DUE: THURSDAY, MAY 21, 2020

1. Define the following functions:

$$A_0(m^2) \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 - m^2 + i\varepsilon}\,,$$

$$B_0(p^2; m_1^2, m_2^2) \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m_1^2 + i\varepsilon)[(q+p)^2 - m_2^2 + i\varepsilon]}\,,$$

$$B^\mu(p; m_1^2, m_2^2) \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n} \frac{q^\mu}{(q^2 - m_1^2 + i\varepsilon)[(q+p)^2 - m_2^2 + i\varepsilon]}\,,$$

where ε is a positive infinitesimal quantity, where m, m_1 and m_2 are real parameters.

(a) Compute A_0 and B_0 explicitly using dimensional regularization. Expand your results about $n = 4$ and drop all terms that vanish as $n \to 4$. Using the notation

$$\Delta \equiv \frac{1}{\epsilon} - \gamma + \ln 4\pi\,,$$

where $n = 4 - 2\epsilon$ and γ is Euler's constant, express your result in each case as a the sum of two terms: one term involving Δ and a second term that is finite as $n \to 4$.

NOTE: Since B_0 is a Lorentz scalar function, it can only depend on the (real) four-vector p^μ through the scalar quantity $p^2 \equiv p^\mu p_\mu$.

(b) Show that B^μ takes the following form

$$B^\mu(p; m_1^2, m_2^2) = p^\mu\, B_1(p^2, m_1^2, m_2^2)\,.$$

Find an expression for the scalar function B_1 in terms of B_0 and A_0 evaluated at the appropriate arguments.

(c) In analyzing a one-loop triangle graph, the following loop integral arises,

$$C_0(p_1^2, p_2^2, p^2; m_1^2, m_2^2, m_3^2)$$
$$\equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m_1^2 + i\varepsilon)[(q+p_1)^2 - m_2^2 + i\varepsilon][(q+p_1+p_2)^2 - m_3^2 + i\varepsilon]}\,,$$

where $p + p_1 + p_2 = 0$, with all external four-momenta pointing into the triangle

Find an explicit expression for $C_0(0,0,0; m_1^2, m_2^2, m_3^2)$ under the assumption that all masses m_i are distinct. Repeat your analysis in two special cases: (i) $m_1 = m_2 \neq m_3$ and (ii) $m_1 = m_2 = m_3$.

2. In QED, the renormalization group functions are:

$$\beta(e) = \mu \frac{de_R}{d\mu}\,,$$

$$\delta(e) = \mu \frac{da_R}{d\mu}\,,$$

$$m_R \gamma_m(e) = \mu \frac{dm_R}{d\mu}\,,$$

$$\gamma_i(e) = \tfrac{1}{2}\mu \frac{\partial}{\partial\mu} \ln Z_i \qquad (i = 2, 3)\,.$$

(a) Compute $\beta(e)$, $\delta(e)$, $\gamma_m(e)$, $\gamma_2(e)$ and $\gamma_3(e)$ in the one-loop approximation, using the MS renormalization scheme.

(b) The running coupling constant in QED can be written as:

$$\overline{\alpha}(Q) = \frac{3\pi}{\ln(\Lambda^2/Q^2)}\,,$$

in the one loop approximation. Using the boundary condition $\overline{\alpha}(\mu) \equiv e_R^2/4\pi$, express Λ in terms of μ and e_R. Show that Λ is a renormalization group invariant; that is, $\mu\, d\Lambda/d\mu = 0$. Evaluate Λ numerically. What is the physical significance of Λ?

(c) Find the relation between the $\overline{\text{MS}}$ mass parameter, m_R, and the physical electron mass m_e (i.e., the pole mass) in the one-loop approximation.

3. In this problem, you will investigate the behavior of the renormalization group functions in QED under a change of renormalization scheme. You should assume throughout the problem that you are working in a class of renormalization schemes that are mass-independent. In particular, if e_1 and e_2 are coupling constants defined in two different schemes, then I can expand one in the other, e.g.,

$$e_1 = e_2 + Ae_2^3 + \ldots$$

for some appropriate mass-independent coefficient A.

(a) Show that there is a one-to-one correspondence between the fixed points [i.e., the zeros of $\beta(e)$] of both schemes, and the value of the first derivative of $\beta(e)$ at the corresponding fixed points is independent of scheme.

(b) Show that the values of γ_m and γ_i $(i = 2, 3)$ at the corresponding fixed points [as defined in part (a)] are independent of scheme.

(c) One can compute $\beta(e)$ as a power series in e in perturbation theory. Show that the coefficients of the first two terms are independent of scheme, but the coefficient of all succeeding terms are scheme-dependent.

(d) Likewise, if one computes γ_m and γ_i $(i = 2, 3)$ in perturbation theory, show that only the leading terms are scheme-independent, whereas all higher order terms are scheme-dependent.

4. Consider QED coupled to a neutral scalar field:

$$\mathscr{L} = \mathscr{L}_{QED} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - g\overline{\psi}\psi\phi\,.$$

Define a separate β-function of each coupling constant: β_e, β_g and β_λ.

(a) Is the QED Ward identity, $Z_1 = Z_2$, modified in this theory? At one-loop, will β_e be the same or different from what you obtained in problem 2?

(b) Compute β_g and β_λ, assuming that λ is of order g^2. Work consistently to lowest nontrivial order in perturbation theory.

(c) The equations for β_e, β_g and β_λ form a set of coupled differential equations for the three running coupling constants. Identify the fixed points of these equations, and discuss their significance.