcsin\big(\tfrac{1}{2}\sqrt{z}\big)\big]^2\,, \qquad \text{for } 0 \leq z < 4\,, \tag{19}$$

Note that an equivalent form for eq. (19) is,

$$F(z) = -2\left[\frac{\pi}{2} - \arccos\big(\tfrac{1}{2}\sqrt{z}\big)\right]^2\,, \qquad \text{for } 0 \leq z < 4\,. \tag{20}$$

An alternative derivation of eq. (19) is given in Appendix A.

The case of $z = 4$ can be treated separately. In this case, $1 - 4x(1-x) = (1-2x)^2$, in which case we can again drop the $-i\epsilon$ term in eq. (1). It then follows that

$$F(z=4) = \int_0^1 \frac{dx}{x}\,\ln\big[(1-2x)^2\big] = 2\int_0^{1/2}\frac{dx}{x}\,\ln(1-2x) + 2\int_{1/2}^1 \frac{dx}{x}\,\ln(2x-1)\,. \tag{21}$$

In the first integral on the right hand side of eq. (21), we substitute $y = 2x$, and in the second integral on the right hand side of eq. (21), we substitute $y = 2x - 1$. Hence,

$$F(z=4) = 2\int_0^1 \frac{dy}{y}\,\ln(1-y) + 2\int_0^1 \frac{dy}{1+y}\,\ln y\,. \tag{22}$$

The two integrals above are well known [5],

$$\int_0^1 \frac{dy}{y}\,\ln(1-y) = -\frac{\pi^2}{6}\,, \qquad \int_0^1 \frac{dy}{1+y}\,\ln y = -\frac{\pi^2}{12}\,. \tag{23}$$

Hence, it follows that

$$F(z=4) = -\tfrac{1}{2}\pi^2\,. \tag{24}$$

In light of eq. (17), $\lim_{\theta\to 0} F(\theta) = \lim_{z\to 4} F(z) = -\frac{1}{2}\pi^2$. Hence, it follows that we can extend the result of eq. (19) to include the endpoint $z = 4$.

If $z > 4$, then $\operatorname{Im} F(z) \neq 0$ and is given explicitly in eq. (6). In order to compute $\operatorname{Re} F(z)$ when $z > 4$, it is convenient to define,

$$z = 4\cosh^2 w\,, \qquad \text{for } 0 < w < \infty\,. \tag{25}$$

In light of eq. (5),

$$\operatorname{Re} F(z) = \int_0^1 \frac{dx}{x}\,\ln\big|1 - zx(1-x)\big|\,. \tag{26}$$

After employing eq. (25), we take the derivative of $\operatorname{Re} F$ with respect to w,

$$\frac{d}{dw}\operatorname{Re} F = \frac{d}{dw}\int_0^1 \frac{dx}{x}\,\ln\bigl|1-4x(1-x)\cosh^2 w\bigr| = -4\sinh 2w\,\mathrm{P}\int_0^1 \frac{(1-x)dx}{1-4x(1-x)\cosh^2 w}\,, \tag{27}$$

where P indicates the principal value prescription. In obtaining this result, we have made use of the relation obtained on p. 26 of Ref. [6],

$$\frac{d}{dy}\,\ln|y| = \mathrm{P}\,\frac{1}{y}\,. \tag{28}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1-4x(1-x)\cosh^2 w = 4\cosh^2 w(x-x_+)(x-x_-)\,, \qquad \text{where } x_\pm \equiv \frac{e^{\pm w}}{2\cosh w}. \tag{29}$$

Hence, it follows that

$$\frac{d}{dw}\operatorname{Re} F = -\frac{\sinh 2w}{\cosh^2 w}\,\mathrm{P}\int_0^1 \frac{(1-x)dx}{(x-x_+)(x-x_-)} = -\frac{2\tanh w}{x_+-x_-}\,\mathrm{P}\int_0^1\left(\frac{1-x_+}{x-x_+}-\frac{1-x_-}{x-x_-}\right)dx\,. \tag{30}$$

