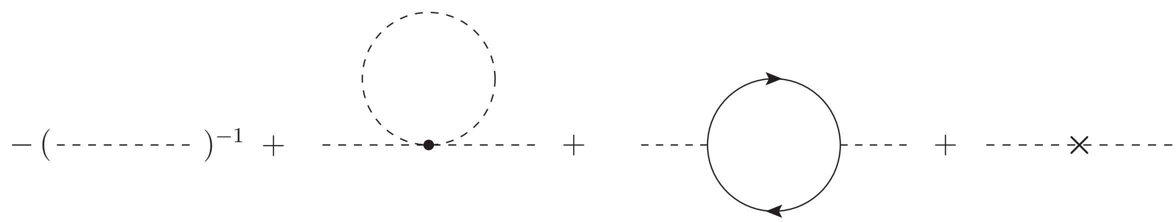
1. Consider a quantum field theory of interacting real scalar and Dirac fermion of mass m_R and M respectively. The interaction Lagrangian is given by,

$$\mathscr{L}_{\rm int} = -g\bar{\psi}\psi\phi - \frac{\lambda}{4!}\phi^4 \,.$$

(a) Compute the wave function renormalization constant of the scalar field using $\overline{\rm MS}$ renormalization, in the one loop approximation.

To compute the wave function renormalization, we must evaluate the diagrams that contribute to the renormalized two-point 1PI Green function, $\Gamma_R^{(2)}(p)$. In a quantum field theory of interacting real scalar and Dirac fermion of mass m_R and M respectively, the Feynman diagram representation of $i\Gamma_R^{(2)}(p)$, in the one loop approximation, consists of the following diagrams:

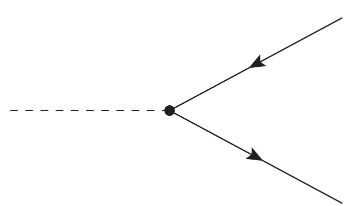


where scalars are represented by dashed lines, fermions are represented by solid lines (with arrows denoting the direction of flow of fermion number), and the dashed line with the $\times$ indicates the counterterm.

In class, we have already evaluated the diagrams involving scalars alone. The result obtained in class was given by,

$$i\Gamma_R^{(2)}(p^2) = i\left\{p^2 Z_\phi - m_R^2\left[Z_m Z_\phi - \frac{\lambda_R}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) + 1 + \ln\left(\frac{\mu^2}{m_R^2}\right)\right)\right]\right\}\,. \quad (1)$$

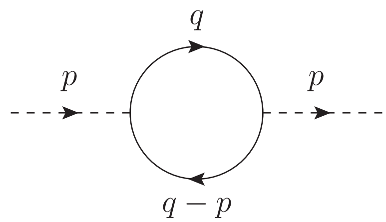
We must now compute the fermion loop contribution to $i\Gamma_R^{(2)}(p)$, which is new. We shall employ the Feynman rule for the scalar-fermion-fermion vertex,



$-i\mu^\epsilon g_R$

where the factor of μ^ϵ appears so that g_R is dimensionless in $n = 4-2\epsilon$ spacetime dimensions. This interaction vertex is obtained by writing $g = Z_g\mu^\epsilon g_R$, $\phi = Z_\phi^{1/2}\phi_R$ and $\psi = Z_\psi^{1/2}\psi_R$ and then separating $\mathscr{L}$ into terms that involve renormalized fields, masses and couplings plus counterterms. However, the new counterterms do not contribute to $i\Gamma_R^{(2)}(p)$ in the one-loop approximation.

Thus, we now focus on the computation of the fermion loop graph, denoted by $-i\Sigma(p^2)$,



where the four-momenta of the external and internal lines are specified. Using the Feynman rules to evaluate the above graph, we obtain

$$
\begin{aligned}
-i\Sigma(p^2) &= -(-i\mu^\epsilon g_R)^2 \int \frac{d^nq}{(2\pi)^n} \frac{i^2\,\mathrm{Tr}\big\{(\not q + M)(\not q - \not p + M)\big\}}{(q^2 - M^2 + i\varepsilon)\big[(q-p)^2 - M^2 + i\varepsilon\big]} \\
&= -4\mu^{2\epsilon} g_R^2 \int \frac{d^nq}{(2\pi)^n} \frac{M^2 + q\cdot(q-p)}{(q^2 - M^2 + i\varepsilon)\big[(q-p)^2 - M^2 + i\varepsilon\big]}\,, \qquad (2)
\end{aligned}
$$

where we are distinguishing the infinitesimal parameter ε that appears in the Feynman rule for the fermion propagator from $\epsilon = 2 - \frac{1}{2}n$.

Introducing Feynman parameters,

$$
\begin{aligned}
-i\Sigma(p^2) &= -4\mu^{2\epsilon} g_R^2 \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \frac{M^2 + q\cdot(q-p)}{(1-x)(q^2 - M^2 + i\varepsilon) + x[(q-p)^2 - M^2 + i\varepsilon]} \\
&= -4\mu^{2\epsilon} g_R^2 \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \frac{M^2 + q\cdot(q-p)}{\big(q^2 - 2xq\cdot p + xp^2 - M^2 + i\varepsilon\big)^2}\,. \qquad (3)
\end{aligned}
$$

The integral over q can be evaluating by using the formulae given in the class handout entitled, *Useful formulae for computing one-loop integrals*,[1]

$$
\int \frac{d^nq}{(2\pi)^n} \frac{1}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^r} = i(-1)^r (p^2 + m^2)^{2-\epsilon-r} (4\pi)^{\epsilon-2} \frac{\Gamma(\epsilon + r - 2)}{\Gamma(r)}\,, \qquad (4)
$$

$$
\int \frac{d^nq}{(2\pi)^n} \frac{q^\mu}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^r} = -i(-1)^r (p^2 + m^2)^{2-\epsilon-r} (4\pi)^{\epsilon-2} \frac{\Gamma(\epsilon + r - 2)}{\Gamma(r)}\, p^\mu\,, \qquad (5)
$$

$$
\int \frac{d^nq}{(2\pi)^n} \frac{q^\mu q^\nu}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^r} = i(-1)^r (p^2 + m^2)^{2-\epsilon-r} (4\pi)^{\epsilon-2} \frac{\Gamma(\epsilon + r - 3)}{\Gamma(r)}\,. \qquad (6)
$$

It follows that

$$
\begin{aligned}
\int \frac{d^nq}{(2\pi)^n} \frac{M^2 + q\cdot(q-p)}{\big(q^2 - 2xq\cdot p + xp^2 - M^2 + i\varepsilon\big)^2} &= i(4\pi)^{\epsilon-2}\Gamma(\epsilon)\big[M^2 - p^2x(1-x)\big]^{-\epsilon} \\
&\quad \times \left[\left(1 + \frac{2-\epsilon}{1-\epsilon}\right)(M^2 - xp^2) + \left(\frac{3-2\epsilon}{1-\epsilon}\right)x^2p^2\right] \\
&= i(4\pi)^{\epsilon-2}\Gamma(\epsilon)\left(\frac{3-2\epsilon}{1-\epsilon}\right)\big[M^2 - p^2x(1-x)\big]^{1-\epsilon}\,. \qquad (7)
\end{aligned}
$$

[1]On the right hand side of eqs. (4)–(6), one should really write $m^2 - i\varepsilon$ in place of m^2. The $i\varepsilon$ factors can be restored later if necessary.

Hence,

$$-i\Sigma(p^2) = -\frac{ig_R^2}{4\pi^2}(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{3-2\epsilon}{1-\epsilon}\right)\int_0^1\left(\frac{M^2-p^2x(1-x)}{\mu^2}\right)^{-\epsilon}\left[M^2-p^2x(1-x)\right]dx\,. \quad (8)$$

Finally, expanding about $\epsilon = 0$,

$$-i\Sigma(p^2) = -\frac{3ig_R^2}{4\pi^2}\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)\right)\left(1+\frac{\epsilon}{3}\right)\left[1+\epsilon\ln\left(\frac{\mu^2}{M^2}\right)\right]$$
$$\times\int_0^1\left[M^2-p^2x(1-x)\right]\left[1-\epsilon\ln\left(1-\frac{p^2x(1-x)}{M^2}\right)\right]dx+\mathcal{O}(\epsilon)\,. \quad (9)$$

Separating out the terms proportional to M^2 and p^2 yields,

$$-i\Sigma(p^2) = -\frac{ig_R^2}{4\pi^2}\Bigg\{3M^2\left[\frac{1}{\epsilon}-\gamma+\ln(4\pi)+\ln\left(\frac{\mu^2}{M^2}\right)+\frac{1}{3}-\int_0^1\ln\left(1-\frac{p^2x(1-x)}{M^2}\right)dx\right]$$
$$-\frac{p^2}{2}\left[\frac{1}{\epsilon}-\gamma+\ln(4\pi)+\ln\left(\frac{\mu^2}{M^2}\right)+\frac{1}{3}-6\int_0^1 x(1-x)\ln\left(1-\frac{p^2x(1-x)}{M^2}\right)dx\right]\Bigg\}. \quad (10)$$

Consequently, adding the result of eq. (10) to eq. (1) yields,

$$i\Gamma_R^{(2)}(p^2) = i\Bigg\{p^2\left[Z_\phi+\frac{g_R^2}{8\pi^2}\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)\right)\right]$$
$$-m_R^2\left[Z_mZ_\phi-\left(\frac{\lambda_R}{32\pi^2}-\frac{3g_R^2M^2}{4\pi^2m_R^2}\right)\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)\right)\right]\Bigg\}+\text{finite terms}\,. \quad (11)$$

Writing $Z_\phi = 1+\delta Z_\phi$, $\overline{\rm MS}$ renormalization consists of completely absorbing the term proportional to $\epsilon^{-1}-\gamma+\ln(4\pi)$ into δZ_ϕ. Hence, we conclude that

$$Z_\phi = 1-\frac{g_R^2}{8\pi^2}\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)\right)\,. \quad (12)$$

One can also determine Z_mZ_ϕ in a similar manner, which then yields an expression for Z_m after employing the result of eq. (12). This computation is left as an exercise for the student.

(b) The renormalized spectral function is defined by $\sigma_R(m^2) \equiv Z_\phi^{-1}\sigma(m^2)$. Then, the Källén-Lehmann representation for the renormalized two-point function reads:

$$G_R^{(2)}(p^2) = \frac{i}{p^2-m_R^2+i\varepsilon}+\int_{4M^2}^\infty dm^2\,\sigma_R(m^2)\,\frac{i}{p^2-m^2+i\varepsilon}\,. \quad (13)$$

Assuming that $m_R < 2M$, compute the contribution to the spectral function, σ_R, of the scalar two-point function due to a fermion loop in the one-loop approximation. Note that in this approximation, the lower limit of integration of $4M^2$ is appropriate (why?). What is the behavior of $\sigma_R(m^2)$ as $m^2\to\infty$?

The two-point 1PI scalar Green function has the form,

$$i\Gamma_R^{(2)}(p^2) = i\big(p^2 - m_R^2 + i\varepsilon - \Sigma_R(p^2)\big)\,, \tag{14}$$

where $-i\Sigma_R(p^2)$ is the sum of scalar self-energy graphs (i.e., 1PI self-energy diagrams involving at least one loop). One can then construct the corresponding connected two-point Green function by summing up the geometric series,

$$\begin{aligned} G_R^{(2)}(p^2) &= \frac{i}{p^2 - m_R^2 + i\varepsilon} + \frac{i}{p^2 - m_R^2 + i\varepsilon}\big(-i\Sigma_R(p^2)\big)\frac{i}{p^2 - m_R^2 + i\varepsilon} + \cdots \\ &= \frac{i}{p^2 - m_R^2 - \Sigma_R(p^2) + i\varepsilon} = \big[i\Gamma_R^{(2)}(p^2)\big]^{-1}\,. \end{aligned} \tag{15}$$

Hence, it follows that

$$i\big[G_R^{(2)}(p^2)\big]^{-1} = p^2 - m_R^2 - \Sigma_R(p^2) = \left[\frac{1}{p^2 - m_R^2 + i\varepsilon} + \int_{4M^2}^{\infty} dm^2\,\sigma_R(m^2)\,\frac{1}{p^2 - m^2 + i\varepsilon}\right]^{-1}. \tag{16}$$

In the one loop approximation in which only the fermion loop contribution is taken into account, $\Sigma_R(p^2)$ is of $\mathcal{O}(g_R^2)$. It follows that $\sigma_R(m^2)$ is also of $\mathcal{O}(g_R^2)$. Thus, we can expand the inverse on the right hand side of eq. (16) and keep only the first two terms of the series. Denoting the integral on the right hand side of eq. (16) by J, it then follows that

$$\left[\frac{1}{p^2 - m_R^2 + i\varepsilon} + J\right]^{-1} = p^2 - m_R^2 + i\varepsilon - (p^2 - m_R^2 + i\varepsilon)^2 J + \mathcal{O}(g_R^4)\,. \tag{17}$$

Since one can always safely take $\varepsilon \to 0$ in factors of $p^2 - m_R^2 + i\varepsilon$ that appear in the numerator, eqs. (16) and (17) yield,

$$\Sigma_R(p^2) = (p^2 - m_R^2)^2 \int_{4M^2}^{\infty} dm^2\,\sigma_R(m^2)\,\frac{1}{p^2 - m^2 + i\varepsilon} + \mathcal{O}(g_R^4)\,. \tag{18}$$

To determine $\sigma_R(m^2)$, we shall employ the Sokhotski-Plemelj formula [see the class handout entitled *Generalized Functions for Physics*],

$$\frac{1}{p^2 - m^2 + i\varepsilon} = \mathrm{P}\,\frac{1}{p^2 - m^2} - i\pi\delta(p^2 - m_R^2)\,. \tag{19}$$

Inserting this result into eq. (18) and taking the imaginary part of both sides of the equation, the end result is,[2]

$$\sigma_R(p^2)\Theta(p^2 - 4M^2) = -\frac{1}{\pi(p^2 - m_R^2)^2}\,\mathrm{Im}\,\Sigma_R(p^2)\,. \tag{20}$$

One can now make use of the results of part (a) to evaluate $\sigma_R(p^2)$. However, it is instructive to repeat the computation of part (a) using the on-shell renormalization scheme. In this scheme, the square of the physical mass, m_R^2 corresponds to pole of $G_R^{(2)}(p^2)$ with residue equal to 1. Hence, it is convenient to expand $\Sigma_R(p^2)$ around $p^2 = m_R^2$,

$$\Sigma_R(p^2) = \Sigma_R(m_R^2) + (p^2 - m_R^2)\Sigma_R'(m_R^2) + (p^2 - m_R^2)^2 R(p^2)\,, \tag{21}$$

[2]Note that we have inserted the step function in eq. (20), since the integration over the delta function yields a nonzero result only if the value of p^2 lies within the integration range $4M^2 \leq m^2 < \infty$.

where $\Sigma'_R(m_R^2) \equiv (d\Sigma/dp^2)_{p^2=m_R^2}$ and the last term above is the remainder term. Given that m_R^2 is a pole of $G_R^{(2)}(p^2)$, it follows that

$$\Sigma_R(m_R^2) = 0\,, \tag{22}$$

Hence, eq. (15) takes the form,

$$G_R^{(2)}(p^2) = \frac{i}{(p^2 - m_R^2 + i\varepsilon)\big[1 - \Sigma'_R(m_R^2) - (p^2 - m_R^2)R(p^2)\big]}\,. \tag{23}$$

Moreover, since the residue at the pole of the renormalized two-point function is equal to 1, it follows that

$$\Sigma'_R(m_R^2) = 0\,. \tag{24}$$

Consequently, eqs. (21), (22) and (24) yield,

$$\Sigma_R(p^2) = (p^2 - m_R^2)^2 R(p^2)\,, \tag{25}$$

and eq. (20) simplifies to,

$$\sigma_R(p^2)\Theta(p^2 - 4M^2) = -\frac{1}{\pi}\,\mathrm{Im}\,R(p^2)\,. \tag{26}$$

Using the results of part (a), the fermion loop contribution to $i\Gamma_R^{(2)}(p^2)$ is given by [cf. eqs. (10) and (11)],

$$\begin{aligned} i\Gamma_R^{(2)}(p^2) = i\Bigg\{ & p^2\left[Z_\phi + \frac{g_R^2}{8\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) + f_1(p^2)\right)\right] \\ & - m_R^2\left[Z_m Z_\phi + \frac{3g_R^2 M^2}{4\pi^2 m_R^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) + f_2(p^2)\right)\right]\Bigg\}\,, \end{aligned} \tag{27}$$

where

$$f_1(p^2) \equiv \ln\left(\frac{\mu^2}{M^2}\right) + \frac{1}{3} - 6\int_0^1 x(1-x)\ln\left(1 - \frac{p^2 x(1-x)}{M^2}\right)dx\,, \tag{28}$$

$$f_2(p^2) \equiv \ln\left(\frac{\mu^2}{M^2}\right) + \frac{1}{3} - \int_0^1 \ln\left(1 - \frac{p^2 x(1-x)}{M^2}\right)dx\,. \tag{29}$$

Using eq. (14), it follows that

$$\begin{aligned} -i\Sigma_R^{(2)}(p^2) = i\Bigg\{ & p^2\left[\delta Z_\phi + \frac{g_R^2}{8\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) + f_1(p^2)\right)\right] \\ & - m_R^2\left[\delta Z_m + \delta Z_\phi + \frac{3g_R^2 M^2}{4\pi^2 m_R^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) + f_2(p^2)\right)\right]\Bigg\}\,, \end{aligned} \tag{30}$$

after writing,

$$\delta Z_\phi \equiv Z_\phi - 1\,, \qquad \delta Z_m \equiv Z_m - 1\,, \tag{31}$$

and dropping the term proportional to $\delta Z_\phi \delta Z_m$, which is of $\mathcal{O}(g_R^4)$.

