Conformal transformations, conformal groups, and conformal Lie algebras

Kuroush Allameh

Presentation for Group Theory and Modern Physics

Outlook

- Conformal transformations
- Conformal groups
- Lie algebras associated with conformal transformations (Witt algebra)
- Quantization of conformal symmetry and Virasoro algebra

- In the following references [1], [2] refer to:

    - [1] M. Schottenloher, "A mathematical introduction to conformal field theory," Lect. Notes Phys. 759, 1-237 (2008) doi:10.1007/978-3-540-68628-6
    - [2] P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory," Springer-Verlag, 1997, ISBN 978-0-387-94785-3, 978-1-4612-7475-9 doi:10.1007/978-1-4612-2256-9

Semi-Riemannian manifold

- *Definition: A semi-Riemannian manifold is a pair* (M, g) *consisting of a smooth manifold* M *of dimension* d *and a non-degenerate metric* g *such that*

$$g_a(X, Y) = g_{\mu\nu}(a) X^{\mu}(a) Y^{\nu}(a), \qquad \forall X, Y \in TM, \qquad \mathrm{a} \in M.$$

  The metric is not needed to be positive definite. If $g(a)$ *has* p *negative eigenvalues and* q *positive eigenvalues we denote the manifold by* $M^{\mathrm{p},\mathrm{q}}$.

- Example: $\mathbb{R}^{1,3}$ with the metric $[g_{\mu\nu}] = \mathrm{diag}(-1, +1, +1, +1)$, i.e. the Minkowski space.

Conformal transformations

- *Definition: A smooth map* $\varphi: M \to M$ *on a semi-Riemannian manifold is called a conformal transformation if there exists a function* $\Omega: M \to R_+$ *such that*

$$g'_{\mu\nu}\big(\varphi(x)\big) = \Omega^2(x) g_{\mu\nu}(x).$$

Since g *is a covariant tensor:*

$$\Omega^2 g_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \varphi^\mu} \frac{\partial x^\beta}{\partial \varphi^\nu}.$$

Conformal transformations – examples

- Isometries $g \to g$ are conformal with $\Omega = 1$. For example, the Lorentz group $O(1,3)$, the group of isometries of the Minkowski space.

- Scale transformations $x^\mu \to \lambda\, x^\mu$ for $\mathbb{R}^{p,q}$ are conformal with $\Omega = \lambda$:

$$ds^2 = -\sum_{i=1}^{p} d\left(x^i\right)^2 + \sum_{j=1}^{q} d\left(x^j\right)^2 \to ds'^2 = \lambda^2\left(-\sum_{i=1}^{p} d\left(x^i\right)^2 + \sum_{j=1}^{q} d\left(x^j\right)^2\right)$$

Classification of conformal transformations for $\mathbb{R}^{0,2}$

- *Theorem: Every holomorphic function*

$$\varphi = u + iv: M \to \mathbb{R}^{0,2} \cong \mathbb{C},$$

on an open subset $M \subset \mathbb{R}^{0,2}$ *is conformal map with conformal factor* $\Omega^2 = \partial_x^2 u + \partial_y^2 u$. *Converse is also true.*

Proof: Take a function $\varphi(x+iy) = u(x,y) + iv(x,y)$. This function is conformal if

$$\begin{aligned} ds'^2 &= du^2 + dv^2 \\ &= (\partial_x^2 u + \partial_x^2 v)dx^2 + \left(\partial_y^2 u + \partial_y^2 v\right)dy^2 + \left(\partial_x u \partial_y u + \partial_x v \partial_y v\right)dxdy \\ &= \Omega^2(dx^2 + dy^2) \end{aligned}$$

$$\Rightarrow \partial_x u \partial_y v + \partial_y u \partial_x v = 0, \qquad \partial_x^2 u + \partial_x^2 v = \Omega^2 = \partial_y^2 u + \partial_y^2 v$$

$$\Rightarrow \partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v, \qquad \partial_x^2 u + \partial_y^2 u = \Omega^2 = \partial_x^2 v + \partial_y^2 v.$$