Using eq. (29), it follows that

$$x_+-x_- = \tanh w\,, \qquad x_++x_- = 1\,. \tag{31}$$

Moreover,

$$\mathrm{P}\int_0^1 \frac{dx}{x-x_-} = \mathrm{P}\int_0^1 \frac{dx}{x-1+x_+} = -\mathrm{P}\int_0^1 \frac{dx}{x-x_+}\,, \tag{32}$$

after changing the integration variable $x \to 1-x$ in the final step above. Using the definition of the principal value prescription,

$$\begin{aligned}
\mathrm{P}\int_0^1 \frac{dx}{x-x_+} &= \lim_{\delta\to 0^+}\left\{\int_0^{x_+-\delta}\frac{dx}{x-x_+}+\int_{x_++\delta}^1\frac{dx}{x-x_+}\right\} \\
&= \lim_{\delta\to 0^+}\left\{\ln(x_+-x)\Bigr|_0^{x_+-\delta}+\ln(x-x_+)\Bigr|_{x_++\delta}^1\right\} \\
&= \lim_{\delta\to 0^+}\bigl\{\ln\delta-\ln x_++\ln(1-x_+)-\ln\delta\bigr\} \\
&= \ln\left(\frac{1-x_+}{x_+}\right) = \ln\left(\frac{x_-}{x_+}\right) = -2w\,,
\end{aligned} \tag{33}$$

after making use of eqs. (29) and (31). Finally, after employing eqs. (31)–(33), one can simplify eq. (30) to obtain,

$$\frac{d}{dw}\operatorname{Re} F = -2(2-x_+-x_-)\,\mathrm{P}\int_0^1 \frac{dx}{x-x_+} = 4w\,. \tag{34}$$

Integrating both sides of eq. (34) and using eq. (24) to determine the constant of integration, we obtain

$$\mathrm{Re}\,F(w) = 2w^2 - \tfrac{1}{2}\pi^2\,. \tag{35}$$

From eq. (25), $\cosh w = \tfrac{1}{2}\sqrt{z}$. Hence, it follows that

$$\mathrm{Re}\,F(z) = 2\big[\operatorname{arccosh}\big(\tfrac{1}{2}\sqrt{z}\big)\big]^2 - \tfrac{1}{2}\pi^2\,, \qquad \text{for real } z \geq 4\,, \tag{36}$$

where the principal value of the arccosh function for real positive values of $z \geq 4$ is

$$\operatorname{arccosh}\big(\tfrac{1}{2}\sqrt{z}\big) = \ln\left(\frac{\sqrt{z}}{2} + \sqrt{\frac{z}{4} - 1}\right)\,. \tag{37}$$

Note that eqs. (6) and (37) imply that

$$\mathrm{Im}\,F(z) = -2\pi\operatorname{arccosh}\big(\tfrac{1}{2}\sqrt{z}\big)\,, \qquad \text{for real } z \geq 4\,. \tag{38}$$

Combining eqs. (36) and (38) yields,

$$F(z) = -2\left[\frac{\pi}{2} + i\operatorname{arccosh}\big(\tfrac{1}{2}\sqrt{z}\big)\right]^2\,, \qquad \text{for } z > 4\,. \tag{39}$$

An alternative derivation of eq. (39) is given in Appendix A.

In summary,

$$F(z) = \begin{cases} -2\big[\arcsin\big(\tfrac{1}{2}\sqrt{z}\,\big)\big]^2\,, & \text{for } 0 \leq z \leq 4\,, \\ -2\left[\tfrac{\pi}{2} + i\operatorname{arccosh}\big(\tfrac{1}{2}\sqrt{z}\big)\right]^2\,, & \text{for } z > 4\,. \end{cases} \tag{40}$$