One can now determine δZ_ϕ and δZ_m in the on-shell renormalization scheme by imposing the conditions given in eqs. (22) and (24),

$$\begin{aligned} 0 = \Sigma_R(m_R^2) &= m_R^2\left[\delta Z_m + \frac{g_R^2}{8\pi}\left(\frac{6M^2}{m_R^2}-1\right)\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)\right)+\frac{g_R^2}{8\pi}\left(\frac{6M^2}{m_R^2}f_2(m_R^2)-f_1(m_R^2)\right)\right], \\ 0 = \Sigma_R'(m_R^2) &= \frac{g^2 m_R^2}{8\pi^2}\left[\frac{6M^2}{m_R^2}f_2'(m_R^2)-f_1'(m_R^2)\right]-\delta Z_\phi-\frac{g_R^2}{8\pi^2}\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)+f_1(m_R^2)\right), \end{aligned} \tag{32}$$

where $f'(m_R^2) \equiv (df/dp^2)_{p^2=m_R^2}$. Thus, we can conclude that

$$\begin{aligned} \delta Z_m &= -\frac{g_R^2}{8\pi}\left(\frac{6M^2}{m_R^2}-1\right)\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)\right)-\frac{g_R^2}{8\pi}\left(\frac{6M^2}{m_R^2}f_2(m_R^2)-f_1(m_R^2)\right), \\ \delta Z_\phi &= \frac{g^2 m_R^2}{8\pi^2}\left[\frac{6M^2}{m_R^2}f_2'(m_R^2)-f_1'(m_R^2)\right]-\frac{g_R^2}{8\pi^2}\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)+f_1(m_R^2)\right). \end{aligned} \tag{33}$$

Plugging these results back into eq. (30) yields,

$$\begin{aligned} \Sigma_R^{(2)}(p^2) = \frac{g_R^2}{8\pi^2}\Big\{&(p^2-m_R^2)\big[m_R^2 f_1'(m_R^2)-6M^2 f_2'(m_R^2)\big]-p^2\big[f_1(p^2)-f_1(m_R^2)\big] \\ &+6M^2\big[f_2(p^2)-f_2(m_R^2)\big]\Big\}. \end{aligned} \tag{34}$$

As a check of eq. (34), note that if we expand,

$$f_1(p^2)-f_1(m_R^2) \simeq (p^2-m_R^2)f_1'(m_R^2), \qquad f_2(p^2)-f_2(m_R^2) \simeq (p^2-m_R^2)f_2'(m_R^2), \tag{35}$$

and plug these expressions back into eq. (34), we obtain

$$\Sigma_R^{(2)}(p^2) \simeq -\frac{g_R^2}{8\pi^2}(p^2-m_R^2)^2 f_1'(m_R^2). \tag{36}$$

which shows that $\Sigma_R^{(2)}(p^2)$ does have the expected form of eq. (25).

Using eqs. (28) and (29), we can write an explicit integral representation for $\Sigma_R^{(2)}(p^2)$. Note that

$$\begin{aligned} &-p^2\big[f_1(p^2)-f_1(m_R^2)\big]+6M^2\big[f_2(p^2)-f_2(m_R^2)\big] \\ &\qquad = -6\int_0^1\big[M^2-p^2x(1-x)\big]\ln\left(\frac{M^2-p^2x(1-x)}{M^2-m_R^2x(1-x)}\right)dx, \end{aligned} \tag{37}$$

and

$$\begin{aligned} m_R^2 f_1'(m_R^2)-6M^2 f_2'(m_R^2) &= 6m_R^2\int_0^1\frac{x^2(1-x)^2}{M^2-m_R^2x(1-x)}-6M^2\int_0^1\frac{x(1-x)}{M^2-m_R^2x(1-x)} \\ &= -6\int_0^1 x(1-x)dx = -1. \end{aligned} \tag{38}$$

Hence,

$$\Sigma_R^{(2)}(p^2) = -\frac{3g^2}{4\pi^2}\left[\int_0^1 \left[M^2 - p^2 x(1-x)\right] \ln\left(\frac{M^2 - p^2 x(1-x)}{M^2 - m_R^2 x(1-x)}\right) + \tfrac{1}{6}(p^2 - m_R^2)\right], \qquad (39)$$

and eq. (25) yields,

$$R(p^2) = -\frac{3g_R^2}{4\pi^2(p^2 - m_R^2)^2}\left[\int_0^1 \left[M^2 - p^2 x(1-x)\right] \ln\left(\frac{M^2 - p^2 x(1-x)}{M^2 - m_R^2 x(1-x)}\right) dx + \tfrac{1}{6}(p^2 - m_R^2)\right]. \qquad (40)$$

To complete our calculation, we must compute $\operatorname{Im} R(p^2)$. Thus, we focus our attention on

$$\operatorname{Im} \ln\bigl(M^2 - p^2 x(1-x) - i\varepsilon\bigr) = -\pi\, \Theta\bigl(p^2 x(1-x) - M^2\bigr), \qquad (41)$$

where we have restored the correct $i\varepsilon$ factor by replacing $M^2 \to M^2 - i\varepsilon$ (where ε is a positive infinitesimal). By doing so, we can identify the correct sign of the imaginary part of the logarithm of a negative number. The argument of the logarithm is negative when $p^2 > 4M^2$ and $0 < x_- < x < x_+ < 1$, where $x_\pm$ are the roots of the quadratic polynomial, $M^2 - p^2 x(1-x) = 0$. Explicitly,

$$x_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4M^2}{p^2}}\,\right]. \qquad (42)$$

Moreover, $m_R^2 < 4M^2$ by assumption of the statement of part (b) of this problem. Consequently, $M^2 - m_R^2 x(1-x) > 0$ for $0 \leq x \leq 1$. It then follows that for $p^2 > 4M^2$,

$$\begin{aligned}
\operatorname{Im} \int_0^1 \left[M^2 - p^2 x(1-x)\right] \ln\left(\frac{M^2 - p^2 x(1-x) - i\varepsilon}{M^2 - m_R^2 x(1-x)}\right) &= -\pi \int_{x_-}^{x_+} \left[M^2 - p^2 x(1-x)\right] dx \\
&\quad -\pi\Bigl\{M^2(x_+ - x_-) - p^2\left[\tfrac{1}{2}(x_+^2 - x_-^2) - \tfrac{1}{3}(x_+^3 - x_-^3)\right]\Bigr\} \\
&= -\pi\sqrt{1 - \frac{4M^2}{p^2}}\left[\tfrac{2}{3}M^2 - \tfrac{1}{6}p^2\right] \\
&= \tfrac{1}{6}\pi p^2\left(1 - \frac{4M^2}{p^2}\right)^{3/2}. \qquad (43)
\end{aligned}$$

In obtaining the above result, we noted that

$$x_+^2 - x_-^2 = (x_+ + x_-)(x_+ - x_-) = \sqrt{1 - \frac{4M^2}{p^2}}, \qquad (44)$$

$$x_+^3 - x_-^3 = (x_+ - x_-)\left[(x_+ + x_-)^2 - x_+ x_-\right] = \sqrt{1 - \frac{4M^2}{p^2}}\left(1 - \frac{M^2}{p^2}\right), \qquad (45)$$

and employed the relations,

$$x_+ + x_1 = 1, \qquad x_+ x_- = \frac{M^2}{p^2}, \qquad x_+ - x_- = \sqrt{1 - \frac{4M^2}{p^2}}. \qquad (46)$$

Hence, eq. (40) yields,

$$\operatorname{Im} R(p^2) = -\frac{g_R^2}{8\pi}\frac{p^2}{(p^2-m_R^2)^2}\left(1-\frac{4M^2}{p^2}\right)^{3/2}\Theta(p^2-4M^2)\,. \tag{47}$$

Inserting this result back into eq. (26) yields,

$$\boxed{\sigma_R(m^2) = \frac{g_R^2}{8\pi^2}\frac{m^2}{(m^2-m_R^2)^2}\left(1-\frac{4M^2}{m^2}\right)^{3/2}\,,\qquad \text{for } m_R^2 < 4M^2.} \tag{48}$$

It follows from eq. (48) that $\sigma_R(m^2) \sim 1/m^2$ as $m^2 \to \infty$. An alternative derivation of eq. (48) is given in the Appendix below.

REMARKS:

1. The significance of the $m^2 \to \infty$ behavior of $\sigma_R(m^2)$ can be understood as follows. Recall that in class we derived the relation,

$$Z_\phi = 1 - \int_0^\infty \sigma(m^2)\,dm^2\,. \tag{49}$$

In terms of the renormalized spectral function, $\sigma_R(m^2) = Z_\phi^{-1}\sigma(m^2)$, one can multiply eq. (49) by Z_ϕ^{-1} to obtain,

$$Z_\phi^{-1} = 1 + \int_0^\infty \sigma_R(m^2)\,dm^2\,. \tag{50}$$

If $\sigma_R(m^2) \sim 1/m^2$ as $m^2 \to \infty$, then the integral in eq. (50) diverges logarithmically at the upper end of the integration range, which implies that Z_ϕ^{-1} diverges. It would then follow that $Z_\phi = 0$ if the one-loop asymptotic behavior of $\sigma_R(m^2)$ were reliable. In fact, based on a fixed-order *perturbative* computation, one should really reinterpret eq. (50) as,

$$Z_\phi = \frac{1}{1+\int_0^\infty \sigma_R(m^2)\,dm^2} \simeq 1 - \int_0^\infty \sigma_R(m^2)\,dm^2\,, \tag{51}$$

which demonstrates that in the context of the one-loop analysis, Z_ϕ diverges logarithmically, which is of course is the actual behavior of Z_ϕ in the one-loop approximation. Thus, we see that even though a nonperturbative treatment yields $0 \leq Z_\phi \leq 1$, this constraint is not necessarily in contradiction to a perturbative analysis that yields a logarithmically divergent wave function renormalization constant.

2. One may be curious why the condition $m_R^2 < 4M^2$ was imposed in part (b) of this problem. In the case of $m_R^2 \geq 4M^2$, the decay of the scalar to a fermion-antifermion pair is kinematically allowed. Consequently, the scalar is unstable and the physical squared-mass parameter is shifted away from the real axis. In particular, the imaginary part of $\Sigma_R(p^2)$ is related to the decay width of the unstable scalar. Moreover, in such a case, one must reassess the derivation of the Källen-Lehmann representation of the propagator, since an unstable scalar is not an asymptotic state of the theory. To avoid this complexity, we imposed the condition $m_R^2 < 4M^2$ to ensure that the scalar particle is stable.

3. Since the ultraviolet divergences appear in the real part of the Green functions, one can solve part (b) of this problem by considering the imaginary part of the unrenormalized 1PI Green function, expressed in terms of the bare parameters of the theory. Moreover, one does not need to specify the renormalization scheme for the mass and couplings. Starting from the spectral representation of the unrenormalized two-point function,

$$G^{(2)}(p^2) = \frac{iZ_\phi}{p^2 - m_P^2 + i\varepsilon} + \int_{4M^2}^{\infty} dm^2\, \sigma(m^2)\, \frac{i}{p^2 - m^2 + i\varepsilon}\,, \tag{52}$$

where $m_P^2 = m_B^2 + \Sigma(m_B^2)$ is the physical (renormalized) squared-mass of the scalar and $Z_\phi^{-1} = 1 + \Sigma'(m_B^2)$, expressed in terms of the bare squared-mass parameter m_B^2. We use the notation m_P^2 to denote the square of the so-called pole mass, which differs from the renormalized squared mass parameter m_R^2 defined in a renormalization scheme other than the on-shell scheme. A calculation analogous to the one presented in eqs. (16)–(20) yields,

$$\sigma(p^2)\Theta(p^2 - 4M^2) = -\frac{1}{\pi(p^2 - m_P^2)^2}\,\mathrm{Im}\,\Sigma(p^2)\,, \tag{53}$$

where $-i\Sigma(p^2)$ is the unrenormalized one-loop contribution to the self-energy of the scalar. Since $\mathrm{Im}\,\Sigma(p^2)$ is already of $\mathcal{O}(g^2)$, one can consistently replace bare parameters with renormalized parameters and Z_ϕ with 1 in the one-loop approximation, since the difference between employing bare and renormalized parameters in eq. (53) is formally of higher order in perturbation theory.

The contribution of the fermion loop to $-i\Sigma(p^2)$ was given in eq. (10). Thus, in light of eqs. (52) and (53),

$$\begin{aligned}\sigma(p^2)\Theta(p^2 - 4M^2) &= -\frac{3g_R^2}{4\pi^2(p^2 - m_P^2)^2}\,\mathrm{Im}\int_0^1 \big[M^2 - p^2x(1-x)\big]\ln\left(\frac{M^2 - p^2x(1-x) - i\varepsilon}{M^2}\right)dx \\ &= \frac{3g_R^2}{4\pi(p^2 - m_P^2)^2}\,\mathrm{Im}\int_{x_-}^{x_+}\big[M^2 - p^2x(1-x)\big] \\ &= \frac{g_R^2}{8\pi^2}\frac{p^2}{(p^2 - m_P^2)^2}\left(1 - \frac{4M^2}{m^2}\right)^{3/2}\,,\end{aligned} \tag{54}$$

after making use of eqs. (42) and (43). Since $\sigma_R(p^2) \equiv Z_\phi^{-1}\sigma(p^2) = \sigma(p^2) + \mathcal{O}(g_R^4)$, it follows that in the one-loop approximation, one can also replace $\sigma(p^2)$ with $\sigma_R(p^2)$. Hence, we have recovered the result of eq. (48) without specifying the renormalization scheme for the mass and coupling and without determining the one-loop expression for Z_ϕ.

APPENDIX: An alternative derivation of $\boldsymbol{\sigma_R(m^2)}$

In this Appendix, we shall find a way to massage the expression for $R(p^2)$ given in eq. (40) so that it exhibits the form given by [cf. eqs. (18) and (25)],

$$R(p^2) = \int_{4M^2}^{\infty} dm^2\, \sigma_R(m^2)\, \frac{i}{p^2 - m^2 + i\varepsilon}\,. \tag{55}$$

The method presented below is inspired by Chapter 8 of K. Nishijima, *Fields and Particles: Field Theory and Dispersion Relations* (W.A. Benjamin, Inc., Reading, MA, 1974).