Classification of conformal transformations for $\mathbb{R}^{1,1}$

- *Theorem: A smooth map*

$$\varphi = (u, v): M \to \mathbb{R}^{1,1},$$

on an open subset $M \subset \mathbb{R}^{1,1}$ *is a conformal map if and only if*

$$\partial_x^2 u > \partial_x^2 v, \qquad and \qquad \partial_x u = \partial_y v, \partial_y u = \partial_x v \quad or \quad \partial_x u = -\partial_y v, \partial_y u = -\partial_x v$$

Proof: Take a function $\varphi(x, y) \to (u(x, y), v(x, y))$. This function is conformal if

$$\begin{aligned} ds'^2 &= -du^2 + dv^2 \\ &= -(\partial_x^2 u - \partial_x^2 v)dx^2 - \left(\partial_y^2 u - \partial_y^2 v\right)dy^2 + \left(-\partial_x u \partial_y u + \partial_x v \partial_y v\right)dxdy \\ &= \Omega^2(-dx^2 + dy^2) \end{aligned}$$

$$\Rightarrow -\partial_x u \partial_y u + \partial_x v \partial_y v = 0, \qquad \partial_x^2 u - \partial_x^2 v = \Omega^2 = -\left(\partial_y^2 u - \partial_y^2 v\right).$$

Global definability of conformal transformations

- In the previous theorems we described the conformal transformations on open subsets of $\mathbb{R}^{p,q}$.
- Extending the definition to the whole $\mathbb{R}^{p,q}$ is tricky.
- Take

$$z \to \frac{1}{z}, \qquad z \in \mathbb{C}\backslash\{0\}.$$

- It is not well defined on $z = 0$.

- The trick is to include the points at infinity into the manifold.

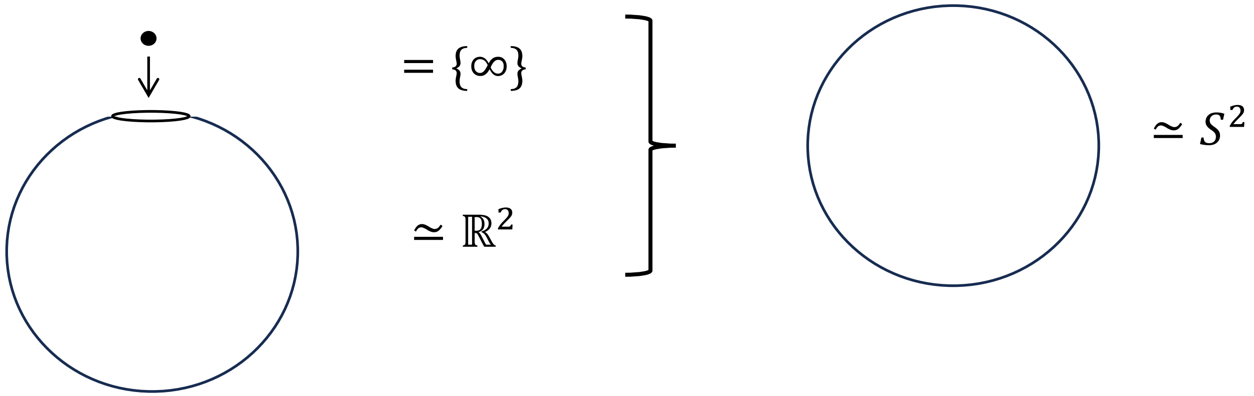
Conformal compactifications

- *Definition: A conformal compactification of a manifold* M *is a larger compact space* $\bar{M}$ *containing* M *as an open dense subset, such that the metric on* M *is related to a metric on* $\bar{M}$ *by a Weyl rescaling:*

$$g_M = \Omega^2(x) g_{\bar{M}}$$

Conformal compactifications – examples

- For $\mathbb{R}^{0,2}$ we have the Riemann sphere S^2 with the same metric as $\mathbb{R}^{0,2}$