In the literature, one often rewrites the expression for $F(z)$ when $z > 4$ in one of the two following equivalent forms,

$$F(z) = -2\left[\frac{\pi}{2} + i\ln\left(\frac{\sqrt{z}}{2} + \sqrt{\frac{z}{4} - 1}\right)\right]^2 = -\frac{1}{2}\left[\pi + i\ln\left(\frac{1 + \sqrt{1 - \frac{4}{z}}}{1 - \sqrt{1 - \frac{4}{z}}}\right)\right]^2\,, \qquad \text{for } z > 4, \tag{41}$$

after employing the identity,

$$\operatorname{arccosh} x = \ln\big(x + \sqrt{x^2 - 1}\,\big)\,, \qquad \text{for } x \geq 1. \tag{42}$$

A similar method to the one presented in these notes for evaluating $F(z)$ has been given in Refs. [8,9]. In these two references, dF/dz is evaluated first and then the result is integrated to obtain $F(z)$. However, the final integration of dF/dz is more difficult as compared to the derivation given in these notes.

It is straightforward to show that the two expressions on the right hand side of eq. (40) are analytic continuations of one another. This statement is proven at the end of Appendix A.

For completeness, I provide below a derivation of $F(z)$ for $z < 0$. In this case, it is convenient to define,

$$z = -4\sinh^2 w\,, \qquad \text{for } 0 < w < \infty. \tag{43}$$

Next, we take the derivative of F with respect to w,

$$\frac{dF}{dw} = \frac{d}{dw}\int_0^1 \frac{dx}{x}\ln\bigl[1+4x(1-x)\sinh^2 w\bigr] = 4\sinh 2w\int_0^1 \frac{(1-x)dx}{1+4x(1-x)\sinh^2 w}\,. \tag{44}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1+4x(1-x)\sinh^2 w = -4\sinh^2 w(x-x_+)(x-x_-)\,, \qquad \text{where } x_\pm = \pm\frac{e^{\pm w}}{2\sinh w}\,. \tag{45}$$

Hence, it follows that

$$\frac{dF}{dw} = -\frac{\sinh 2w}{\sinh^2 w}\int_0^1 \frac{(1-x)dx}{(x-x_+)(x-x_-)} = -\frac{2\cosh w}{\sinh w(x_+-x_-)}\int_0^1\left(\frac{1-x_+}{x-x_+}-\frac{1-x_-}{x-x_-}\right)dx\,. \tag{46}$$

Note that $x_+ \in (1,\infty)$ and $x_- \in (-\infty, 0)$. Hence, the integrands above are not singular for $0 \leq x \leq 1$, and the corresponding integrals are well defined.

Using eq. (45), it follows that

$$x_+ - x_- = \frac{\cosh w}{\sinh w}\,, \qquad x_+ + x_- = 1\,. \tag{47}$$

It follows that

$$\begin{aligned}\frac{dF}{dw} &= -2\int_0^1\left(\frac{x_-}{x-x_+}-\frac{x_+}{x-x_-}\right)dx = -2x_-\ln\left(\frac{x_+-1}{x_+}\right)+2x_+\ln\left(\frac{x_1-1}{x_-}\right)\\ &= -2x_-\ln\left(-\frac{x_-}{x_+}\right)+2x_+\ln\left(-\frac{x_+}{x_-}\right) = 2\ln\left(-\frac{x_+}{x_-}\right) = 4w\,,\end{aligned} \tag{48}$$

after employing eqs. (45) and (47). Using the boundary condition, $F(0) = 0$, one can solve the differential equation above to obtain $F(w) = 2w^2$. In light of eq. (43), $w = \operatorname{arcsinh}\frac{1}{2}\sqrt{-z}$, and we end up with

$$F(z) = 2\bigl[\operatorname{arcsinh}\bigl(\tfrac{1}{2}\sqrt{-z}\bigr)\bigr]^2\,, \qquad \text{for } z \leq 0, \tag{49}$$

which is the analytic continuation of eq. (19) into the region of negative values of z. Employing the identity,

$$\operatorname{arcsinh} x = \ln\bigl(x+\sqrt{1+x^2}\bigr)\,, \tag{50}$$

one can obtain another form of eq. (49), which we can write in two different ways,

$$F(z) = 2\ln^2\left(\frac{\sqrt{-z}}{2}+\sqrt{1-\frac{z}{4}}\right) = \frac{1}{2}\ln^2\left(\frac{\sqrt{1-\frac{4}{z}}+1}{\sqrt{1-\frac{4}{z}}-1}\right)\,, \qquad \text{for } z \leq 0. \tag{51}$$