We begin with eq. (40) under the assumption that $m_R^2 \leq 4M^2$, which we reproduce below,

$$R(p^2) = -\frac{3g_R^2}{4\pi^2(p^2-m_R^2)^2}\left[\int_0^1 \bigl[M^2-p^2x(1-x)\bigr]\ln\left(\frac{M^2-p^2x(1-x)-i\varepsilon}{M^2-m_R^2x(1-x)}\right)dx+\tfrac{1}{6}(p^2-m_R^2)\right], \tag{56}$$

where the $-i\varepsilon$ factor (via $M^2 \to M^2 - i\varepsilon$) has been restored in the numerator of the argument of the logarithm, as it is needed to obtain the correct imaginary part of the logarithm for values of p^2 and x where $M^2 - p^2x(1-x)$ is negative.[3] Next, we note the following identify,

$$\ln\left(\frac{M^2-p^2x(1-x)-i\varepsilon}{M^2-m_R^2x(1-x)}\right) = \int_{M^2}^\infty \left(\frac{1}{\kappa^2-m_R^2x(1-x)}-\frac{1}{\kappa^2-p^2x(1-x)-i\varepsilon}\right)d\kappa^2\,. \tag{57}$$

We now consider the following identities,

$$\frac{1}{a+b}=\frac{1}{b}-\frac{a}{b(a+b)}=\frac{1}{b}-\frac{a}{b^2}+\frac{a^2}{b^2(a+b)} \tag{58}$$

It is convenient to make the following choices for a and b,

$$a \equiv -(p^2-m_R^2)x(1-x)-i\varepsilon\,, \qquad b \equiv \kappa^2 - m_R^2x(1-x)\,, \tag{59}$$

which implies that $a + b = \kappa^2 - p^2x(1-x) - i\varepsilon$. Then, as a consequence of eq. (58),

$$\begin{aligned}\frac{1}{\kappa^2-p^2x(1-x)-i\varepsilon}-\frac{1}{\kappa^2-m_R^2x(1-x)} &= \frac{(p^2-m_R^2)x(1-x)}{\bigl[\kappa^2-m_R^2x(1-x)\bigr]\bigl[\kappa^2-p^2x(1-x)-i\varepsilon\bigr]}\\ &= \frac{(p^2-m_R^2)x(1-x)}{\bigl[\kappa^2-m_R^2x(1-x)\bigr]^2}+\frac{(p^2-m_R^2)^2x^2(1-x)^2}{\bigl[\kappa^2-m_R^2x(1-x)\bigr]^2\bigl[\kappa^2-p^2x(1-x)-i\varepsilon\bigr]}\,,\end{aligned} \tag{60}$$

where it is safe to take $\varepsilon \to 0$ in any numerator factor. In light of eqs. (57) and (60), and noting that the left hand side of eq. (60) is the negative of the integrand of eq. (57), it follows that

$$\begin{aligned}&\bigl[M^2-p^2x(1-x)\bigr]\ln\left(\frac{M^2-p^2x(1-x)-i\varepsilon}{M^2-m_R^2x(1-x)}\right)\\ &\quad= \bigl[M^2-m_R^2x(1-x)\bigr]\int_{M^2}^\infty\left(\frac{1}{\kappa^2-m_R^2x(1-x)}-\frac{1}{\kappa^2-p^2x(1-x)-i\varepsilon}\right)d\kappa^2\\ &\qquad\qquad -(p^2-m_R^2)x(1-x)\int_{M^2}^\infty\left(\frac{1}{\kappa^2-m_R^2x(1-x)}-\frac{1}{\kappa^2-p^2x(1-x)-i\varepsilon}\right)d\kappa^2\\ &\quad= -\bigl[M^2-m_R^2x(1-x)\bigr](p^2-m_R^2)x(1-x)\int_{M^2}^\infty\frac{d\kappa^2}{\bigl[\kappa^2-m_R^2x(1-x)\bigr]^2}\\ &\qquad -\bigl[M^2-m_R^2x(1-x)\bigr](p^2-m_R^2)^2x^2(1-x)^2\int_{M^2}^\infty\frac{d\kappa^2}{\bigl[\kappa^2-m_R^2x(1-x)\bigr]^2\bigl[\kappa^2-p^2x(1-x)-i\varepsilon\bigr]}\\ &\qquad +(p^2-m_R^2)^2x^2(1-x)^2\int_{M^2}^\infty\frac{d\kappa^2}{\bigl[\kappa^2-m_R^2x(1-x)\bigr]\bigl[\kappa^2-p^2x(1-x)-i\varepsilon\bigr]}\,.\end{aligned} \tag{61}$$

[3] In contrast, $M^2 - m_R^2x(1-x) > 0$ for $0 < x < 1$ since $m_R^2 < 4M^2$. Hence, it is safe to take $\varepsilon \to 0$ in the denominator of the argument of the logarithm (as well as in the prefactor that multiplies the logarithm).

Note that the last line of eq. (61) was obtained by making use of the first line of eq. (60), whereas the previous two lines of eq. (61) made use of the second line of eq. (60).

One integral can be carried out immediately,

$$\int_{M^2}^{\infty} \frac{d\kappa^2}{\big[\kappa^2 - m_R^2 x(1-x)\big]^2} = \frac{1}{M^2 - m_R^2 x(1-x)}\,. \tag{62}$$

Hence,

$$-\int_0^1 dx\big[M^2 - m_R^2 x(1-x)\big](p^2 - m_R^2)x(1-x)\int_{M^2}^{\infty} \frac{d\kappa^2}{\big[\kappa^2 - m_R^2 x(1-x)\big]^2} = -\tfrac{1}{6}(p^2 - m_R^2)\,, \tag{63}$$

which exactly cancels the factor of $\frac{1}{6}(p^2 - m_R^2)$ in eq. (56). Hence, we are left with

$$\begin{aligned} R(p^2) &= \frac{3g_R^2}{4\pi^2}\int_0^1 x^2(1-x)^2 dx \int_{M^2}^{\infty} \frac{d\kappa^2}{\big[\kappa^2 - m_R^2 x(1-x)\big]\big[\kappa^2 - p^2 x(1-x) - i\varepsilon\big]}\left[\frac{M^2 - m_R^2 x(1-x)}{\kappa^2 - m_R^2 x(1-x)} - 1\right] \\ &= \frac{3g_R^2}{4\pi^2}\int_0^1 x^2(1-x)^2 dx \int_{M^2}^{\infty} \frac{(M^2-\kappa^2)\,d\kappa^2}{\big[\kappa^2 - m_R^2 x(1-x)\big]^2\big[\kappa^2 - p^2 x(1-x) - i\varepsilon\big]} \\ &= -\frac{3g_R^2}{4\pi^2}\int_0^1 x(1-x) dx \int_{M^2}^{\infty} \frac{(M^2-\kappa^2)\,d\kappa^2}{\big[\kappa^2 - m_R^2 x(1-x)\big]^2\big[p^2 - \frac{\kappa^2}{x(1-x)} + i\varepsilon\big]} \end{aligned} \tag{64}$$

One more trick suffices to write $R(p^2)$ in the form given by eq. (55). An equivalent form of eq. (64) is

$$R(p^2) = -\frac{3g_R^2}{4\pi^2}\int_0^1 \frac{dx}{x(1-x)}\int_{M^2}^{\infty} d\kappa^2 \int_{4M^2}^{\infty} \frac{M^2-\kappa^2}{(m^2-m_R^2)^2(p^2-m^2+i\varepsilon)}\delta\left(m^2 - \frac{\kappa^2}{x(1-x)}\right) dm^2\,, \tag{65}$$

where the limits of the integration over m^2 ensure that $\kappa^2 = m^2 x(1-x)$ for some choice of $M^2 \leq \kappa < \infty$ and $0 \leq x \leq 1$. Using

$$\delta\left(m^2 - \frac{\kappa^2}{x(1-x)}\right) = x(1-x)\delta\big(\kappa^2 - m^2 x(1-x)\big)\,, \tag{66}$$

and interchanging the order of integration,

$$R(p^2) = -\frac{3g_R^2}{4\pi^2}\int_{4M^2}^{\infty} \frac{dm^2}{(m^2-m_R^2)^2(p^2-m^2+i\varepsilon)}\int_{M^2}^{\infty}(M^2-\kappa^2)\,d\kappa^2\int_0^1 \delta\big(\kappa^2 - m^2 x(1-x)\big)dx\,. \tag{67}$$

Thus, comparing with eq. (55), it follows that

$$\sigma_R(m^2) = -\frac{3g_R^2}{4\pi^2(m^2-m_R^2)^2}\int_{M^2}^{\infty}(M^2-\kappa^2)\,d\kappa^2\int_0^1 \delta\big(\kappa^2 - m^2 x(1-x)\big)dx\,. \tag{68}$$

The argument of the delta function vanishes when $\kappa^2 - m^2 x(1-x) = 0$. The roots of this equation are

$$x_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4\kappa^2}{m^2}}\,\right]\,. \tag{69}$$

We demand that $0 \leq x_\pm \leq 1$; otherwise the argument of the delta function does not vanish over the range of integration over x. This condition is satisfied if $0 \leq \kappa^2 \leq \frac{1}{4}m^2$. Hence, we can write,

$$\begin{aligned}
\delta\big(\kappa^2 - m^2 x(1-x)\big) &= \frac{1}{m^2(x_+ - x_-)}\big[\delta(x-x_+) + \delta(x-x_-)\big] \\
&= \frac{1}{m^2}\left(1 - \frac{4\kappa^2}{m^2}\right)^{-1/2}\big[\delta(x-x_+) + \delta(x-x_-)\big]\,. \qquad (70)
\end{aligned}$$

After integrating eq. (68) over x,

$$\sigma_R(m^2) = -\frac{3g_R^2}{2\pi^2 m^2(m^2-m_R^2)^2}\int_{M^2}^{m^2/4}\left(1-\frac{4\kappa^2}{m^2}\right)^{-1/2}(M^2-\kappa^2)\,d\kappa^2\,. \qquad (71)$$

The remaining integral is elementary. Defining $u = 1 - 4\kappa^2/m^2$, it follows that

$$\begin{aligned}
\sigma_R(m^2) &= -\frac{3g_R^2}{32\pi^2(m^2-m_R^2)^2}\int_0^{1-4M^2/m^2}\frac{du}{\sqrt{u}}\big[4M^2 - m^2 + m^2 u\big] \\
&= \frac{g_R^2}{8\pi^2}\frac{m^2}{(m^2-m_R^2)^2}\left(1-\frac{4M^2}{m^2}\right)^{3/2}, \qquad (72)
\end{aligned}$$

in agreement with eq. (48).

2. Consider the function of a *real* parameter z

$$F(z) \equiv \int_0^1 dx\,\ln\big[1 - zx(1-x) - i\epsilon\big]\,, \qquad (73)$$

where ϵ is a positive infinitesimal quantity. The function $F(z)$ appears in the computation of the one-loop correction to the 4-point Green function in scalar field theory.

(a) Evaluate $\operatorname{Im} F(z)$. For what values of z does $\operatorname{Im} F$ vanish?

We shall denote the argument of the logarithm in eq. (73) by the function,

$$f(x) \equiv zx^2 - zx + 1 \geq 0\,.$$

First, we note that $f(0) = f(1) = 1$. Next, we compute the first and second derivatives,

$$f'(x) = z(2x-1)\,, \qquad f''(x) = 2z\,,$$

Thus, $f(x)$ has an extremum at $x = \frac{1}{2}$. Since $f''(\frac{1}{2}) = 2z$, it follows that $x = \frac{1}{2}$ is a maximum if $z < 0$ and $x = \frac{1}{2}$ is a minimum if $z > 0$. At $z = 0$, we have $f(x) = 1$ for all x. Moreover, for $z > 0$, the minimum value of $f(x)$ is equal to $f(\frac{1}{2}) = 1 - \frac{1}{4}z$. That is, for values of $0 \leq z \leq 4$, the minimum value of $f(x)$ is nonnegative for all $0 \leq x \leq 1$. Moreover, for values of $z \leq 0$, we have $f(x) \geq 1$ in the region where $0 \leq x \leq 1$.

Observe that $\operatorname{Im} F(z)=0$ if $f(x)>0$ for $0\leq x\leq 1$, which implies that $\operatorname{Im} F(z)=0$ if $z<4$. When $z>4$, the minimum value of $f(x)$ at $x=\frac{1}{2}$ is negative. Since $f(0)=f(1)=1$, it follows that $f(x)<0$ for values of $x_-<x<x_+$, where $x_\pm$ are the roots of $f(x)$,

$$x_\pm=\tfrac{1}{2}\left[1\pm\sqrt{1-\frac{4}{z}}\,\right]. \tag{74}$$

Thus,

$$\operatorname{Im} F(z)=\Theta(z-4)\int_{x_-}^{x_+}dx\,\operatorname{Im}\ln\bigl[1-zx(1-x)-i\epsilon\bigr]\,, \tag{75}$$

where we have explicitly included the step function to enforce the condition that $\operatorname{Im} F(z)=0$ if $z<4$. To evaluate the imaginary part of the logarithm, we employ the principal value of the complex-valued logarithm, with the branch cut taken along the negative real axis. In particular, assuming that x is a non-zero real number and ϵ is a *positive* infinitesimal,

$$\ln(x-i\epsilon)=\ln|x|-i\pi\Theta(-x)\,. \tag{76}$$

It follows that $\operatorname{Im}\ln(x-i\epsilon)=-\pi\Theta(-x)$. Employing this result in eq. (75),

$$\operatorname{Im} F(z)=-\Theta(z-4)\pi\int_{x_-}^{x_+}dx=-\Theta(z-4)\pi\bigl(x_+-x_-\bigr)=-\Theta(z-4)\pi\sqrt{1-\frac{4}{z}}\,, \tag{77}$$

after using the explicit form for $x_\pm$ given in eq. (74). Note that $\operatorname{Im} F(z=4)=0$ at the boundary between the regions where $\operatorname{Im} F(z)$ is nonzero and where it vanishes.

(b) Evaluate $\operatorname{Re} F(z)$. Consider separately the cases of $0\leq z<4$ and $z>4$.

Assume first that $0\leq z<4$. In this case, the argument of the logarithm in eq. (73) is positive, in which case we can drop the $-i\epsilon$ term and write

$$F(z)\equiv\int_0^1 dx\,\ln\bigl[1-zx(1-x)\bigr]\,,\qquad\text{for } 0\leq z<4. \tag{78}$$

Let us set $x=\frac{1}{2}(1-y)$. Then, $1-x=\frac{1}{2}(1+y)$ and $x(1-x)=\frac{1}{4}(1-y^2)$. Thus,

$$F(z)=\tfrac{1}{2}\int_{-1}^{1}\ln\bigl[1-\tfrac{1}{4}z(1-y^2)\bigr]\,dy=\int_0^1\ln\bigl[1-\tfrac{1}{4}z(1-y^2)\bigr]\,dy\,,\qquad\text{for } 0\leq z<4, \tag{79}$$

after noting that the integrand is an even function of y. Integrating by parts, we take $u=\ln\bigl[1-\frac{1}{4}z(1-y^2)\bigr]$ and $dv=dy$, which yields,

$$\begin{aligned}F(z)&=y\ln\bigl[1-\tfrac{1}{4}z(1-y^2)\bigr]\Big|_0^1-\int_0^1\frac{\frac{1}{2}zy^2\,dy}{1-\frac{1}{4}z+\frac{1}{4}zy^2}\\&=-2\int_0^1\frac{y^2\,dy}{\frac{4}{z}-1+y^2}=-2\left[1-\left(\frac{4}{z}-1\right)\int_0^1\frac{dy}{\frac{4}{z}-1+y^2}\right].\end{aligned} \tag{80}$$

The remaining integral is elementary, and we end up with,

$$F(z)=-2+2\left(\frac{4}{z}-1\right)^{1/2}\arctan\left(\frac{1}{\sqrt{\frac{4}{z}-1}}\right),\qquad\text{for } 0\leq z\leq 4. \tag{81}$$

One can rewrite eq. (81) in a slightly different form. In light of the relation,

$$\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right), \qquad \text{for } x^2 < 1, \tag{82}$$

it follows that

$$\boxed{F(z) = 2\left\{\left(\frac{4}{z}-1\right)^{1/2}\arcsin\left(\tfrac{1}{2}\sqrt{z}\right)-1\right\}, \qquad \text{for } 0 \le z \le 4.} \tag{83}$$

As indicated in eqs. (81) and (83), the above results are also applicable at $z = 4$, since eq. (79) yields

$$F(z=4) = \int_0^1 \ln(y^2)\,dy = -2\,, \tag{84}$$

in agreement with eqs. (81) and (83) in the limit of $z \to 4$.