- For $\mathbb{R}^{1,1}$ we have $S^1 \times S^1$:

$$ds^2 = -dt^2 + dx^2 \xrightarrow{u=t+x,\, v=t-x} ds'^2 = -du\, dv \xrightarrow{u=\tan U,\, v=\tan V} ds''^2 = -\frac{dU dV}{\cos^2 U \cos^2 V}, \qquad U, V \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Conformal group

- *Definition: The conformal group* $Conf(\mathbb{R}^{p,q})$ *is the connected component containing the identity in the group of conformal diffeomorphisms of the conformal compactification of* $R^{p,q}$.

- *Theorem: The conformal group* $Conf(\mathbb{R}^{0,2})$ *is the group of Mobius transformations* [1]*:*

$$PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm I\} = \left\{\varphi(z) = \frac{az+b}{bz+d}, \qquad z \in S^2, \qquad ad - bc = 1\right\}/\{\pm I\}$$

- *Theorem: The conformal group* $Conf(\mathbb{R}^{1,1})$ *is*

$$Conf(\mathbb{R}^{1,1}) = Diff_+(S^1) \times Diff_+(S^1)$$

where $Diff_+(S^1)$ *is the group of orientation preserving diffeomorphisms on* S^1 [1].

Is the conformal group infinite dimensional?

- It is widely stated in the physics literature that the conformal group is infinite-dimensional. For example, we quote the original 1984 paper by Belavin, Polyakov, and Zamolodchikov:

> The situation is somewhat better in two dimensions. The main reason is that the conformal group is infinite-dimensional in this case, it consists of the conformal analytical transformations. To describe this group, it is convenient to introduce the complex coordinates
>
> $$z = \xi^1 + i\xi^2, \qquad \bar{z} = \xi^1 - i\xi^2, \tag{1 7}$$
>
> the metric having the form
>
> $$ds^2 = dz\,d\bar{z} \tag{1 8}$$
>
> The conformal group of the two-dimensional space which will be denoted by $\mathfrak{G}$, consists of all substitutions of the form
>
> $$z \to \zeta(z), \qquad \bar{z} \to \bar{\zeta}(\bar{z}), \tag{1 9}$$
>
> where ζ and $\bar{\zeta}$ are arbitrary analytical functions

Is the conformal group infinite dimensional?

- The conformal group of $\mathbb{R}^{0,2}$, $PSL(2,\mathbb{C})$ is not infinite-dimensional, however the conformal group of $\mathbb{R}^{1,1}$ being $Diff_+(S^1)\times Diff_+(S^1)$ is.

- However, the actual meaning of the infinite-dimensionality is the fact that the infinitesimal conformal transformations (in $d = 2$) lead to an infinite-dimensional Lie Algebra.

- In practice in the study of conformal quantum field theories, one does not care about the global behavior of the group, but rather the infinitesimal behavior is of interest.

- We will study the Lie algebras relating to the conformal symmetry in $d = 2$.

Witt alegbra

- Let us elaborate on the infinite dimensional Lie algebra governing the classical conformal symmetry.
- *Theorem: The Lie algebra* $\mathtt{Lie}\big(Diff_+(S^1)\big) \coloneqq Vect(S^1)$ *is the space of smooth vector fields over* S^1 [1].
- A generic element $X \in Vect(S^1)$ is given by $X = f\frac{d}{d\theta}$ where f is a smooth function over $(0,2\pi)$.
- *Theorem:* $Vect(S^1)$ *is generated by* $\frac{d}{d\theta}, \cos(n\theta)\frac{d}{d\theta}, \sin(n\theta)\frac{d}{d\theta}$ [1].
- *Theorem: The complexification of* $Vect(S^1)$ *denoted by* $Vect^{\mathbb{C}}(S^1)$ *is generated by* [1]

$$L_n \coloneqq -ie^{-in\theta}\frac{d}{d\theta} = -iz^{-n}\frac{d}{d\theta} = z^{1-n}\frac{d}{dz}$$