Appendix A: Alternative derivation of $F(z)$

For $0 \leq z < 4$, $\operatorname{Im} F(z) = 0$. One can therefore drop the factor of $-i\epsilon$ in eq. (1) and write,

$$F(z) \equiv \int_0^1 \frac{dx}{x} \ln\bigl[1 - zx(1-x)\bigr]\,, \qquad \text{for } 0 \leq z < 4. \tag{52}$$

Noting that $0 \leq zx(1-x) < 1$ for all $0 \leq x \leq 1$ and $0 \leq z < 4$, we can employ a series expansion for the logarithm,

$$\ln(1-w) = -\sum_{n=1}^{\infty} \frac{w^n}{n}\,, \qquad \text{for } -1 \leq w < 1. \tag{53}$$

Setting $w = zx(1-x)$ in eq. (52) and interchanging the order of integration and summation,

$$F(z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^1 x^{n-1}(1-x)^n\, dx\,, \qquad \text{for } 0 \leq z < 4. \tag{54}$$

We recognize the integral above as a beta function,

$$B(n, n+1) = \int_0^1 x^{n-1}(1-x)^n\, dx = \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)} = \frac{(n-1)!\, n!}{(2n)!}\,. \tag{55}$$

Plugging this result back into eq. (54) yields,

$$F(z) = -\sum_{n=1}^{\infty} \frac{\bigl[(n-1)!\bigr]^2}{(2n)!} z^n\,, \qquad \text{for } 0 \leq z < 4. \tag{56}$$

Comparing this result with eq. (71) of Appendix B, we conclude that

$$F(z) = -2\bigl[\arcsin\bigl(\tfrac{1}{2}\sqrt{z}\bigr)\bigr]^2\,, \qquad \text{for } 0 \leq z < 4\,, \tag{57}$$

where arcsin is the principal value of the arcsine function, which satisfies $|\arcsin x| \leq \frac{1}{2}\pi$ for real values of x. Thus, we have confirmed the result of eq. (19).

One can now employ the method of analytic continuation to obtain $F(z)$ in the region where $z > 4$. Note that an equivalent form for eq. (57) is,

$$F(z) = -2\left[\frac{\pi}{2} - \arccos\bigl(\tfrac{1}{2}\sqrt{z}\bigr)\right]^2\,, \qquad \text{for } 0 \leq z < 4\,. \tag{58}$$

To analytically continue into the region of real $z > 4$, we employ eqs. (4.23.24) and (4.37.19) of Ref. [7], which imply that for a positive infinitesimal ϵ,

$$\lim_{\epsilon \to 0^+} \arccos(x + i\epsilon) = -i \operatorname{arccosh} x = -i \ln\bigl(x + \sqrt{x^2 - 1}\bigr)\,, \qquad \text{for } 1 < x < \infty. \tag{59}$$

Consequently, for $z > 4$,

$$F(z) = \lim_{\epsilon \to 0^+} F(z + i\epsilon) = -2\left[\frac{\pi}{2} - \arccos\bigl(\tfrac{1}{2}\sqrt{z} + i\epsilon\bigr)\right]^2 = -2\left[\frac{\pi}{2} + i \operatorname{arccosh}\bigl(\tfrac{1}{2}\sqrt{z}\bigr)\right]^2\,, \tag{60}$$

in agreement with eq. (39). As expected, both eqs. (58) and (60) yield the same result at their common boundary, $F(z=4)=-\frac{1}{2}\pi^2$.

Similarly, the analytic continuation of eq. (57) into the region of $z<0$ yields eq. (49) in light of the relation, $\arcsin(ix)=i\,\operatorname{arcsinh} x$.

A careful treatment of the analytic continuation of $F(z)$ is also given in Ref. [9].