Second, we assume that $z > 4$. In this case, $\operatorname{Im} F(z) \neq 0$ and is given explicitly in eq. (77). In light of eq. (76) it follows that

$$\operatorname{Re} F(z) = \int_0^1 dx\, \ln\bigl|1 - zx(1-x)\bigr|\,. \tag{85}$$

We can again employ eq. (79), which yields

$$\operatorname{Re} F(z) = \int_0^1 \ln\bigl|1 - \tfrac{1}{4}z(1-y^2)\bigr|\,dy\,, \qquad \text{for } z > 4\,. \tag{86}$$

Integrating by parts, we take $u = \ln\bigl|1 - \frac{1}{4}z(1-y^2)\bigr|$ and $dv = dy$. In the computation of du, we shall use the relation given in eq. (2) of the class handout entitled *Generalized Functions for Physics*,

$$\frac{d}{dx}\ln|x| = \mathrm{P}\,\frac{1}{x}\,, \tag{87}$$

where P indicates the principal value prescription. Hence, the integration by parts yields

$$\begin{aligned}\operatorname{Re} F(z) &= y\ln\bigl|1 - \tfrac{1}{4}z(1-y^2)\bigr|\Bigr|_0^1 - \mathrm{P}\int_0^1 \frac{\frac{1}{2}zy^2\,dy}{1-\frac{1}{4}z+\frac{1}{4}zy^2} = -2\,\mathrm{P}\int_0^1 \frac{y^2\,dy}{\frac{4}{z}-1+y^2} \\ &= -2\left[1 - \left(\frac{4}{z}-1\right)\mathrm{P}\int_0^1 \frac{dy}{\frac{4}{z}-1+y^2}\right],\end{aligned} \tag{88}$$

Using the definition of the principal value prescription, it follows that for $0 < a < 1$,

$$\begin{aligned}\mathrm{P}\int_0^1 \frac{dy}{y^2-a^2} &= -\frac{1}{2a}\,\mathrm{P}\int_0^1\left(\frac{1}{y+a}-\frac{1}{y-a}\right)dy = -\frac{1}{2a}\left[\ln\left(\frac{1+a}{a}\right) - \mathrm{P}\int_0^1 \frac{dy}{y-a}\right] \\ &= -\frac{1}{2a}\left[\ln\left(\frac{1+a}{a}\right) - \lim_{\delta\to 0^+}\left\{\int_0^{a-\delta}\frac{dy}{y-a} + \int_{a+\delta}^1 \frac{dy}{y-a}\right\}\right] \\ &= -\frac{1}{2a}\left[\ln\left(\frac{1+a}{a}\right) - \lim_{\delta\to 0^+}\left\{\ln\left(\frac{\delta}{a}\right) + \ln\left(\frac{1-a}{\delta}\right)\right\}\right] \\ &= -\frac{1}{2a}\ln\left(\frac{1+a}{1-a}\right).\end{aligned} \tag{89}$$

Hence, after setting $a = (1-4/z)^{1/2}$, eqs. (77), (88) and (89) yield,

$$F(z) = -2 + \sqrt{1-\frac{4}{z}}\left[\ln\left(\frac{1+\sqrt{1-\frac{4}{z}}}{1-\sqrt{1-\frac{4}{z}}}\right) - i\pi\right], \qquad \text{for } z \geq 4. \tag{90}$$

In light of eq. (84), we see that eq. (90) is also valid at the boundary where $z = 4$. One can also rewrite eq. (90) in a slightly different form. Employing the relation,

$$\operatorname{arccosh} x = \ln\bigl(x+\sqrt{x^2-1}\,\bigr) = \tfrac{1}{2}\ln\left(\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\right) = \tfrac{1}{2}\ln\left(\frac{1+\sqrt{1-\frac{1}{x^2}}}{1-\sqrt{1-\frac{1}{x^2}}}\right), \qquad \text{for } x \geq 1, \tag{91}$$

it follows that

$$\boxed{F(z) = 2\left\{\left(1-\frac{4}{z}\right)^{1/2}\bigl[\operatorname{arccosh}\bigl(\tfrac{1}{2}\sqrt{z}\bigr) - \tfrac{1}{2}i\pi\bigr] - 1\right\}, \qquad \text{for } z \geq 4.} \tag{92}$$

As a final check, we can compute the asymptotic form of $F(z)$ in the limit of $z \to \infty$. Starting from eq. (73), the leading behavior as $z \to \infty$ can be obtained simply by ignoring the 1 inside the argument of the logarithm. Hence,

$$F(z) \sim \ln(-z - i\epsilon) + \int_0^1 \ln\bigl[x(1-x)\bigr]\,dx = \ln z - i\pi - 2\,, \qquad \text{as } z \to \infty. \tag{93}$$

Indeed, in the limit of $z \to \infty$, it is straightforward to use eq. (90) to obtain,

$$F(z) = -2 + \ln z - i\pi + \mathcal{O}\left(\frac{1}{z}\right), \qquad \text{as } z \to \infty, \tag{94}$$

in agreement with eq. (93).

REMARKS:

One can easily verify that eqs. (83) and (92) are analytic continuations of each other by keeping track of the $i\epsilon$ factors. In particular, note that $F(z) \equiv \lim_{\epsilon\to 0^+} F(z+i\epsilon)$. Thus, for real values of $z > 4$,

$$\lim_{\epsilon\to 0^+}\sqrt{\frac{4}{z+i\epsilon}-1} = \lim_{\epsilon\to 0^+}\sqrt{\frac{4}{z}-1-i\epsilon} = -i\sqrt{1-\frac{4}{z}}\,, \tag{95}$$

since we are evaluating the square root of a number that lies just *below* the branch cut of the complex square root function that runs along the negative real axis. Hence, if we analytically continue the expression given by eq. (83) into the parameter regime where $z > 4$,[4] we recover the result previously obtained in eq. (92),

[4]The principal value of the complex arccosine function, $\arccos(x+iy)$, is defined in the cut complex plane, where the cuts comprise the real intervals $(-\infty,-1]\cup[1,\infty)$. For example, for values of $x+i\epsilon$ where $x \geq 1$ and ϵ is a positive infinitesimal, $\lim_{\epsilon\to 0^+}\arccos(x+i\epsilon) = -i\operatorname{arccosh} x$, which has been employed in obtaining the final result of eq. (96) [cf. eqs. (4.23.24) and (4.37.19) of F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark, editors, *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, UK, 2010)].

$$\lim_{\epsilon\to 0^+} F(z+i\epsilon) = \lim_{\epsilon\to 0^+} -2 + 2\left(\frac{4}{z+i\epsilon}-1\right)^{1/2} \arcsin\bigl(\tfrac{1}{2}\sqrt{z}+i\epsilon\bigr)$$
$$= -2 - 2i\left(1-\frac{4}{z}\right)^{1/2} \lim_{\epsilon\to 0^+}\left[\tfrac{1}{2}\pi - \arccos\bigl(\tfrac{1}{2}\sqrt{z}+i\epsilon\bigr)\right]$$
$$= -2 + 2\left(1-\frac{4}{z}\right)^{1/2}\left[-\tfrac{1}{2}i\pi + \operatorname{arccosh}\bigl(\tfrac{1}{2}\sqrt{z}\bigr)\right]\,. \tag{96}$$

For completeness, we provide an evaluation $F(z)$ in the region of $z < 0$. Note that eqs. (78) and (80) are valid if $z < 0$. Hence, if we denote $a^2 \equiv 1 - 4/z$ with $a > 1$ then

$$F(z) = -2 - 2a^2\int_0^1 \frac{dy}{y^2-a^2} = -2 + a\int_0^1\left(\frac{1}{y+a}-\frac{1}{y-a}\right)dy = -2 + a\ln\left(\frac{a+1}{a-1}\right)\,. \tag{97}$$

Hence, we end up with

$$F(z) = -2 + \sqrt{1-\frac{4}{z}}\,\ln\left(\frac{\sqrt{1-\frac{4}{z}}+1}{\sqrt{1-\frac{4}{z}}-1}\right)\,, \qquad \text{for } z<0. \tag{98}$$

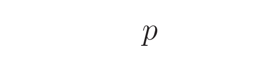
One can check that $\lim_{z\to 0} F(z) = 0$ which implies that $F(z)$ is continuous at $z = 0$. Moreover,

$$F(z) = -2 + \ln(-z) + \mathcal{O}\left(\frac{1}{z}\right)\,, \qquad \text{as } z \to -\infty. \tag{99}$$

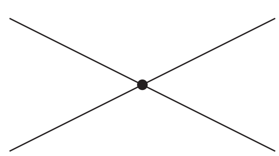
Eq. (99) also applies in the case of $z \to +\infty$ after restoring the $i\varepsilon$ factor [cf. eq. (93)] via $z \to z + i\varepsilon$ in the case of large positive z. This observation is not surprising given that eqs. (90) and (98) are analytic continuations of each other.

(c) Consider the unrenormalized 1PI 4-point Green function, $\Gamma^{(4)}(p_1, \ldots p_4)$, where all four-momenta p_i are on mass shell, in a field theory of a real scalar field with mass m and an interaction Lagrangian density, $\mathscr{L}_I = -\lambda\phi^4/4!$. Using the Feynman rules for this theory, write down an integral expression for the full $\mathcal{O}(\lambda^2)$ contribution to $\Gamma^{(4)}$. From the integral expression, evaluate $\operatorname{Im}\Gamma^{(4)}$, up to order λ^2 by making use of the Cutkosky cutting rules.[5]

The Feynman rules for the scalar propagator and the 4-point scalar interaction are



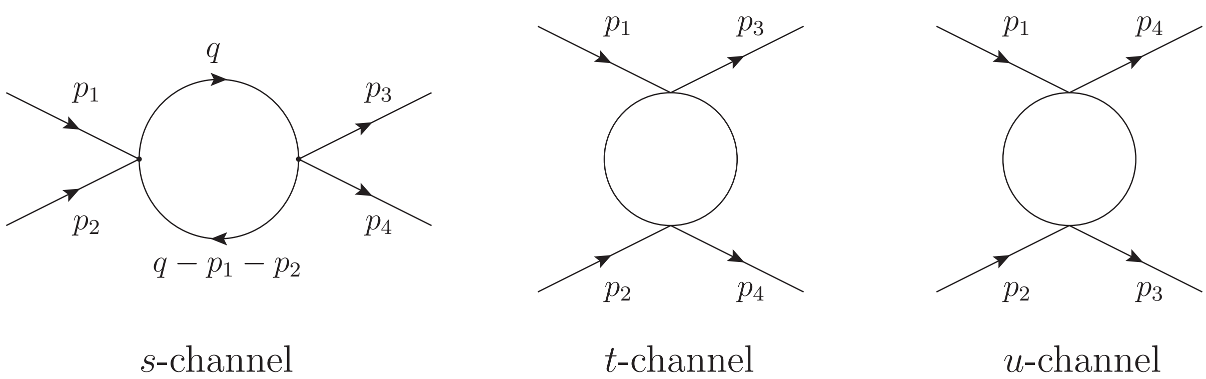
$$\frac{i}{p^2 - m^2 + i\epsilon}$$



$$-i\lambda$$

[5]See, e.g. Section 24.1.2 [pp. 456–459] of Matthew Schwartz, *Quantum Field Theory and the Standard Model* (Cambridge University Press, 2014).

The Feynman rules are used to compute $i\Gamma^{(4)}$, where $\Gamma^{(4)}$ is the 1PI 4-point Green function. At tree level, $i\Gamma^{(4)} = -i\lambda$. The one-loop contributions to $i\Gamma^{(4)}$ are obtained by using the Feynman rules to evaluate the one-loop diagrams that are exhibited below.



Thus, employing the Feynman rules (and recalling the symmetry factor of $\frac{1}{2}$ for each of the diagrams above), it follows that including all terms up to $\mathcal{O}(\lambda^2)$,

$$i\Gamma^{(4)} = -i\lambda+\tfrac{1}{2}(-i\lambda)^2\int\frac{d^4q}{(2\pi)^4}\left\{\frac{i}{q^2-m^2+i\epsilon}\,\frac{i}{(q-p_1-p_2)^2-m^2+i\epsilon}+(p_2\to p_3)+(p_2\to p_4)\right\},$$

where the second and third terms above in the integrand are given by the first term with the momentum substitutions indicated. That is, the three terms exhibited in $i\Gamma^{(4)}$ correspond to the s-channel, t-channel and u-channel diagrams, respectively. Thus,

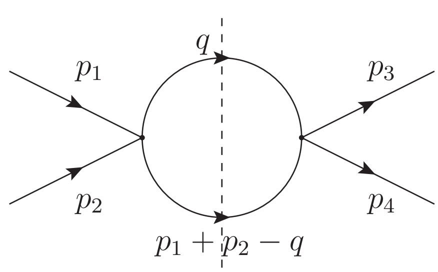
$$\Gamma^{(4)} = -\lambda-\tfrac{1}{2}i\lambda^2\int\frac{d^4q}{(2\pi)^4}\left\{\frac{1}{q^2-m^2+i\epsilon}\,\frac{1}{(q-p_1-p_2)^2-m^2+i\epsilon}+(p_2\to p_3)+(p_2\to p_4)\right\}.$$

We shall focus first on the s-channel diagram. We expect that the singularity structure in the complex s plane to have a branch point at the threshold for the $2\to 2$ scattering process at threshold, $s=4m^2$, and a branch cut extending to ∞ along the positive real axis.[6]

By definition, the discontinuity of $\Gamma^{(4)}(s)$ across the branch cut is

$$\operatorname{Disc}\Gamma^{(4)}(s)\equiv\Gamma^{(4)}(s+i\epsilon)-\Gamma^{(4)}(s-i\epsilon)\,,$$

where ϵ is a positive infinitesimal. The cutting rules state that $\operatorname{Disc}\Gamma^{(4)}(s)$ is obtained by cutting the Feynman diagram



[6]Note that $s=(p_1+p_2)^2=2(m^2+p_1\cdot p_2)=2(m^2+E_1E_2-\vec{\boldsymbol{p}}_1\cdot\vec{\boldsymbol{p}}_2)$. At threshold, $\vec{\boldsymbol{p}}_1=\vec{\boldsymbol{p}}_2=0$ and $E_1=E_2=m$, which implies that $s=4m^2$ at threshold.

and replacing the "cut" propagators by:

$$\frac{1}{q^2 - m^2 + i\epsilon} \longrightarrow -2\pi i \delta(q^2 - m^2)\Theta(q_0)\,.$$

The discontinuity $\text{Disc}\,\Gamma^{(4)}(s)$ is related to $\text{Im}\,\Gamma^{(4)}(s)$ as follows. First, we observe that the reflection principle of complex analysis implies that[7]

$$\Gamma^{(4)}(s - i\epsilon) = \Gamma^{(4)}(s + i\epsilon)^*\,. \tag{100}$$

It then follows that

$$\text{Disc}\,\Gamma^{(4)}(s) \equiv \Gamma^{(4)}(s + i\epsilon) - \Gamma^{(4)}(s + i\epsilon)^* = 2i\,\text{Im}\,\Gamma^{(4)}(s)\,,$$

where $\Gamma^{(4)}(s) \equiv \lim_{\epsilon\to 0}\Gamma^{(4)}(s + i\epsilon)$. Applying the cutting rules to the s-channel one-loop diagram (shown above),

$$2i\,\text{Im}\,\Gamma^{(4)}(s) = -\tfrac{1}{2}i\lambda^2(-2\pi i)^2 \int \frac{d^4q}{(2\pi)^4}\,\delta(q^2 - m^2)\Theta(q_0)\delta\big((q - p_1 - p_2)^2 - m^2\big)\Theta(p_{10} + p_{20} - q_0)\,. \tag{101}$$

It should be noted that the form of the Θ-function corresponds to placing a cut propagator line *on mass shell*. To evaluate the integral in eq. (101), note that

$$\begin{aligned}
\int d^4q\,\delta(q^2 - m^2)\Theta(q_0) &= \int d^3q\,dq_0\,\delta(q_0^2 - |\vec{\boldsymbol{q}}|^2 - m^2)\Theta(q_0) \\
&= \int d^3q\,dq_0\,\frac{1}{2\sqrt{|\vec{\boldsymbol{q}}|^2 + m^2}}\left[\delta\big(q_0 - \sqrt{|\vec{\boldsymbol{q}}|^2 + m^2} + \delta\big(q_0 + \sqrt{|\vec{\boldsymbol{q}}|^2 + m^2}\right]\Theta(q_0) \\
&= \int \frac{d^3q}{2\sqrt{|\vec{\boldsymbol{q}}|^2 + m^2}}\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int \frac{d^4q}{(2\pi)^4}\,\delta(q^2 - m^2)\Theta(q_0)\delta\big((q - p_1 - p_2)^2 - m^2\big)\Theta(p_{10} + p_{20} - q_0) \\
&\qquad = \frac{1}{(2\pi)^4}\int \frac{d^3q}{2\sqrt{|\vec{\boldsymbol{q}}|^2 + m^2}}\delta\big((q - p_1 - p_2)^2 - m^2\big)\Theta(p_{10} + p_{20} - q_0)\bigg|_{q_0 = \sqrt{|\vec{\boldsymbol{q}}|^2 + m^2}}\,,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\int \frac{d^4q}{(2\pi)^4}\,\delta(q^2 - m^2)\Theta(q_0)\delta\big((q - p_1 - p_2)^2 - m^2\big)\Theta(p_{10} + p_{20} - q_0) \\
&\qquad = \frac{1}{(2\pi)^4}\int \frac{d^3q}{2\sqrt{|\vec{\boldsymbol{q}}|^2 + m^2}}\delta\big(s - 2q\cdot(p_1 + p_2)\big)\Theta(p_{10} + p_{20} - q_0)\bigg|_{q_0 = \sqrt{|\vec{\boldsymbol{q}}|^2 + m^2}}\,,
\end{aligned} \tag{102}$$

after using $s \equiv (p_1 + p_2)^2$ and noting that $q^2 - m^2 = 0$ is equivalent to $q_0 = \sqrt{|\vec{\boldsymbol{q}}|^2 + m^2}$.

[7]See the Appendix at the end of this solution. In addition, a very nice discussion can be found in Paul Roman, *Introduction to Quantum Field Theory* (John Wiley & Sons, Inc., New York, NY, 1969) pp. 440–441.