- *Theorem:* $Vect^{\mathbb{C}}(S^1)$ *generators satisfy the Witt algebra commutation relations* [1]

$$[L_n, L_m] = (n-m)L_{n+m}$$

Quantization of symmetries and Wigner's theorem

- In order to quantize a classical theory, one needs to replace the classical phase space of the theory with a suitable Hilbert space $\mathbb{H}$.
- In quantum mechanics on is interested in the subspace $\mathbb{P}$ of the Hilbert space $\mathbb{H}$ of unit rays:

$$\psi, \phi: \qquad \langle \psi, \phi \rangle = 1.$$

- If a symmetry group G acting on the classical phase space is to reduce to a symmetry of the quantum theory, then it should preserve the transition probabilities:

$$|\langle T\psi, T\phi \rangle|^2 = |\langle \psi, \phi \rangle|^2, \qquad \forall \psi, \phi \in \mathbb{P}$$

- *Theorem (Wigner 1931): Every transformation T that preserves the transition probabilities, is either a projective unitary or anti-unitary representation of G. We denote these representations by $U(\mathbb{P})$ and they satisfy:*

$$U_g U_h = \omega(g,h) U_{gh}, \qquad \omega(g,h) \in U(1), \qquad g, h \in G.$$

Quantization of symmetries and central extension of groups

- Wigner's theorem provides us with a homomorphism $G \to U(\mathbb{P})$ but does this lead to also a homomorphism on to the unitary/anti-unitary group on the full Hilbert space, i.e. $G \to U(\mathbb{H})$?
- The answer is no in general: The phase $\omega(g,h)$ is not equal to 1 for all groups G.
- However, there is a way to get a genuine representation.
- *Theorem: Every projective representation of the group G lifts to an ordinary representation of E defined by*

$$E = U(1)\times_\omega G = \{(e^{i\alpha}, g) \,|\, \alpha \in \mathbb{R}, g \in G\},$$
$$(e^{i\alpha}, g)(e^{i\beta}, h) = (e^{i(\alpha+\beta)}\omega(g,h), gh).$$

Proof: define $\tilde{U}(e^{i\alpha}, g) = e^{i\alpha}U_g$ for $U_g \in U(\mathbb{P})$. Then

$$\tilde{U}(e^{i\alpha}, g)\tilde{U}(e^{i\beta}, h) = e^{i(\alpha+\beta)}U_g U_h = e^{i(\alpha+\beta)}\omega(g,h)U_{gh} = U\left((e^{i\alpha}, g)(e^{i\beta}, h)\right)$$

- E is called a central extension of the group G.

Quantization of symmetries and central extension of algebras

- *Theorem: The Lie algebra associated to* $E = U(1)\times_\omega G$ *is related to the Lie algebra* $\mathfrak{g}$ *of* G *by*

  $$\mathfrak{e} = \mathfrak{g} \oplus \mathbb{R}Z,$$

  where Z *is a central generator:* $[Z, X]_g = 0, \;\; \forall X \in \mathfrak{g}$, *and the bracket is defined by*

  $$[X, Y]_{\mathfrak{e}} = [X, Y]_{\mathfrak{g}} + c(X, Y)Z,$$

  for some function $c(X, Y)$ [1].
- To quantize the symmetries, one also needs to centrally extend the Lie algebras.