Appendix B: Power series of $(\arcsin x)^2$

One method for deriving a power series of a function is to develop a differential equation (with appropriate initial conditions) whose solution is the function in question. This differential equation can then be solved by the series expansion method. This technique was used by Ref. [10] to derive the Taylor series for $(\arcsin x)^2$ about the origin.[1] Inspired by the computation of Ref. [10], we first consider the function,

$$y=\frac{\arcsin x}{\sqrt{1-x^2}}\,, \tag{61}$$

where the principal value of the arcsine function is employed such that $|\arcsin x|\leq\frac{1}{2}\pi$ for real values of x. We can derive the Taylor series of eq. (61) about $x=0$ by the following technique. Taking the derivative of eq. (61) yields

$$\frac{dy}{dx}=\frac{1}{1-x^2}+\frac{x\,\arcsin x}{(1-x^2)^{3/2}}\,. \tag{62}$$

It follows that eq. (61) is the solution to the following first order differential equation,

$$(1-x^2)\frac{dy}{dx}-xy=1\,,\qquad\text{where } y(x=0)=0\,. \tag{63}$$

Note that setting $x=0$ in eq. (61) yields $y=0$ which fixes the initial condition for eq. (63).

One can solve eq. (63) using a series solution,

$$y=\sum_{n=0}^{\infty}c_nx^n\,. \tag{64}$$

Plugging eq. (64) back into eq. (63) yields,

$$\sum_{n=1}^{\infty}nc_nx^{n-1}-\sum_{n=1}^{\infty}nc_nx^{n+1}-\sum_{n=0}^{\infty}c_nx^{n+1}=1\,. \tag{65}$$

Equating coefficients of x^n on both sides of eq. (65) and imposing $y(x=0)=0$ yields $c_0=0$, $c_1=1$ and

$$c_{n+1}=\frac{n}{n+1}c_{n-1}\,,\quad\text{for } n=1,2,3,\ldots \tag{66}$$

[1]Other methods for obtaining the Taylor series for $(\arcsin x)^2$ about $x=0$ can be found in Refs. [11–15].

It immediately follows that

$$c_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3} = \frac{2^n n!}{(2n+1)!!}\,, \qquad\qquad c_{2n} = 0\,, \qquad \text{for } n = 0, 1, 2, \ldots \tag{67}$$

Hence we conclude that

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!!}\, x^{2n+1}\,, \qquad \text{for } |x| < 1. \tag{68}$$

In eq. (68), we have noted that the convergence of the sum requires that $|x| < 1$.

In light of

$$\frac{d}{dx}\,(\arcsin x)^2 = \frac{2\arcsin x}{\sqrt{1-x^2}}\,,$$

it follows that

$$(\arcsin x)^2 = 2\int_0^x \frac{\arcsin t}{\sqrt{1-t^2}}\, dt\,. \tag{69}$$

Inserting the series obtained in eq. (68) on the right hand side of eq. (69) yields,

$$\begin{aligned}(\arcsin x)^2 &= 2\sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!!}\int_0^x t^{2n+1} = \sum_{n=0}^{\infty} \frac{2^n n!}{(n+1)(2n+1)!!}\, x^{2n+2} \\ &= \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)!}{n(2n-1)!!}\, x^{2n}\,, \qquad \text{for } |x| \leq 1\,. \end{aligned} \tag{70}$$

One can check that the series on the right hand side of eq. (70) converges at all points on the boundary of the circle of convergence.

Note that

$$(2n)! = (2n)!!\,(2n-1)!! = 2^n n!\,(2n-1)!!\,.$$

Hence,

$$(2n-1)!! = \frac{(2n)!}{2^n n!}\,.$$

Inserting this result into eq. (70) yields,

$$(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\big[(n-1)!\big]^2}{(2n)!}\,(2x)^{2n}\,, \qquad \text{for } |x| \leq 1\,. \tag{71}$$

Using $(n-1)! = n!/n$ and introducing the central binomial coefficient,

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}\,,$$

one can rewrite eq. (71) as

$$(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2\binom{2n}{n}}\,, \qquad \text{for } |x| \leq 1\,. \tag{72}$$

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