Outer Automorphisms as a Pathway to Discrete Symmetries

Giorgos Demetriou 9/06/2026

	0	1	2
X0	1	1	1
X1	1	w	w^2
X2	1	w^2	W

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X0	1	1	1
X1	1	w^2	w
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X0	1	1	1
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What are Automorphisms

What are Outer Automorphisms

$$f : G \mapsto G$$

Inner Automorphisms

$f_h : g \mapsto hgh^{-1}$

Outer Automorphisms

~~$f_h : g \mapsto hgh^{-1}$~~

$$Out(G) \equiv \frac{Aut(G)}{Inn(G)} = \left[\phi\right] = \{\phi \circ conj_h | \forall h \epsilon H\}$$

Automorphisms of Finite Groups

Groups $\mathbb{Z}_n$ and D_n

$$\mathbb{Z}_n = \langle g \mid g^n = e \rangle$$

Inner Automorphisms are trivial

$$f_h(g) = hgh^{-1} = g$$

Automorphisms are Isomorphisms

$$f(g^n) = f(g)^n = e$$

Under f elements preserve their order

$$ord(g^a) = \frac{a}{gcd(n, a)}$$

$$Aut(G) = \{f(g) = g^a \mid gcd(a, n) = 1\}$$

$$Out(\mathbb{Z}_n) = \frac{Aut(\mathbb{Z}_n)}{Inn(\mathbb{Z}_n)} \equiv (\mathbb{Z}/n\mathbb{Z})^\times$$

Automorphisms of Finite Groups

Groups $\mathbb{Z}_n$ and D_n

$$\mathbb{D}_n = \langle r, s \,|\, r^n = e, srs^{-1} = r^{-1} \rangle$$

Generators Mappings

$$f(r^n) = f(r)^n = e \Rightarrow f_a(r) = r^a \,|\, gcd(a, n) = 1$$

$$f(s^2) = f(s)^2 = e \Rightarrow f_b(s) = r^b s$$

$$Aut(D_n) = \{f_{a,b}(r, s) = r^a(r^b s) \,|\, a \epsilon (\mathbb{Z}/n\mathbb{Z})^{\times}, b \epsilon \mathbb{Z}/n\mathbb{Z}\}$$

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N odd

$$Inn(D_n) = \{f_{\pm 1,b}(r,s) \,|\, b \epsilon \mathbb{Z}_n\}$$

$$\downarrow$$

$$Out(D_n) = \frac{(\mathbb{Z}/n\mathbb{Z})^\times}{\mathbb{Z}_2}$$

N even

$$Inn(D_n) = \{f_{\pm 1,b}(r,s) \,|\, b \epsilon 2\mathbb{Z}_n\}$$

$$\downarrow$$

$$Out(D_n) = \frac{\mathbb{Z}_n \rtimes (\mathbb{Z}/n\mathbb{Z})^\times}{2\mathbb{Z}_n \rtimes \mathbb{Z}_2}$$

DIVIDE AND CONQUER

FROM THE FINITE TO THE LIE

THE SAME INTUITION APPLIES TO LIE ALGEBRAS

Match the order

$$f(g) \mapsto h \,|\, ord(g) = ord(h)$$

Cartan Algebra maps to Cartan Algebra

$$f : \mathfrak{h} \mapsto \mathfrak{h}$$

Simple Roots map to Simple Roots

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FROM THE FINITE TO THE LIE

THE SAME INTUITION APPLIES TO LIE ALGEBRAS

Action on generators

$$f_{a,b}(r, s) \mapsto r^a(r^b s)$$

Action on root spaces

$$f : \mathfrak{g}_a \mapsto \mathfrak{g}_{f \cdot a} | f \cdot a = a \circ f^{-1}$$