The simplest way to evaluate the integral above is to work in the center-of-mass frame of the system, where $p_1+p_2=\bigl(\sqrt{s}\,;\,\vec{\boldsymbol{0}}\bigr)$, where $\sqrt{s}$ is defined to be the *positive* square root of s. In this case, $2q\cdot(p_1+p_2)=2q_0\sqrt{s}=2\sqrt{s}\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}$, and eq. (102) reduces to

$$\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m^2)\Theta(q_0)\delta\bigl((q-p_1-p_2)^2-m^2\bigr)\Theta(p_{10}+p_{20}-q_0)$$

$$=\frac{1}{(2\pi)^4}\int\frac{d^3q}{2\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}}\delta\bigl(s-2\sqrt{s}\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\bigr)\Theta\bigl(\sqrt{s}-\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\bigr)\,. \qquad (103)$$

The delta function enforces $\sqrt{s}=2\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}$, which means that the argument of the step function is positive so that $\Theta\bigl(\sqrt{s}-\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\bigr)=1$. Hence,

$$\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m^2)\Theta(q_0)\delta\bigl((q-p_1-p_2)^2-m^2\bigr)\Theta(p_{10}+p_{20}-q_0)$$

$$=\frac{1}{(2\pi)^4\sqrt{s}}\int d^3q\,\delta\bigl(s-2\sqrt{s}\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\bigr)\,. \qquad (104)$$

To evaluate the above integral, use spherical coordinates, $d^3q=|\vec{\boldsymbol{q}}|^2d|\vec{\boldsymbol{q}}|\,d\Omega=4\pi|\vec{\boldsymbol{q}}|^2d|\vec{\boldsymbol{q}}|$, since there is no dependence on the direction of $\vec{\boldsymbol{q}}$ in the integrand above. It is convenient to change the integration variable to $E\equiv\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}$, in which case $|\vec{\boldsymbol{q}}|d|\vec{\boldsymbol{q}}|=EdE$. Hence,

$$\begin{aligned}\int d^3q\,\delta\bigl(s-2\sqrt{s}\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\bigr)&=4\pi\int_m^\infty|\vec{\boldsymbol{q}}|\,E\,dE\,\delta\bigl(s-2\sqrt{s}E\bigr)\\&=\frac{2\pi}{\sqrt{s}}\int_m^\infty E(E^2-m^2)^{1/2}\delta\bigl(E-\tfrac{1}{2}\sqrt{s}\bigr)\,dE\\&=\frac{\pi\sqrt{s}}{2}\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\bigl(\sqrt{s}-2m\bigr)\,.\end{aligned} \qquad (105)$$

The presence of the Θ-function in eq. (105) is due to the observation that if $\sqrt{s}<2m$, then the argument of the delta function is never zero over the range of integration from $m\leq E<\infty$, in which case the delta function must be set to zero. Moreover, since $\sqrt{s}$ is positive by definition, one can rewrite $\Theta\bigl(\sqrt{s}-2m\bigr)$ as follows,

$$\Theta(\sqrt{s}-2m)=\Theta\bigl((\sqrt{s}-2m)(\sqrt{s}+2m)\bigr)=\Theta(s-4m^2)\,, \qquad (106)$$

Indeed, the δ function and Θ function conditions in eq. (105) are satisfied if and only if $s\geq 4m^2$. In light of eqs. (104)–(106), we end up with

$$\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m^2)\Theta(q_0)\delta\bigl((q-p_1-p_2)^2-m^2\bigr)\Theta(p_{10}+p_{20}-q_0)=\frac{1}{32\pi^3}\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\bigl(s-4m^2\bigr)\,. \qquad (107)$$

Inserting the above expression into eq. (101) yields our final result,

$$\mathrm{Im}\,\Gamma^{(4)}(s)=\frac{\lambda^2}{32\pi}\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\bigl(s-4m^2\bigr)\,. \qquad (108)$$

So far, we have only examined the s-channel piece of the above expression. If we now include the t-channel and u-channel diagrams, it is clear that the only change in our analysis is to replace s with t and u, respectively. Thus,

$$\operatorname{Im}\Gamma^{(4)}(p_1,p_2,p_3,p_4) = \frac{\lambda^2}{32\pi}\left[\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\bigl(s-4m^2\bigr)\right.$$
$$\left.+\left(1-\frac{4m^2}{t}\right)^{1/2}\Theta\bigl(t-4m^2\bigr)+\left(1-\frac{4m^2}{u}\right)^{1/2}\Theta\bigl(u-4m^2\bigr)\right]. \tag{109}$$

The physical region of scattering corresponds to $s \geq 4m^2$, $t < 0$ and $u < 0$. Thus, the last two terms on the right hand side of eq. (109) do not survive in the physical scattering amplitude, in which case

$$\operatorname{Im}\Gamma^{(4)}(p_1,p_2,p_3,p_4) = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}}\,. \tag{110}$$

(d) An explicit one-loop computation of $\Gamma^{(4)}$ yields

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\lambda - \frac{\lambda^2}{32\pi^2}\left[F\left(\frac{s}{m^2}\right)+F\left(\frac{t}{m^2}\right)+F\left(\frac{u}{m^2}\right)+G(m^2)\right], \tag{111}$$

where all momenta point into the vertex, $s \equiv (p_1+p_2)^2$, $t \equiv (p_1+p_3)^2$, $u \equiv (p_1+p_4)^2$ are Lorentz-invariant kinematic variables, the function F is defined in part (a), and the function $G(m^2)$ is a real function.[8] Using eq. (111) and the results of part (a), compute $\operatorname{Im}\Gamma^{(4)}$ and check that your calculation in part (b) is correct.

Using eq. (77) with $z = s/m^2$ yields

$$\operatorname{Im} F\left(\frac{s}{m^2}\right) = -\theta(s-4m^2)\pi\sqrt{1-\frac{4m^2}{s}}\,.$$

In eq. (111), only $F(s/m^2)$ has an imaginary part in the physical region corresponding to $s \geq 4m^2$, $t < 0$ and $u < 0$. Taking the imaginary part of eq. (111) therefore yields

$$\operatorname{Im}\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\frac{\lambda^2}{32\pi^2}\operatorname{Im} F\left(\frac{s}{m^2}\right) = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}}\,.$$

Indeed, we have reproduced the result of the cutting rules given by eq. (110).

APPENDIX: The reflection principle of complex analysis and its implications

If $f(z)$ is an analytic function in some region of the complex plane, then so is $f^*(z^*)$. If $f(z)$ is a real valued function in a region of the complex plane that includes part of the real axis, then $f(z) = f^*(z^*)$ along that part of the real axis (since $z = z^*$ on the real axis). Thus,

[8] In fact, the function G is infinite, but this infinity can be removed by renormalization. Since we are only interested here in $\operatorname{Im}\Gamma^{(4)}$, we can safely ignore any details associated with the renormalization procedure.

$f^*(z^*)$ and $f(z)$ are analytic continuations of one another. As long as no singularities are encountered, it follows that $f(z) = f^*(z^*)$, which implies that $f^*(z) = f(z^*)$. That is, we have proven the reflection principle of complex analysis,

Theorem (Reflection principle): If $f(z)$ is real and analytic on a continuous part of the real axis, then $f^*(z) = f(z^*)$ at all points in the complex plane where $f(z)$ is analytic.

That is, the reflection principle is a consequence of the principle of analytic continuation. As an application of the reflection principle, we can show that $\operatorname{Disc}\Gamma^{(4)}(s)$ is related to $\operatorname{Im}\Gamma^{(4)}(s)$. In particular, applying the reflection principle to $\Gamma^{(4)}(s+i\epsilon)$ yields

$$\Gamma^{(4)}(s-i\epsilon) = \Gamma^{(4)}(s+i\epsilon)^* \,, \tag{112}$$

which was quoted in eq. (100). We can therefore conclude that

$$\operatorname{Disc}\Gamma^{(4)}(s) \equiv \Gamma^{(4)}(s+i\epsilon) - \Gamma^{(4)}(s+i\epsilon)^* = 2i\operatorname{Im}\Gamma^{(4)}(s)\,,$$

where we have defined

$$\Gamma^{(4)}(s) \equiv \lim_{\epsilon\to 0}\Gamma^{(4)}(s+i\epsilon)\,.$$

The upshot of this discussion is that the cutting rules can be employed to compute $\operatorname{Im}\Gamma^{(4)}(s)$.

3. The photon vacuum polarization function is defined to be:

$$\Pi^{\mu\nu}(q) = (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi(q^2)\,.$$

In class, we evaluated this function at one-loop in the $\overline{\text{MS}}$ scheme. Consider a second scheme, called the *on-shell scheme*, in which we define $\Pi(q^2=0)\equiv 0$.

(a) Evaluate Z_3 in this scheme.

In class, we used the method of counterterms to derive the renormalized vacuum polarization in terms of the renormalized coupling e_R and mass m_R. To simplify the typography, we shall henceforth drop the subscripts R. The end result obtained in class was,

$$\begin{aligned}\Pi(q^2) &= \frac{2\alpha}{\pi}(4\pi)^\epsilon\Gamma(\epsilon)\int_0^1 dx\, x(1-x)\left[\frac{m^2-q^2x(1-x)}{\mu^2}\right]^{-\epsilon} + Z_3 - 1\\ &= \frac{\alpha}{3\pi}\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi\right) - \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2-q^2x(1-x)}{\mu^2}\right] + Z_3 - 1,\end{aligned} \tag{113}$$

after dropping terms of $\mathcal{O}(\epsilon)$ and higher. In the on-shell scheme, $\Pi(q^2=0)=0$. That is,

$$\frac{\alpha}{3\pi}\left[\frac{1}{\epsilon}-\gamma+\ln 4\pi - \ln\left(\frac{m^2}{\mu^2}\right)\right] + Z_3 - 1 = 0\,.$$

Solving for Z_3, we find

$$Z_3 = 1 - \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon}-\gamma+\ln 4\pi - \ln\left(\frac{m^2}{\mu^2}\right)\right]\,. \tag{114}$$

(b) Obtain asymptotic forms for $\Pi(q^2)$ in two limiting cases: (i) $q^2 \to 0$, and (ii) $q^2 \to \infty$.

Inserting the expression for Z_3 given in eq. (114) back into eq. (113) yields

$$\Pi(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 - q^2x(1-x)}{m^2}\right],$$

in the on-shell scheme. Consider first the $q^2 \to 0$ limit. Expanding the logarithm,

$$\ln\left[\frac{m^2 - q^2x(1-x)}{m^2}\right] \simeq -\frac{q^2}{m^2}\, x(1-x)\,.$$

Thus,

$$\Pi(q^2)\bigg|_{q^2\to 0} \simeq \frac{2\alpha q^2}{\pi m^2}\int_0^1 x^2(1-x)^2\, dx = \frac{\alpha q^2}{15\pi m^2}\,. \tag{115}$$

Next, we consider the $q^2 \to \infty$ limit. In this case, we need to restore the positive infinitesimal ε back into the argument of the logarithm using $m^2 \to m^2 - i\varepsilon$. In the $q^2 \to \infty$ limit, we can drop the m^2 in the numerator of the argument of the logarithm, in which case,

$$\Pi(q^2)\bigg|_{q^2\to\infty} = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left\{\ln\left(-\frac{q^2}{m^2} - i\varepsilon\right) + \ln\bigl[x(1-x)\bigr]\right\}.$$

Performing the elementary integrations yields,

$$\Pi(q^2)\bigg|_{q^2\to\infty} = -\frac{\alpha}{3\pi}\ln\left(-\frac{q^2}{m^2} - i\varepsilon\right) + \frac{5\alpha}{9\pi}\,. \tag{116}$$

<u>*JUST FOR FUN*</u>:

One can compute $\Pi(q^2)$ exactly (in the one-loop approximation) by evaluating the integral,

$$G(z) = \int_0^1 x(1-x)\ln\bigl[1 - zx(1-x) - i\varepsilon\bigr]\, dx\,. \tag{117}$$

Following the steps of the derivation of $F(z)$ in part (b) of problem 2, we first consider the case of $0 \leq z < 4$. After changing the integration variable from x to y and integrating by parts as in eqs. (79) and (81), we obtain

$$G(z) = -\frac{1}{2}\int_0^1 \frac{y^2\bigl(1 - \frac{1}{3}y^2\bigr)\, dy}{\frac{4}{z} - 1 + y^2}\,. \tag{118}$$

Defining $a^2 \equiv -1 + 4/z$ with $a > 0$, we evaluate the following integrals,

$$\int_0^1 \frac{y^2\, dy}{y^2 + a^2} = 1 - a^2\int_0^1 \frac{dy}{y^2 + a^2} = 1 - a\arctan\left(\frac{1}{a}\right),$$

$$\int_0^1 \frac{y^4\, dy}{y^2 + a^2} = \int_0^1 y^2\, dy - a^2\int_0^1 \frac{y^2\, dy}{y^2 + a^2} = \frac{1}{3} - a^2 + a^3\arctan\left(\frac{1}{a}\right). \tag{119}$$

Hence,

$$G(z) = -\tfrac{4}{9} - \tfrac{1}{6}a^2 + \tfrac{1}{6}a(a^2+3)\arctan\left(\frac{1}{a}\right) . \tag{120}$$

Finally, after employing eq. (82),

$$G(z) = -\frac{5}{18} - \frac{2}{3z} + \frac{1}{3}\left(\frac{2}{z}+1\right)\left(\frac{4}{z}-1\right)^{1/2} \arcsin\left(\tfrac{1}{2}\sqrt{z}\right) , \qquad \text{for } 0 \le z \le 4. \tag{121}$$

We can check eq. (121) in the limits as $z \to 0$ and $z \to 4$. The behavior of $G(z)$ as $z \to 0$ requires employing the expansion of the arcsine function,

$$\arcsin x = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \mathcal{O}(x^7) . \tag{122}$$

A straightforward but tedious exercise (which I have worked out by hand) yields,

$$G(z) = -\frac{z}{30} + \mathcal{O}(z^2) . \tag{123}$$

Indeed $G(0) = 0$ is a consequence of setting $z = 0$ in eq. (117). The behavior of $G(z)$ as $z \to 0$ was obtained in eq. (115) and is consistent with the result shown in eq. (123). Finally, $G(4)$ can be immediately deduced from eq. (118),

$$G(4) = -\tfrac{1}{2}\int_0^1 \left(1 - \tfrac{1}{3}y^2\right) = -\tfrac{4}{9} , \tag{124}$$

in agreement with eq. (121).

The case of $z > 4$ can be obtained from eq. (121) by analytic continuation using the results given in eqs. (91), (95) and (96). The end result is,

$$G(z) = -\frac{4}{9} + \frac{1}{6}\left(1-\frac{4}{z}\right) + \frac{1}{6}\left(1+\frac{2}{z}\right)\left(1-\frac{4}{z}\right)^{1/2}\left[\ln\left(\frac{1+\sqrt{1-\frac{4}{z}}}{1-\sqrt{1-\frac{4}{z}}}\right) - i\pi\right] , \quad \text{for } z \ge 4. \tag{125}$$

One can easily confirm the result for $\operatorname{Im} G(z)$ following the method presented in part (a) of problem 2. The behavior of $G(z)$ as $z \to \infty$ was obtained in eq. (116) and is consistent with the result,

$$G(z) \sim -\tfrac{5}{18} + \tfrac{1}{6}[\ln z - i\pi] , \quad \text{as } z \to \infty , \tag{126}$$

obtained from eq. (125).

Finally, the case of $z < 0$ is easily treated starting from eq. (118), and results in

$$G(z) = -\frac{4}{9} + \frac{1}{6}\left(1-\frac{4}{z}\right) + \frac{1}{6}\left(1+\frac{2}{z}\right)\left(1-\frac{4}{z}\right)^{1/2} \ln\left(\frac{\sqrt{1-\frac{4}{z}}+1}{\sqrt{1-\frac{4}{z}}-1}\right) , \quad \text{for } z < 0. \tag{127}$$

It is straightforward to check that eq. (127) yields $\lim_{z\to 0} G(z) = 0$, which implies that $G(z)$ is continuous at $z = 0$. Moreover, eq. (123) is valid for both signs of z. Finally,

$$G(z) = -\tfrac{5}{18} + \tfrac{1}{6}\ln(-z) + \mathcal{O}\left(\frac{1}{z}\right) , \qquad \text{as } z \to -\infty. \tag{128}$$

Eq. (128) also applies in the case of $z \to +\infty$ after restoring the $i\varepsilon$ factor via $z \to z + i\varepsilon$ in the case of large positive z. This observation is not surprising given that eqs. (125) and (127) are analytic continuations of each other.

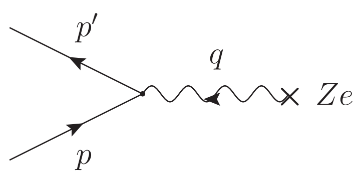
In summary, in the on-shell renormalization scheme in the one-loop approximation,

$$\Pi(q^2) = -\frac{2\alpha}{\pi} G(q^2/m^2)\,, \tag{129}$$

where the function $G(q^2/m^2)$ is given by eq. (127) [for $q^2 < 0$], eq. (121) [for $0 \leq q^2 \leq 4m^2$] and eq. (125) [for $q^2 \geq 4m^2$]. The case of $q^2 < 0$ will be treated further in part (c) below.

(c) Using the $q^2 \to 0$ limit of part (b), compute the $\mathcal{O}(\alpha)$ correction to the Coulomb potential. *OPTIONAL:* Compute the $\mathcal{O}(\alpha)$ correction to the Coulomb potential without making the approximation of small q^2. Examine explicitly the limiting cases $m_e r \gg 1$ and $m_e r \ll 1$.

