DUE: TUESDAY, MAY 5, 2020

1. Consider a quantum field theory of interacting real scalar and Dirac fermion of mass m_R and M respectively. The interaction Lagrangian is given by,

$$\mathscr{L}_{\rm int} = -g\bar{\psi}\psi\phi - \frac{\lambda}{4!}\phi^4 \,.$$

(a) Compute the wave function renormalization constant of the scalar field using $\overline{\rm MS}$ renormalization, in the one loop approximation.

(b) The renormalized spectral function is defined by $\sigma_R(m^2) \equiv Z_\phi^{-1}\sigma(m^2)$. Then, the Källén-Lehmann representation for the renormalized two-point function reads:

$$G_R^{(2)}(p^2) = \frac{i}{p^2 - m_R^2 + i\epsilon} + \int_{4M^2}^{\infty} dm^2\, \sigma_R(m^2)\, \frac{i}{p^2 - m^2 + i\epsilon}\,. \tag{1}$$

Assuming that $m_R < 2M$, compute the contribution to the spectral function, σ_R, of the scalar two-point function due to a fermion loop in the one-loop approximation. Note that in this approximation, the lower limit of integration of $4M^2$ is appropriate (why?). What is the behavior of $\sigma_R(m^2)$ as $m^2 \to \infty$?

HINT: First, evaluate the renormalized 1PI two-point Green function, $\Gamma_R^{(2)}(p^2)$ and then obtain the corresponding two-point Green function, $G_R^{(2)}(p^2)$, due to a fermion loop in the one-loop approximation. Perform this computation in the on-shell scheme, where $G_R^{(2)}(p^2)$ has a pole at the square of the physical mass m_R^2 with unit residue. Comparing the result of this calculation with eq. (1), determine $\sigma_R(m^2)$ by considering the imaginary part of the $\mathcal{O}(g_R^2)$ contribution to $G_R^{(2)}(p^2)$. You should recall the identity:

$$\frac{1}{z+i\epsilon} = {\rm P}\,\frac{1}{z} - i\pi\delta(z)\,,$$

where ϵ is a real positive infinitesimal quantity.

2. Consider the function of a *real* parameter z

$$F(z) \equiv \int_0^1 dx\, \ln\bigl[1 - zx(1-x) - i\epsilon\bigr]\,, \tag{2}$$

where ϵ is a positive infinitesimal quantity. The function $F(z)$ appears in the computation of the one-loop correction to the 4-point Green function in scalar field theory.

(a) Evaluate ${\rm Im}\,F(z)$. For what values of z does ${\rm Im}\,F$ vanish?

HINT: First, determine the imaginary part of the integrand. Note that $\ln\bigl[1 - zx(1-x) - i\epsilon\bigr]$ should be interpreted as the principal value of the complex-valued logarithm, with the branch cut along the negative real axis.

(b) Evaluate ${\rm Re}\,F(z)$. Consider separately the cases of $0 < z < 4$ and $z > 4$.

(c) Consider the unrenormalized 1PI 4-point Green function, $\Gamma^{(4)}(p_1, \ldots p_4)$, where all four-momenta p_i are on mass shell, in a field theory of a real scalar field with mass m and an interaction Lagrangian density, $\mathscr{L}_I = -\lambda\phi^4/4!$. Using the Feynman rules for this theory, write down an integral expression for the full $\mathcal{O}(\lambda^2)$ contribution to $\Gamma^{(4)}$. From the integral expression, evaluate $\operatorname{Im}\Gamma^{(4)}$, up to order λ^2 by making use of the Cutkosky cutting rules.[1]

(d) An explicit one-loop computation of $\Gamma^{(4)}$ yields

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -\lambda - \frac{\lambda^2}{32\pi^2}\left[F\left(\frac{s}{m^2}\right) + F\left(\frac{t}{m^2}\right) + F\left(\frac{u}{m^2}\right) + G(m^2)\right], \qquad (3)$$

where all momenta point into the vertex, $s \equiv (p_1+p_2)^2$, $t \equiv (p_1+p_3)^2$, $u \equiv (p_1+p_4)^2$ are Lorentz-invariant kinematic variables, the function F is defined in part (a), and the function G is a real function.[2] Using eq. (3) and the results of part (a), compute $\operatorname{Im}\Gamma^{(4)}$ and check that your calculation in part (b) is correct.