Quantum mechanical conformal symmetry and Virasoro alegbra

- The classical algebra governing the conformal symmetries was the Witt algebra W:

$$[L_n, L_m] = (n - m)L_{n+m}$$

- To quantize we apply the theorem in the last slide to a complexified Lie algebra:
- *Theorem: The Lie algebra associated with the quantum conformal symmetry is the Virasoro algebra given by*

$$Vir = W \oplus \mathbb{C}Z,$$

and it has the following commutations relations [1]

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m}\frac{n}{12}(n^2 - 1)Z,$$
$$[L_n, Z] = 0, \qquad \forall n, m \in \mathbb{Z}$$

Does there exist a Virasoro group

- A natural though pathological question to ask is:

  Does there exist a complex Lie group $\mathcal{G}$ with Virasoro algebra Vir as its Lie algebra?

- *Theorem: There does not exist a complex Lie group* $\mathcal{G}$ *with* $Lie\ \mathcal{G} = Vir$ [1].

Virasoro algebra in physics

- Consider a set of decoupled free scalar fields over $\mathbb{R}^{0,2} \cong \mathbb{C}$:

$$S = \sum_{i}^{n} \int d^2 z \, \partial\phi_i \bar{\partial}\phi_i$$

- One can easily quantize this gaussian theory using the path integral method

$$Z = \int [D\phi_i] \, e^{-S}$$

- The OPE's for the scalars can be calculated exactly [2]

$$\phi_i(z_1, \bar{z}_1)\phi_j(z_2, \bar{z}_2) = -\frac{\delta_{ij}}{4\pi} \log|z_1 - z_2|^2 + reg$$

Virasoro algebra in physics

- For this theory only two components of stress tensor will be non-zero

$$T_{zz} = -2\pi \sum_i \partial\phi_i \partial\phi_i \,, \qquad T_{\bar{z}\bar{z}} = -2\pi \sum_i \bar{\partial}\phi_i \bar{\partial}\phi_i$$

- One can then define the Virasoro generators as contour integrals around the origin of the complex plane

$$L_n = \oint \frac{dz}{2\pi i z} z^{n+2} T_{zz}, \qquad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i \bar{z}} \bar{z}^{n+2} T_{\bar{z}\bar{z}}$$

- Using the OPE $\phi_i(z_1, \bar{z}_1)\phi_j(z_2, \bar{z}_2) = -\frac{1}{4\pi}\log|z_1 - z_2|^2 + reg$ [2]:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n}{12}(n^3 - n)\delta_{n+m}$$
$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{n}{12}(n^3 - n)\delta_{n+m}$$

Virasoro algebra in physics

- In general similar analysis for a generic conformal field theory leads to

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}$$

- c is called the central charge.
- Neatly enough as we mentioned the extra term involving the central charge c came from quantization of symmetries. In physics one can explicitly see that the anomalies in conformal symmetry are governed by the central charge reinforcing the idea of central extending for quantization.

References and further reading

- [1] M. Schottenloher, “A mathematical introduction to conformal field theory,” Lect. Notes Phys. 759, 1-237 (2008) doi:10.1007/978-3-540-68628-6
  - Chapters 1,2,3,4,5 for what we discussed, Chapter 6 for representation theory of Virasoro algebra
- [2] P. Di Francesco, P. Mathieu and D. Senechal, “Conformal Field Theory,” Springer-Verlag, 1997, ISBN 978-0-387-94785-3, 978-1-4612-7475-9 doi:10.1007/978-1-4612-2256-9
  - Chapters 6,7,8 for the representation theory of Virasoro algebra
- [3] S. Weinberg, “The Quantum theory of fields. Vol. 1: Foundations,” Cambridge University Press, 2005, ISBN 978-0-521-67053-1, 978-0-511-25204-4 doi:10.1017/CBO9781139644167
  - Chapter 2 discusses Wigner's theorem and projective representations
- [4] M. Gillioz, “Conformal field theory for particle physicists,” Springer, 2023, ISBN 978-3-031-27085-7, 978-3-031-27086-4 doi:10.1007/978-3-031-27086-4 [arXiv:2207.09474 [hep-th]].