To find the correction to the Coulomb potential, consider the potential felt by an electron due to an infinitely heavy source of charge Ze. In this limit, if p is the initial four-momentum and p' is the final four-momentum, then the three-momentum is conserved but there is no energy transfer. Diagrammatically, we can represent this process by the interaction of an electron with a classical external source (denoted by $\times$ in the diagram below),



The kinematics of this process are:

$$q = p' - p\,, \qquad q_0 = 0\,, \qquad q^2 = (q_0)^2 - |\vec{\boldsymbol{q}}|^2 = -|\vec{\boldsymbol{q}}|^2\,.$$

At tree-level, the matrix element is proportional to the propagator,

$$\mathcal{M} \sim -\frac{Ze^2}{|\vec{\boldsymbol{q}}|^2}\,. \tag{130}$$

Recalling the first-order Born approximation of non-relativistic quantum mechanics, the non-relativistic potential is given by the Fourier transform of the matrix element given in eq. (130),

$$V(r) = -Ze^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}}{|\vec{\boldsymbol{q}}|^2}\,. \tag{131}$$

Here is a quick and dirty way to evaluate the above integral. Employing the identity,

$$\boldsymbol{\nabla}^2 \frac{1}{r} = -4\pi\delta^3(\vec{\boldsymbol{r}})\,,$$

it follows that

$$\frac{1}{r} = -4\pi \frac{1}{\boldsymbol{\nabla}^2} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}} = 4\pi \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}}{|\vec{\boldsymbol{q}}|^2}\,,$$

where we have employed the integral representation of the three-dimensional delta function. We conclude that

$$V(r) = -\frac{Ze^2}{4\pi r}\,,$$

which is the well-known Coulomb potential.

We next examine the effects of vacuum polarization at one-loop. As shown in class, the photon propagator is modified as follows

$$\mathscr{D}_{\mu\nu}(q^2) = \frac{-i}{q^2\bigl[1+\Pi(q^2)\bigr]}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - ia\frac{q_\mu q_\nu}{q^4}\,. \tag{132}$$

It is convenient to work in the Feynman gauge with $a = 1$, in which case,

$$\mathscr{D}_{\mu\nu}(q^2) = \frac{-ig_{\mu\nu}}{q^2\bigl[1+\Pi(q^2)\bigr]}\,.$$

In the static limit (corresponding to the $q^2 \to 0$ limit of part (b), we make use of eq. (115) to obtain

$$\mathcal{M} \sim -\frac{Ze^2}{|\vec{\boldsymbol{q}}|^2}\left(1 + \frac{\alpha|\vec{\boldsymbol{q}}|^2}{15\pi m^2}\right)^{-1}\,,$$

after putting $q^2 = -|\vec{\boldsymbol{q}}|^2$. In the static approximation, $|\vec{\boldsymbol{q}}| \to 0$, and we can expand to first order in $|\vec{\boldsymbol{q}}|$,

$$\mathcal{M} \sim -\frac{Ze^2}{|\vec{\boldsymbol{q}}|^2}\left(1 - \frac{\alpha|\vec{\boldsymbol{q}}|^2}{15\pi m^2}\right)\,.$$

Thus, eq. (131) is modified,

$$V(r) = -Ze^2\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}}{|\vec{\boldsymbol{q}}|^2}\left(1 - \frac{\alpha|\vec{\boldsymbol{q}}|^2}{15\pi m^2}\right) = -\frac{Ze^2}{4\pi r} - \frac{Ze^2\alpha}{15\pi m^2}\delta^3(\vec{\boldsymbol{r}})\,.$$

This is the famous Uehling potential—the correction to the Coulomb potential of a heavy nucleus due to vacuum polarization.

Suppose we do not use the $|\vec{\boldsymbol{q}}| \to 0$ limit of $\Pi(q^2)$. Then, in the static approximation,

$$\mathcal{M} \sim -\frac{Ze^2}{|\vec{\boldsymbol{q}}|^2}\left\{1 + \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 + |\vec{\boldsymbol{q}}|^2 x(1-x)}{m^2}\right]\right\}\,.$$

To simplify the notation, for the rest of this calculation I shall denote $q \equiv |\vec{\boldsymbol{q}}|$. Following our previous analysis,

$$V(r) = -Ze^2\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}}{q^2}\left\{1 + \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 + q^2 x(1-x)}{m^2}\right]\right\}\,. \tag{133}$$

Thus, we need to examine,

$$\begin{aligned}
&\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}}{q^2}\ln\left[1 + \frac{q^2}{m^2}x(1-x)\right] \\
&\qquad = \frac{1}{(2\pi)^2}\int_0^\infty dq\,\ln\left[1 + \frac{q^2}{m^2}x(1-x)\right]\int_{-1}^1 d\cos\theta\, e^{ikr\cos\theta} \\
&\qquad = \frac{1}{2\pi^2 r}\int_0^\infty \frac{dq}{q}\,\sin qr\,\ln\left[1 + \frac{q^2}{m^2}x(1-x)\right]\,.
\end{aligned}$$

Inserting the above result into eq. (133),

$$V(r) = -\frac{Ze^2}{2\pi^2 r}\int_0^\infty \frac{dq}{q}\,\sin qr\left\{1+\frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\ln\left[1+\frac{q^2}{m^2}x(1-x)\right]\right\}. \tag{134}$$

It will prove useful to change of variables, $x=\frac{1}{2}(1-y)$. Then, the resulting y-integration, which now goes from $y=-1$ to 1, is over an even function of y. Hence, we can take the limits of integration to go from $y=0$ to 1 and multiply by 2. Thus,

$$\int_0^1 dx\,x(1-x)\ln\left[1+\frac{q^2}{m^2}x(1-x)\right] = \tfrac{1}{4}\int_0^1 dy\,(1-y^2)\ln\left[1+\frac{q^2(1-y^2)}{4m^2}\right]$$

$$= \tfrac{1}{2}q^2\int_0^1 \frac{y^2\bigl(1-\frac{1}{3}y^3\bigr)\,dy}{4m^2+q^2(1-y^2)}\,.$$

To achieve the last step, we integrated by parts by taking $u=\ln\bigl[1+q^2(1-y^2)/(4m^2)\bigr]$ and $dv=(1-y^2)dy$. Using the above result in eq. (134),

$$V(r) = -\frac{Ze^2}{2\pi^2 r}\int_0^\infty \frac{dq}{q}\,\sin qr\left\{1+\frac{\alpha q^2}{\pi}\int_0^1 \frac{y^2\bigl(1-\frac{1}{3}y^3\bigr)\,dy}{4m^2+q^2(1-y^2)}\right\}.$$

We can perform the integration over q using

$$\int_0^\infty \frac{q\sin qr}{q^2+a^2}\,dq = \tfrac{1}{2}\pi e^{-ar}\,,$$

either using the calculus of residues or by consulting a good table of integrals.

We end up with

$$V(r) = -\frac{Ze^2}{4\pi r}\left\{1+\frac{\alpha}{\pi}\int_0^1 dy\,\frac{y^2\bigl(1-\frac{1}{3}y^3\bigr)}{1-y^2}\exp\left[-\frac{2mr}{\sqrt{1-y^2}}\right]\right\}.$$

We can rewrite the integral over y with another change of variables:

$$u=\frac{1}{\sqrt{1-y^2}}\,,\qquad du=\frac{y\,dy}{(1-y^2)^{3/2}}=u^2\sqrt{u^2-1}\,du\,.$$

Then $y=\sqrt{u^2-1}/u$ and we obtain

$$V(r) = -\frac{Ze^2}{4\pi r}\left\{1+\frac{2\alpha}{3\pi}\int_1^\infty du\,e^{-2mru}\left(1+\frac{1}{2u^2}\right)\frac{\sqrt{u^2-1}}{u^2}\right\}.$$

The analysis of the limiting cases for $mr \ll 1$ and $mr \gg 1$ is described in V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, *Quantum Electrodynamics* (Pergamon Press, Oxford, UK, 1980) pp. 504–508. In the two limiting cases,

$$V(r) = -\frac{Ze^2}{4\pi r}\times\begin{cases}1-\dfrac{2\alpha}{3\pi}\bigl[\ln(mr)+\gamma+\frac{5}{6}+\cdots\bigr]\,, & \text{for } mr\ll 1\,,\\[2ex] 1+\dfrac{\alpha}{4\sqrt{\pi}(mr)^{3/2}}e^{-2mr}+\cdots\,, & \text{for } mr\gg 1\,.\end{cases}$$

(d) Show that the quantity:

$$\alpha_{\text{eff}}(q^2) \equiv \frac{\alpha}{1+\Pi(q^2)} \tag{135}$$

is independent of whether you evaluate this expression using bare or renormalized quantities. As a result, argue that $\alpha_{\text{eff}}(q^2)$ is independent of renormalization scheme. Find the relation between the couplings defined in the $\overline{\text{MS}}$ and on-shell schemes ($\alpha_{\overline{\text{MS}}}$ and α_{OS}, respectively), in the one loop approximation. Sketch a graph of $\alpha_{\text{eff}}(-q^2)$ at one-loop, in the on-shell scheme, i.e. for *negative* values of the argument.

NOTE: In the on-shell scheme, $\alpha_{\text{eff}}(0)$ is the fine structure constant, which is approximately equal to 1/137.

Consider the quantity defined in eq. (135), where α and $\Pi(q^2)$ are the bare coupling and vacuum polarization, respectively. Note that in part (a), the method of counterterms was used to obtain the renormalized vacuum polarization in terms of the renormalized coupling. For typographical simplicity, we omitted the subscript R on all relevant quantities. To distinguish between bare and renormalized quantities, we shall instead include a subscript B on bare quantities. Thus, α_B will denote the bare coupling, $\alpha_B = e_B^2/(4\pi)$, and $\Pi_B(q^2)$ will be the vacuum polarization as computed with the original Lagrangian expressed in terms of bare fields, couplings and masses.

We therefore define

$$\alpha_{\text{eff}}(q^2) \equiv \frac{\alpha_B}{1+\Pi_B(q^2)}\,.$$

The relation between bare and renormalized quantities were obtained in class. In particular,

$$A_B^\mu = Z_3^{1/2}A^\mu\,, \qquad e_B = Z_1 Z_2^{-1} Z_3^{-1/2}\mu^\epsilon e\,, \qquad a_B = Z_a a\,. \tag{136}$$

Using of the Ward-Takahashi identity for gauge invariance to deduce that $Z_1 = Z_2$, it follows that $e_B = Z_3^{-1/2}\mu^\epsilon e$, or in terms of $\alpha \equiv e^2/(4\pi)$,

$$\alpha_B = Z_3^{-1}\mu^{2\epsilon}\alpha\,. \tag{137}$$

The connected 2-point Green function is

$$\mathscr{D}_B^{\mu\nu}(x,y) = \langle\Omega|A_B^\mu(x)A_B^\nu(y)|\Omega\rangle_{\text{conn}} = Z_3\langle\Omega|A^\mu(x)A^\nu(y)|\Omega\rangle_{\text{conn}} = Z_3\mathscr{D}^{\mu\nu}(x,y)\,. \tag{138}$$

Eq. (132) applies to both the bare and the renormalized connected 2-point Green functions. Hence, it follows that

$$\mathscr{D}_B^{\mu\nu}(q^2) = \frac{-i}{q^2\left[1+\Pi_B(q^2)\right]}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) - ia_B\frac{q^\mu q^\nu}{q^4}\,,$$

$$\mathscr{D}^{\mu\nu}(q^2) = \frac{-i}{q^2\left[1+\Pi(q^2)\right]}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) - ia\frac{q^\mu q^\nu}{q^4}\,,$$

In light of eq. (138), it follows that $Z_3 = Z_a$, which was a result previously noted in class. In addition, we conclude that

$$\frac{1}{1+\Pi_B(q^2)} = \frac{Z_3}{1+\Pi(q^2)}\,. \tag{139}$$

In the limit of $\epsilon \to 0$, eq. (137) yields $Z_3 = \alpha/\alpha_B$, from which it follows that

$$\frac{\alpha_B}{1+\Pi_B(q^2)} = \frac{\alpha}{1+\Pi(q^2)}\,. \tag{140}$$

That is, the definition of $\alpha_{\rm eff}$ in eq. (135) does not depend on whether it is computed using bare quantities or renormalized quantities. Moreover, in deriving eq. (140), no specific renormalization scheme was imposed. Thus, $\alpha_{\rm eff}$ is renormalization scheme independent!

Thus, if we denote the minimal subtraction scheme by $\overline{\rm MS}$ and the on-shell scheme by OS, then

$$\alpha_{\rm eff}(q^2) = \frac{\alpha_{\overline{\rm MS}}}{1+\Pi_{\overline{\rm MS}}(q^2)} = \frac{\alpha_{\rm OS}}{1+\Pi_{\rm OS}(q^2)}\,. \tag{141}$$

In class, we derived

$$\Pi(q^2) = -\frac{2\alpha_{\overline{\rm MS}}}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2-q^2x(1-x)}{\mu^2}\right]\,,$$

which should be compared with the one-loop vacuum polarization obtained in part (b),

$$\Pi(q^2) = -\frac{2\alpha_{\rm OS}}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2-q^2x(1-x)}{m^2}\right]\,,$$

Employing eq. (141) to one-loop order, we can expand the denominators,

$$\alpha_{\overline{\rm MS}}\big[1-\Pi_{\overline{\rm MS}}(q^2)\big] = \alpha_{\rm OS}\big[1-\Pi_{\rm OS}(q^2)\big]\,,$$

which then yields

$$\begin{aligned}\alpha_{\overline{\rm MS}}&\left\{1+\frac{2\alpha_{\overline{\rm MS}}}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2-q^2x(1-x)}{\mu^2}\right]\right\}\\ &= \alpha_{\rm OS}\left\{1+\frac{2\alpha_{\rm OS}}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2-q^2x(1-x)}{m^2}\right]\right\}.\end{aligned} \tag{142}$$

We can express $\alpha_{\rm OS}$ as a power series in $\alpha_{\rm MS}$,

$$\alpha_{\rm OS} = \alpha_{\overline{\rm MS}} + \mathcal{O}\big(\alpha^2_{\overline{\rm MS}}\big)\,.$$

Then, eq. (142) yields

$$\begin{aligned}\alpha_{\rm OS} &= \alpha_{\overline{\rm MS}}\left\{1+\frac{2\alpha_{\overline{\rm MS}}}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2-q^2x(1-x)}{\mu^2}\right]\right\} + \mathcal{O}\big(\alpha^3_{\overline{\rm MS}}\big)\\ &= \alpha_{\overline{\rm MS}}\left\{1+\frac{\alpha_{\overline{\rm MS}}}{3\pi}\ln\left(\frac{m^2}{\mu^2}\right)\right\} + \mathcal{O}\big(\alpha^3_{\overline{\rm MS}}\big)\,.\end{aligned}$$

In particular, in the one-loop approximation, we have

$$\alpha_{\rm OS} = \alpha_{\overline{\rm MS}}(\mu = m)\,. \tag{143}$$

Note that for $q^2 > 4m^2$, $\Pi_{\rm OS}(q^2)$ acquires an imaginary part due to the on-shell production of intermediate state e^+e^- pairs. Thus, to avoid this region, we shall sketch a graph of $\alpha_{\rm eff}(-q^2) = \alpha_{\rm OS}/\big[1+\Pi_{\rm OS}(-q^2)\big]$, which is a real function for positive values of q^2 [cf. eq. (141)]. Although the exact expression for $\Pi_{\rm OS}(-q^2)$ was given by eqs. (127) and (129), it suffices to employ the limiting cases obtained in part (b),

$$\text{as } q^2 \to 0, \quad \Pi_{\rm OS}(-q^2) \to -\frac{\alpha_{\rm OS} q^2}{15\pi m^2}\,, \tag{144}$$

$$\text{as } q^2 \to \infty, \quad \Pi_{\rm OS}(-q^2) \to -\frac{\alpha_{\rm OS}}{3\pi}\ln\left(\frac{q^2}{m^2}\right)\,. \tag{145}$$

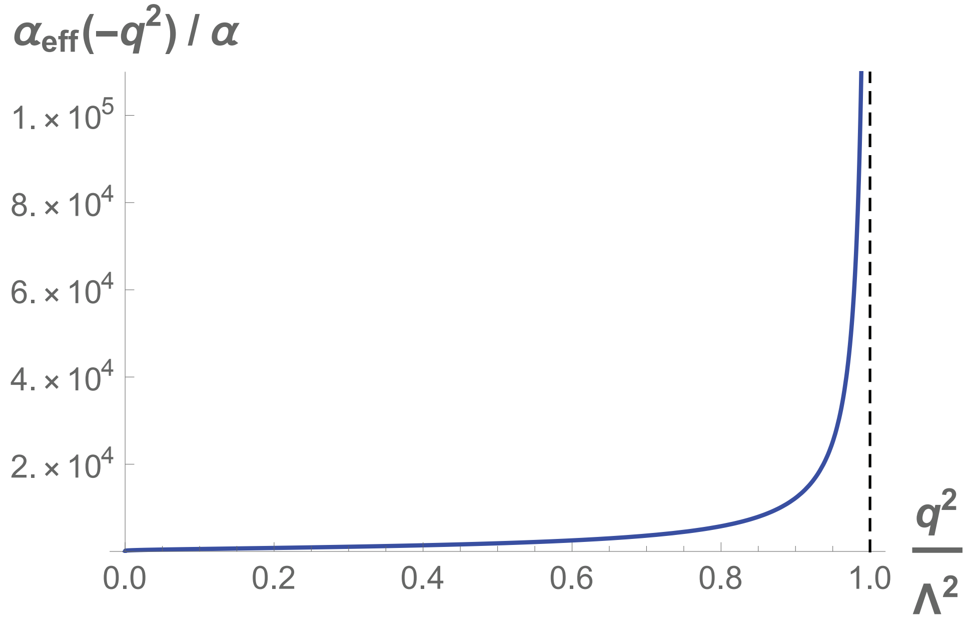
Indeed, the quantity, $-\Pi_{\rm OS}(-q^2)$, monotonically increases with increasing q^2, but its value is small compared to 1 until q^2/m^2 becomes exponentially large (for more details, see the Remark following part (e) below). Thus, $\alpha_{\rm eff}(-q^2)/\alpha_{\rm OS} = [1+\Pi_{\rm OS}(-q^2)]^{-1}$ diverges in a regime where $\ln(q^2/m^2) \gg 1$. In light of eq. (145), it follows that $\alpha_{\rm eff}(-q^2)$ blows up when

$$1 - \frac{\alpha_{\rm OS}}{3\pi}\ln\left(\frac{q^2}{m^2}\right) \simeq 0\,,$$

that is, when $q^2 \simeq m^2\exp\bigl(3\pi/\alpha_{\rm OS}\bigr)$. In particular,

$$\frac{\alpha_{\rm eff}(-q^2)}{\alpha} \simeq 1 - \frac{3\pi}{\alpha\ln(q^2/\Lambda^2)}\,, \tag{146}$$

where $\alpha \equiv \alpha_{\rm OS} = \alpha_{\rm eff}(0)$ and $\Lambda^2 \equiv m^2\exp\bigl(3\pi/\alpha\bigr)$. A sketch of $\alpha_{\rm eff}(-q^2)/\alpha$ vs. q^2/Λ^2 is shown below, Note that $\alpha_{\rm eff}(0) = \alpha \simeq 1/137$ is the standard definition of the QED coupling constant based on the Thomson limit.