3. The photon vacuum polarization function is defined to be:

$$\Pi^{\mu\nu}(q) = (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi(q^2)\,.$$

In class, we evaluated this function at one-loop in the $\overline{\text{MS}}$ scheme. Consider a second scheme, called the *on-shell scheme*, in which we define $\Pi(q^2=0) \equiv 0$.

(a) Evaluate Z_3 in this scheme.

(b) Obtain asymptotic forms for $\Pi(q^2)$ in two limiting cases: (i) $q^2 \to 0$, and (ii) $q^2 \to \infty$.

(c) Using the $q^2 \to 0$ limit of part (b), compute the $\mathcal{O}(\alpha)$ correction to the Coulomb potential. *OPTIONAL:* Compute the $\mathcal{O}(\alpha)$ correction to the Coulomb potential without making the approximation of small q^2. Examine explicitly the limiting cases $m_e r \gg 1$ and $m_e r \ll 1$.

(d) Show that the quantity:

$$\alpha_{\text{eff}}(q^2) \equiv \frac{\alpha}{1+\Pi(q^2)}$$

is independent of whether you evaluate this expression using bare or renormalized quantities. As a result, argue that $\alpha_{\text{eff}}(q^2)$ is independent of renormalization scheme. Outline how you would relate the coupling constants defined in the $\overline{\text{MS}}$ and on-shell schemes. Sketch a graph of $\alpha_{\text{eff}}(-q^2)$ at one-loop, in the on-shell scheme, i.e. for *negative* values of the argument].

NOTE: In the on-shell scheme, $\alpha_{\text{eff}}(0)$ is the fine structure constant, which is approximately equal to 1/137.

(e) Calculate the numerical value of the momentum scale (in GeV units) where $\alpha_{\text{eff}}(-q^2)$ diverges.

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[1] See,e.g. Section 24.1.2 [pp. 456–459] of Matthew Schwartz, *Quantum Field Theory and the Standard Model* (Cambridge University Press, 2014).

[2] In fact, the function G is infinite, but this infinity can be removed by renormalization. Since we are only interested here in $\operatorname{Im}\Gamma^{(4)}$, we can safely ignore any details associated with the renormalization procedure.

4. Consider QED coupled to a neutral scalar field:

$$\mathscr{L} = \mathscr{L}_{\mathrm{QED}} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - g\overline{\psi}\psi\phi\,.$$

(a) Compute the amplitude for the decay $\phi \to \gamma\gamma$, as a function of m_e, m, g, and $\alpha \equiv e^2/(4\pi)$, using perturbation theory at one-loop. Simplify your answer by invoking the kinematics of the problem, i.e. momentum conservation and the on-shell conditions for the external particles. Take care to consider two diagrams which differ only in the direction of flow of electric charge in the loop. Do you need to add a counterterm in order to remove an infinity? Explain.

(b) Denote the amplitude for the scalar decay by $\mathcal{M}_{\mu\nu}$, where μ and ν are the photon Lorentz indices. Gauge invariance implies that $k_1^\mu\mathcal{M}_{\mu\nu} = k_2^\nu\mathcal{M}_{\mu\nu} = 0$, where k_1 and k_2 are the respective photon momenta. Does your amplitude of part (a) respect this requirement?

(c) Work out all integrals explicitly and evaluate the imaginary part of $\mathcal{M}_{\mu\nu}$. For what range of m_e/m is the amplitude purely real? Explain the physical significance of the non-zero imaginary part.

HINT: You may find the following integral useful:

$$\int_0^1 \frac{dy}{y}\,\log\bigl[1 - 4Ay(1-y)\bigr] = -2\left(\sin^{-1}\sqrt{A}\right)^2\,, \tag{4}$$

for $0 \le A \le 1$. For values of A outside this region, you may analytically continue the above result. The imaginary part of this integral is easily computed once the $i\epsilon$ factor is restored in the argument of the logarithm.

EXTRA CREDIT: Derive eq. (4).

(d) Evaluate the leading behavior of $\mathcal{M}_{\mu\nu}$ in the limit of $m_e \to \infty$.