(e) Calculate the numerical value of the momentum scale (in GeV units) where $\alpha_{\rm eff}(-q^2)$ blows up.

As is evident from the above plot, $\alpha_{\rm eff}(-q^2)$ blows up at $q^2 = \Lambda^2 = m^2\exp\bigl(3\pi/\alpha_{\rm OS}\bigr)$. Using $\alpha_{\rm OS} = 1/137$ and $m = m_e = 0.511\times 10^{-3}$ GeV, we obtain[9]

$$\Lambda = (0.511\times 10^{-3}~\text{GeV})e^{3\pi\cdot 137/2} \simeq 10^{277}~\text{GeV}\,, \tag{147}$$

[9]Note that $e^{3\pi\cdot 137/2} = 10^{[3\pi\cdot 137/2]/\ln 10} = 2.4\times 10^{280}$.

which is an incredibly large number (well beyond the Planck scale, $M_{\rm PL} = 10^{19}$ GeV, at which quantum gravitational effects become significant and QED surely must break down).

<u>*REMARK*</u>:

In order to justify in more detail the statement that $-\Pi_{\rm OS}(-q^2)$ monotonically increases with increasing q^2, but its value is small compared to 1 until q^2/m^2 becomes exponentially large, the following plot exhibited below may be illuminating. Recall that [cf. eq. (129)],

$$\Pi(-q^2) = -\frac{2\alpha}{\pi} G(-q^2/m^2)\,, \tag{148}$$

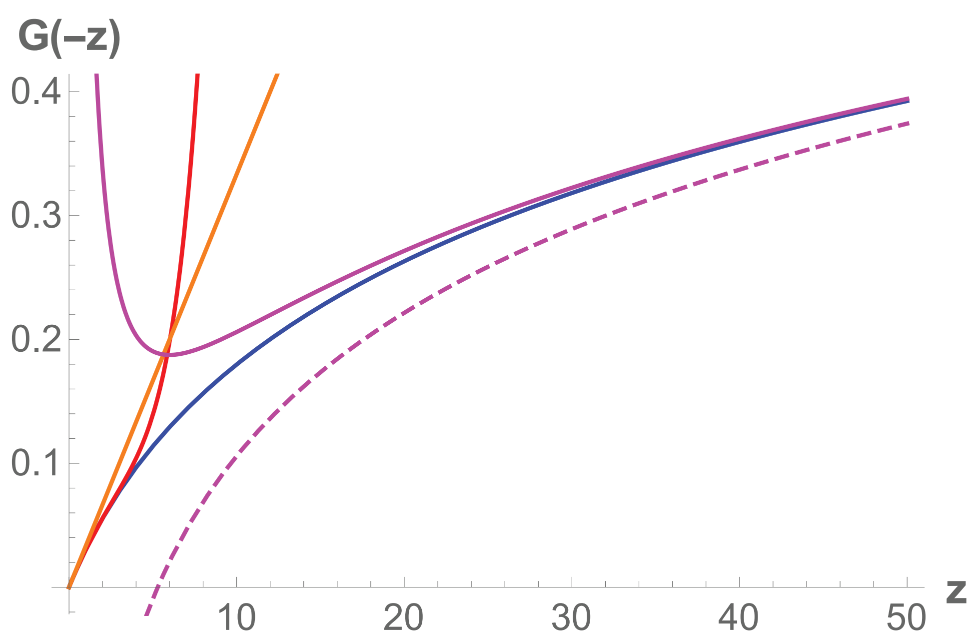
where $G(z)$ is the function whose exact form was given in eq. (127). Mathematica provides the following expansion for $|z| \ll 1$,

$$G(-z) = \frac{z}{30} - \frac{z^2}{280} + \frac{z^3}{1890} - \frac{z^4}{11088} + \frac{z^5}{60060} + \mathcal{O}(z^6)\,. \tag{149}$$

For values of $|z| \gg 1$ [cf. eq. (128)],

$$G(-z) = -\frac{5}{18} + \frac{1}{6}\ln z + \frac{1}{z} + \mathcal{O}\left(\frac{\ln z}{z^2}\right)\,, \qquad \text{as } z \to -\infty. \tag{150}$$

In the plot below, I compare the exact expression for $G(-z)$ [blue curve] with the linear approximation, $G(-z) \simeq z/30$ [orange curve], the small $|z|$ expression given by eq. (149) [red curve], and the large $|z|$ expression given by eq. (150) [magenta curve]. I also added the dashed magenta curve which corresponds to the first two terms of the large $|z|$ expansion.



It should now be clear that $-\Pi(-q^2) = (2\alpha/\pi)G(-z)$ is a monotonically increasing function of $z \equiv q^2/m^2$, but the quantity $(2\alpha/\pi)G(-z)$ attains the value of 1 only for exponentially large values of z where $G(-z)$ is dominated by its logarithmic behavior in z.

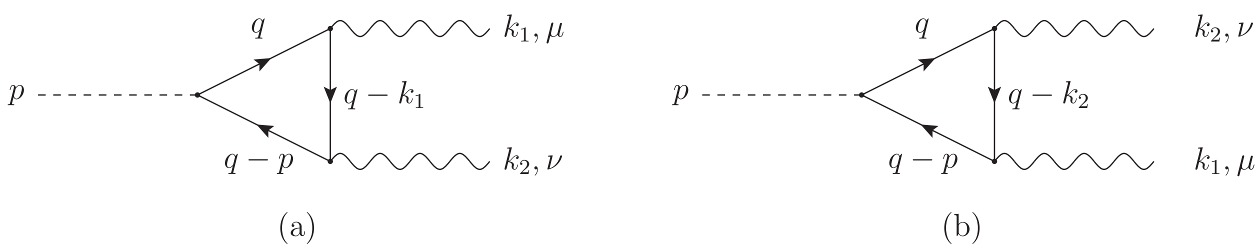
4. Consider QED coupled to a neutral scalar field:

$$\mathscr{L} = \mathscr{L}_{\rm QED} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - g\overline{\psi}\psi\phi\,. \tag{151}$$

(a) Compute the amplitude for the decay $\phi\to\gamma\gamma$, as a function of m_e, m, g, and $\alpha\equiv e^2/(4\pi)$, using perturbation theory at one-loop. Simplify your answer by invoking the kinematics of the problem, i.e. momentum conservation and the on-shell conditions for the external particles. Take care to consider two diagrams which differ only in the direction of flow of electric charge in the loop. Do you need to add a counterterm in order to remove an infinity? Explain.

Because there is no coupling of ϕ to two photons in the bare Lagrangian [cf. eq. (151)], there is no counterterm for the $\phi\gamma\gamma$ vertex. Thus, the renormalizability of the theory implies that the sum of all loop diagrams that contribute to $\phi\to\gamma\gamma$ must be finite.[10]

There are two Feynman diagrams contributing to $\phi\gamma\gamma$ at one loop:



Diagrams (a) and (b) differ in that the outgoing photons are interchanged. Equivalently, one can say that in diagram (b) the flow of electric charge is opposite to that of diagram (a) [by rotating the triangle by 180° out of the plane at the scalar–fermion vertex].

Applying the Feynman rules, and recalling the minus sign for the closed fermion loop,

$$i\mathcal{M}_a = -\int\frac{d^nq}{(2\pi)^n}\,\frac{i^3\,{\rm Tr}\big[(-ig)(\not{q}-\not{p}+m_e)(ie\gamma^\nu)(\not{q}-\not{k}_1+m_e)(ie\gamma^\mu)(\not{q}+m_e)\big]}{(q^2-m_e^2+i\varepsilon)\big[(q-p)^2-m_e^2+i\varepsilon\big]\big[(q-k_1)^2-m_e^2+i\varepsilon\big]}\,\epsilon^*_\mu(k_1,\lambda_1)\epsilon^*_\nu(k_2,\lambda_2)\,, \tag{152}$$

where the factors of i arise from the three fermion propagators. Next, $\mathcal{M}_b$ is obtained from $\mathcal{M}_a$ by interchanging $k_1\to k_2$ and $\mu\to\nu$,

$$i\mathcal{M}_b = -\int\frac{d^nq}{(2\pi)^n}\,\frac{i^3\,{\rm Tr}\big[(-ig)(\not{q}-\not{p}+m_e)(ie\gamma^\mu)(\not{q}-\not{k}_2+m_e)(ie\gamma^\nu)(\not{q}+m_e)\big]}{(q^2-m_e^2+i\varepsilon)\big[(q-p)^2-m_e^2+i\varepsilon\big]\big[(q-k_2)^2-m_e^2+i\varepsilon\big]}\,\epsilon^*_\mu(k_1,\lambda_1)\epsilon^*_\nu(k_2,\lambda_2)\,, \tag{153}$$

We now evaluate the trace that appears in the numerator in eq. (152),

$$\begin{aligned}&{\rm Tr}\big[(\not{q}-\not{p}+m_e)\gamma^\nu(\not{q}-\not{k}_1+m_e)\gamma^\mu(\not{q}+m_e)\big]\\ &\qquad = m_e^3\,{\rm Tr}(\gamma^\mu\gamma^\nu) + m_e\Big\{{\rm Tr}\big[(\not{q}-\not{p})\gamma^\nu(\not{q}-\not{k}_1)\gamma^\mu\big] + {\rm Tr}\big[(\not{q}-\not{p})\gamma^\nu\gamma^\mu\not{q}\big] + {\rm Tr}\big[\gamma^\nu(\not{q}-\not{k}_1)\gamma^\mu\not{q}\big]\Big\}\,.\end{aligned}$$

[10]Indeed, if a term $\phi F_{\mu\nu}F^{\mu\nu}$, which has mass-dimension 5, did appear in eq. (151) [such a term would then contribute at tree-level to $\phi\to\gamma\gamma$], the resulting theory would be non-renormalizable.

The computation is straightforward. The end result is,

$$
\begin{aligned}
&\mathrm{Tr}\big[(\not{q}-\not{p}+m_e)\gamma^\nu(\not{q}-\not{k}_1+m_e)\gamma^\mu(\not{q}+m_e)\big] \\
&\quad = 4m_e^3 g^{\mu\nu} + 4m_e\Big\{(q-p_1)^\mu(q-k_1)^\nu + (q-p_1)^\nu(q-k_1)^\mu - g^{\mu\nu}(q-p)\cdot(q-k_1) \\
&\qquad\quad +(q-p)^\nu q^\mu + g^{\mu\nu}q\cdot(q-p) - (q-p)^\mu q^\nu + (q-k_1)^\nu q^\mu + (q-k_1)^\mu q^\nu - g^{\mu\nu}q\cdot(q-k_1)\Big\} \\
&\quad = 4m_e\Big\{g^{\mu\nu}\big[m_e^2 - q^2 + 2q\cdot k_1 - p\cdot k_1\big] + 4q^\mu q^\nu - 2q^\mu(k_1+p)^\nu - 2q^\nu k_1^\mu + p^\mu k_1^\nu + p^\nu k_1^\mu\Big\}\,.
\end{aligned}
$$

To perform the integral over q, we introduce Feynman parameters. Denoting the resulting denominator factor in eq. (152) by D,

$$D = (1-x-y)(q^2-m_e^2)+\big[(q-p)^2-m_e^2\big]x+\big[(q-k_1)^2-m_e^2\big]y = q^2-2q\cdot(px+k_1y)-m_e^2+p^2x+k_1^2y+i\varepsilon\,.$$

For the physical $\phi\to\gamma\gamma$ decay, we have $p^2=m^2$ and $k_1^2=0$, where m is the mass of the scalar particle. Then,

$$D = q^2 - 2q\cdot(px+k_1y) + m^2x - m_e^2 + i\varepsilon\,.$$

Hence,

$$
\begin{aligned}
\mathcal{M}_a = 8ie^2gm_e\,\epsilon^*_\mu(k_1,\lambda_1)\epsilon^*_\nu(k_2,\lambda_2)\int_0^1 dx\int_0^{1-x} dy \\
\times\int\frac{d^nq}{(2\pi)^n}\frac{g^{\mu\nu}\big[m_e^2-q^2+2q\cdot k_1-p\cdot k_1\big]+4q^\mu q^\nu-2q^\mu(k_1+p)^\nu-2q^\nu k_1^\mu+p^\mu k_1^\nu+p^\nu k_1^\mu}{\big[q^2-2q\cdot(px+k_1y)+m^2x-m_e^2+i\varepsilon\big]^3}\,.
\end{aligned}
$$

It is convenient to isolate the numerator term that is quadratic in q, since this term yields a potential divergence. Let us write

$$\mathcal{M}_a = \big(\mathcal{M}_a^{(1)\,\mu\nu}+\mathcal{M}_a^{(2)\,\mu\nu}\big)\epsilon^*_\mu(k_1,\lambda_1)\epsilon^*_\nu(k_2,\lambda_2)\,,$$

where

$$\mathcal{M}_a^{(1)\,\mu\nu} = 8ie^2gm_e\int_0^1 dx\int_0^{1-x}dy\int\frac{d^nq}{(2\pi)^n}\frac{4q^\mu q^\nu - g^{\mu\nu}q^2}{\big[q^2-2q\cdot(px+k_1y)+m^2x-m_e^2+i\varepsilon\big]^3}$$

$$\mathcal{M}_a^{(2)\,\mu\nu} = 8ie^2gm_e\int_0^1 dx\int_0^{1-x}dy\int\frac{d^nq}{(2\pi)^n}\frac{g^{\mu\nu}(m_e^2+2q\cdot k_1-p\cdot k_1)-2q^\mu(k_1+p)^\nu-2q^\nu k_1^\mu+p^\mu k_1^\nu+p^\nu k_1^\mu}{\big[q^2-2q\cdot(px+k_1y)+m^2x-m_e^2+i\varepsilon\big]^3}\,.$$

Using the formulae given in the class handout entitled, *Useful formulae for computing one-loop integrals* [cf. eq. (6)],

$$\int\frac{d^nq}{(2\pi)^n}\,\frac{4q^\mu q^\nu-g^{\mu\nu}q^2}{\big[q^2-2q\cdot P-M^2+i\varepsilon\big]^3} = \frac{-i\Gamma(\epsilon)(4\pi)^\epsilon}{32\pi^2}(P^2+M^2-i\varepsilon)^{-1-\epsilon}\big[4\epsilon P^\mu P^\nu-\epsilon(2P^2+M^2)g^{\mu\nu}\big]\,,$$

where $\epsilon = 2 - \frac{1}{2}n$. Using $\epsilon\Gamma(\epsilon) = \Gamma(1+\epsilon)$, we see that the integral is finite as $\epsilon \to 0$. Hence taking the $n \to 4$ limit,

$$\lim_{n\to 4}\int \frac{d^n q}{(2\pi)^n}\,\frac{4q^\mu q^\nu - g^{\mu\nu}q^2}{\big[q^2 - 2q\cdot P - M^2 + i\varepsilon\big]^3} = \frac{-i\big[4P^\mu P^\nu - g^{\mu\nu}(2P^2+M^2)\big]}{32\pi^2(P^2+M^2-i\varepsilon)}\,.$$

In computing $\mathcal{M}_a^{(1)\,\mu\nu}$, we identify $P = px + k_1 y$ and $M^2 = m_e^2 - m^2 x$. Hence,

$$P^2 + M^2 = m_e^2 - m^2x(1-x) + 2p\cdot k_1 xy = m_e^2 - m^2 x(1-x-y)\,.$$

At the final step above, we used the kinematic constraints of the $\phi \to \gamma\gamma$ decay[11] to obtain $2p\cdot k_1 = m^2$. Hence,

$$\mathcal{M}_a^{(1)\,\mu\nu} = \frac{e^2 g m_e}{4\pi^2}\int_0^1 dx\int_0^{1-x} dy\left\{-2g^{\mu\nu} + \frac{4(px+k_1y)^\mu(px+k_1y)^\nu + g^{\mu\nu}(m_e^2 - m^2x)}{m_e^2 - m^2x(1-x-y) - i\varepsilon}\right\}. \tag{154}$$

Further simplification can be achieved by using the properties of the photon polarization vectors,

$$k_1^\mu \epsilon_\mu(k_1,\lambda_1) = k_2^\nu \epsilon_\nu(k_2,\lambda_2) = 0\,. \tag{155}$$

By writing $p = k_1 + k_2$ in the numerator of the integrand in eq. (154), we can then omit any terms proportional to k_1^μ and/or k_2^ν. The end result is,

$$\mathcal{M}_a^{(1)\,\mu\nu} = \frac{e^2 g m_e}{4\pi^2}\int_0^1 dx\int_0^{1-x} dy\;\frac{g^{\mu\nu}\big[-m_e^2 + m^2x(1-2x-2y)\big] + 4x(x+y)k_2^\mu k_1^\nu}{m_e^2 - m^2x(1-x-y) - i\varepsilon}\,. \tag{156}$$

To evaluate $\mathcal{M}_a^{(2)\,\mu\nu}$ we can set $\epsilon \to 0$ immediately, since the loop integral is manifestly finite. Using the formulae given previously in eqs. (4) and (5),

$$\mathcal{M}_a^{(2)\,\mu\nu} = \frac{e^2 g m_e}{4\pi^2}\int_0^1 dx\int_0^{1-x} dy$$

$$\times\frac{g^{\mu\nu}\big[m_e^2 - p\cdot k_1 + 2k_1\cdot(px+k_1y)\big] + p^\mu k_1^\nu + p^\nu k_1^\mu - 2(px+k_1y)^\mu(k_1+p)^\nu - 2(px+k_1y)^\nu(k_1+p)^\mu}{m_e^2 - m^2x(1-x-y) - i\varepsilon}\,.$$

We can simplify this result by imposing the kinematical constraints [cf. footnote 11],

$$k_1^2 = k_2^2 = 0\,, \qquad p^2 = m^2 = 2p\cdot k_1\,.$$

In addition, we put $p = k_1 + k_2$ and drop terms proportional to k_1^μ and/or k_2^ν, as noted below eq. (155). The end result it,

$$\mathcal{M}_a^{(2)\,\mu\nu} = \frac{e^2 g m_e}{4\pi^2}\int_0^1 dx\int_0^{1-x} dy\;\frac{g^{\mu\nu}\big[m_e^2 + m^2(x-\frac{1}{2})\big] + (1-4x)k_2^\mu k_1^\mu}{m_e^2 - m^2x(1-x-y) - i\varepsilon}\,. \tag{157}$$

[11]Since momentum conservation implies that $k_2 = p - k_1$, we have

$$0 = k_2^2 = (p-k_1)^2 = p^2 - 2p\cdot k_1 + k_1^2 = m^2 - 2p\cdot k_1\,,$$

after using $k_1^2 = k_2^2 = 0$ and $p^2 = m^2$. Hence, we conclude that $m^2 = 2p\cdot k_1$.

Adding up eqs. (156) and (157) yields,

$$\mathcal{M}_a^{\mu\nu} = \frac{e^2 g m_e}{8\pi^2}\big(m^2 g^{\mu\nu} - 2k_2^\mu k_1^\nu\big)\int_0^1 dx \int_0^{1-x} dy\, \frac{4x(1-x-y)-1}{m_e^2 - m^2 x(1-x-y) - i\varepsilon}\,.$$

We can immediately write down the result for $\mathcal{M}_a^{\mu\nu}$ by interchanging $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. It immediately follows that $\mathcal{M}_a^{\mu\nu} = \mathcal{M}_b^{\mu\nu}$. Hence, the sum of the amplitudes resulting from the two contributing one-loop Feynman diagrams is

$$\mathcal{M} = \frac{\alpha g m_e}{\pi}\big(m^2 g^{\mu\nu} - 2k_2^\mu k_1^\nu\big)\,\epsilon_\mu^*(k_1,\lambda_1)\epsilon_\nu^*(k_2,\lambda_2)\int_0^1 dx \int_0^{1-x} dy\, \frac{4x(1-x-y)-1}{m_e^2 - m^2 x(1-x-y) - i\varepsilon}\,,$$

after writing $\alpha \equiv e^2/(4\pi)$. As advertised, the amplitude is manifestly finite, and no counterterm is required.

(b) Denote the amplitude for the scalar decay by $\mathcal{M}_{\mu\nu}$, where μ and ν are the photon Lorentz indices. Gauge invariance implies that $k_1^\mu \mathcal{M}_{\mu\nu} = k_2^\nu \mathcal{M}_{\mu\nu} = 0$, where k_1 and k_2 are the respective photon momenta. Does your amplitude of part (a) respect this requirement?

The result from part (a) yields

$$\mathcal{M}_{\mu\nu} = \frac{\alpha g m_e}{\pi}\big(m^2 g_{\mu\nu} - 2k_{2\mu} k_{1\nu}\big)\int_0^1 dx \int_0^{1-x} dy\, \frac{4x(1-x-y)-1}{m_e^2 - m^2 x(1-x-y) - i\varepsilon}\,. \tag{158}$$

It is straightforward to verify that $k_1^\mu \mathcal{M}_{\mu\nu} = k_2^\nu \mathcal{M}_{\mu\nu} = 0$. For example,

$$k_1^\mu\big(m^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\big) = (m^2 - 2k_1\!\cdot\! k_2)k_{1\nu} = 0\,,$$

after noting that

$$2k_1\!\cdot\! k_2 = (k_1+k_2)^2 - k_1^2 - k_2^2 = p^2 = m^2\,,$$

where we have used $p = k_1 + k_2$ and $k_1^2 = k_2^2 = 0$. Likewise,

$$k_2^\nu\big(m^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\big) = (m^2 - 2k_1\!\cdot\! k_2)k_{2\mu} = 0\,,$$

(c) Work out all integrals explicitly and evaluate the imaginary part of $\mathcal{M}_{\mu\nu}$. For what range of m_e/m is the amplitude purely real? Explain the physical significance of the non-zero imaginary part.

HINT: You may find the following integral useful:

$$\int_0^1 \frac{dy}{y}\,\log\big[1 - 4Ay(1-y)\big] = -2\left(\sin^{-1}\sqrt{A}\right)^2\,, \tag{159}$$

for $0 \le A \le 1$. For values of A outside this region, you may analytically continue the above result. The imaginary part of this integral is easily computed once the $i\varepsilon$ factor is restored in the argument of the logarithm.

We examine the integral,

$$\mathcal{I}=\int_0^1 dx\int_0^{1-x}dy\,\frac{4x(1-x-y)-1}{1-Rx(1-x-y)-i\varepsilon}\,,\tag{160}$$

where $R\equiv m^2/m_e^2$. Rewrite the numerator as

$$4x(1-x-y)-1=\frac{4\big[Rx(1-x-y)-1\big]+4-R}{R}\,.$$

Then,

$$\begin{aligned}\mathcal{I}&=-\frac{4}{R}\int_0^1 dx\int_0^{1-x}dy+\frac{4-R}{R}\int_0^1 dx\int_0^{1-x}dy\,\frac{1}{1-Rx(1-x-y)-i\varepsilon}\\ &=-\frac{2}{R}+\frac{R-4}{R^2}\int_0^1\frac{dx}{x}\,\ln\big[1-Rx(1-x)-i\varepsilon\big]\,.\end{aligned}\tag{161}$$

Thus, we must now evaluate

$$J\equiv\int_0^1\frac{dx}{x}\,\ln\big[1-Rx(1-x)-i\varepsilon\big]\,.\tag{162}$$

In the case of $0\leq R\leq 4$, the argument of the logarithm is nonnegative, and we can safely take the limit of $\varepsilon\to 0$. Using eq. (159),

$$J=-2\big[\sin^{-1}\big(\tfrac{1}{2}\sqrt{R}\,\big)\big]^2\,,\qquad\text{for } 0\leq R\leq 4\,.\tag{163}$$

To analytically continue beyond $R=4$, we first rewrite eq. (163) by employing an identity,

$$\sin^{-1}\big(\tfrac{1}{2}\sqrt{R}\big)=\tfrac{1}{2}\pi-\cos^{-1}\big(\tfrac{1}{2}\sqrt{R}\big)\,.\tag{164}$$

Hence,

$$J=-2\big[\tfrac{1}{2}\pi-\cos^{-1}\big(\tfrac{1}{2}\sqrt{R}\,\big)\big]^2\,,\qquad\text{for } 0\leq R\leq 4\,.\tag{165}$$

In the case of $R>4$, one cannot neglect the $i\varepsilon$ factor in eq. (162). Consequently, we replace $R\to R+i\varepsilon$ in eq. (165),

$$J=-2\big[\tfrac{1}{2}\pi-\cos^{-1}\big(\tfrac{1}{2}\sqrt{R}+i\varepsilon\,\big)\big]^2\,.\tag{166}$$

For $R>4$, we must evaluate the value of the arccosine function just above the branch cut that runs from $\frac{1}{2}\sqrt{R}=1$ to ∞ along the real axis. To accomplish this, we consult F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark, editors, *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, UK, 2010). In particular, eqs. (4.23.24) and (4.37.19) of this reference state that for a positive infinitesimal ε,

$$\lim_{\varepsilon\to 0^+}\cos^{-1}(x+i\varepsilon)=-i\cosh^{-1}x=-i\ln\big(x+\sqrt{x^2-1}\big)\,,\qquad\text{for } 1<x<\infty.\tag{167}$$

Thus, eq. (166) yields,

$$J = -2\left[\tfrac{1}{2}\pi + i\ln\left(\tfrac{1}{2}\sqrt{R} + \sqrt{\tfrac{1}{4}R - 1}\right)\right]^2\,, \qquad \text{for } R > 4. \tag{168}$$

In order to check this result, we shall verify the sign of $\operatorname{Im} J$ following the procedure of Problem 2. We examine,

$$\operatorname{Im} J = \operatorname{Im}\int_0^1 \frac{dx}{x}\,\ln\bigl[1 - Rx(1-x) - i\epsilon\bigr]\,. \tag{169}$$

Following eqs. (74) and (77), the roots of the argument of the logarithm are given by

$$x_\pm = \tfrac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{R}}\right]\,. \tag{170}$$

Thus,

$$\begin{aligned}
\operatorname{Im} J = \Theta(R-4)\int_{x_-}^{x_+}\frac{dx}{x}\,\operatorname{Im}\ln\bigl[1 - Rx(1-x) - i\epsilon\bigr] &= -\Theta(R-4)\pi\int_{x_-}^{x_+}\frac{dx}{x} \\
= -\pi\ln\left(\frac{x_+}{x_-}\right)\Theta(R-4) = -\pi\ln\left(\frac{1+\sqrt{1-\frac{4}{R}}}{1-\sqrt{1-\frac{4}{R}}}\right)\Theta(R-4) \\
= -\pi\ln\left[\left(\tfrac{1}{2}\sqrt{R} + \sqrt{\tfrac{1}{4}R - 1}\right)^2\right]\Theta(R-4) \\
= -2\pi\ln\left(\tfrac{1}{2}\sqrt{R} + \sqrt{\tfrac{1}{4}R - 1}\right)\Theta(R-4)\,,
\end{aligned} \tag{171}$$

in agreement with the $\operatorname{Im} J$ obtained from eq. (168).

It is common practice to rewrite the argument of the logarithm eq. (168) in a more symmetrical form,

$$J = \begin{cases} -2\bigl[\sin^{-1}\bigl(\tfrac{1}{2}\sqrt{R}\bigr)\bigr]^2\,, & \text{for } 0 \le R \le 4, \\ -\tfrac{1}{2}\left[\pi + i\ln\left(\frac{1+\sqrt{1-\frac{4}{R}}}{1-\sqrt{1-\frac{4}{R}}}\right)\right]^2\,, & \text{for } R > 4\,. \end{cases} \tag{172}$$

Hence, if we introduce a function $f(R)$ defined by

$$f(R) = \begin{cases} \sin^{-1}\bigl(\tfrac{1}{2}\sqrt{R}\bigr)\,, & \text{for } 0 \le R \le 4, \\ \tfrac{1}{2}\left[\pi + i\ln\left(\frac{1+\sqrt{1-\frac{4}{R}}}{1-\sqrt{1-\frac{4}{R}}}\right)\right]\,, & \text{for } R > 4\,, \end{cases} \tag{173}$$

then $J = -2[F(R)]^2$, and eq. (161) yields,

$$\mathcal{I} = -\frac{2}{R}\left\{1 + \left(1 - \frac{4}{R}\right)\bigl[f(R)\bigr]^2\right\}\,.$$

In light of eq. (160), we see that eq. (158) yields

$$\mathcal{M}_{\mu\nu} = -\frac{2\alpha m_e}{\pi m^2}\big(m^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\big)\left\{1 + \left(1 - \frac{4}{R}\right)\big[f(R)\big]^2\right\}. \qquad (174)$$

In particular,

$$\operatorname{Im}\mathcal{M}_{\mu\nu} = -\frac{\alpha m_e}{\pi m^2}\big(m^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\big)\left(1 - \frac{4}{R}\right)\ln\left(\frac{1+\sqrt{1-\frac{4}{R}}}{1-\sqrt{1-\frac{4}{R}}}\right)\Theta(R-4).$$

Thus, $\operatorname{Im}\mathcal{M}_{\mu\nu} \neq 0$ when $R = m^2/m_e^2 > 4$, which corresponds to $m > 2m_e$. In this case, the kinematics allows the scalar particle to decay into an e^+e^- pair. Thus, we can cut the triangle diagrams to reveal the on-shell electron and positron. By the Cutkosky cutting rules, $\operatorname{Disc}\operatorname{Im}\mathcal{M}_{\mu\nu} \neq 0$, and we expect a non-zero imaginary part.

EXTRA CREDIT: Derive eq. (159).

See the class handout entitled, *Evaluating the one-loop function arising in $H \to \gamma\gamma$.*

(d) Evaluate the leading behavior of $\mathcal{M}_{\mu\nu}$ in the limit of $m_e \to \infty$.

The limit of $m_e \to \infty$ corresponds to $R = m^2/m_e^2 \to 0$. Using eq. (173), in the limit of $R \to 0$,

$$\begin{aligned}
1 + \left(1 - \frac{4}{R}\right)\big[f(R)\big]^2 &= 1 + \left(1 - \frac{4}{R}\right)\left[\sin^{-1}\!\left(\tfrac{1}{2}\sqrt{R}\right)\right]^2 \\
&\simeq 1 + \left(1 - \frac{4}{R}\right)\left[\frac{\sqrt{R}}{2} + \frac{1}{6}\left(\frac{\sqrt{R}}{2}\right)^3\right]^2 \\
&\simeq 1 + \left(1 - \frac{4}{R}\right)\frac{R}{4}\left(1 + \frac{R}{24}\right)^2 \\
&\simeq 1 + \left(\frac{R}{4} - 1\right)\left(1 + \frac{R}{12}\right) \simeq \frac{R}{6} + \mathcal{O}(R^2). \qquad (175)
\end{aligned}$$

Hence, eq. (174) yields

$$\mathcal{M}_{\mu\nu}(\phi \to \gamma\gamma)\Big|_{m_e\to\infty} = -\frac{\alpha g}{3\pi m_e}\big(m^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\big).$$

This is an example of the *decoupling theorem*, which states that the effects of very massive particles in internal loops of Feynman diagrams yield contributions to the corresponding amplitude that vanish in the infinite mass limit. The decoupling theorem relies on an assumption that it is possible to take the infinite mass limit while holding all coupling constants of the theory fixed. This assumption is valid in the present application, where it is possible to take $m_e \to \infty$ while holding e and g fixed.