Automorphisms permute the root spaces

Outer Automorphisms Permute Simple Roots While Preserving the Cartan Matrix

Outer Automorphisms are symmetries of the Dynkin Diagrams

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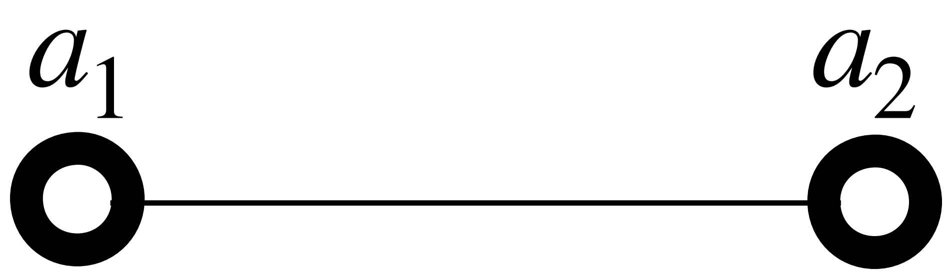
Outer Automorphisms are symmetries of the Dynkin Diagrams

SU(4) $\Rightarrow Out(SU(4)) \equiv \mathbb{Z}_2$

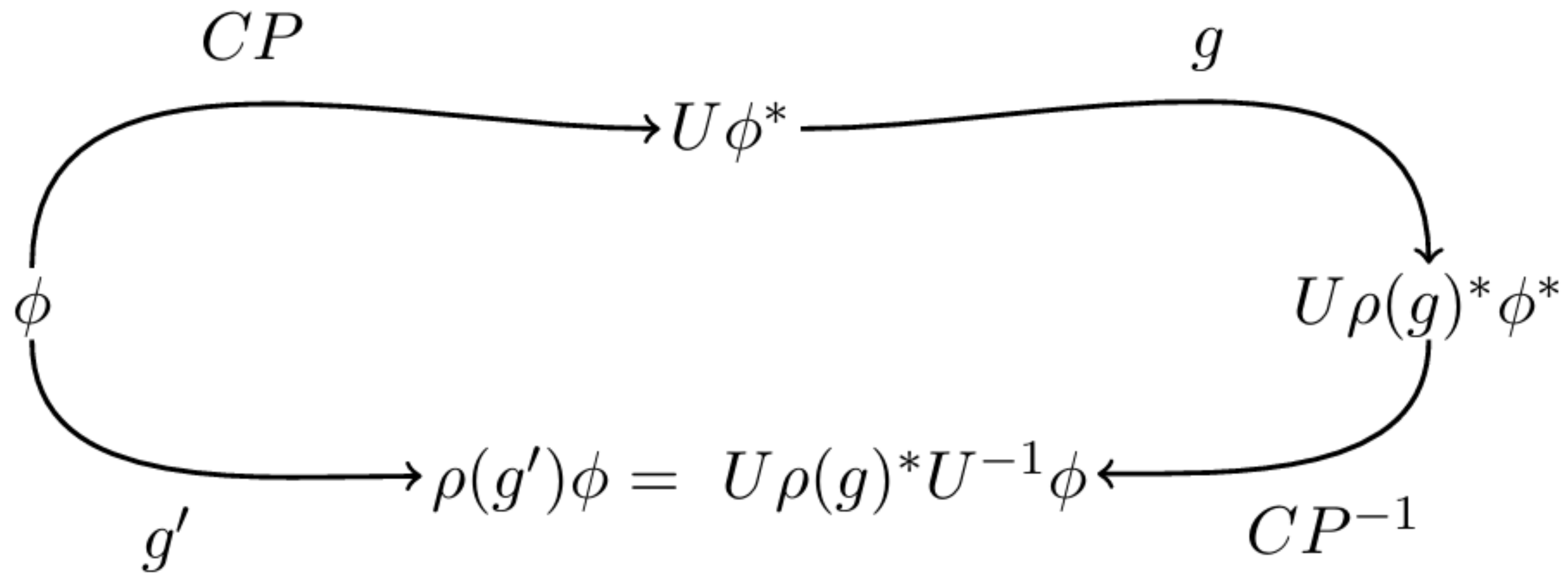
Simple Roots $= a_1 = (e_1 - e_2), a_2 = (e_2 - e_3), a_3 = (e_3 - e_4)$

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \mapsto = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix} \mapsto = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$



How to Use Outer Automorphisms in Physics

Discete Symmetry Compatability with Internal Symmetry Groups



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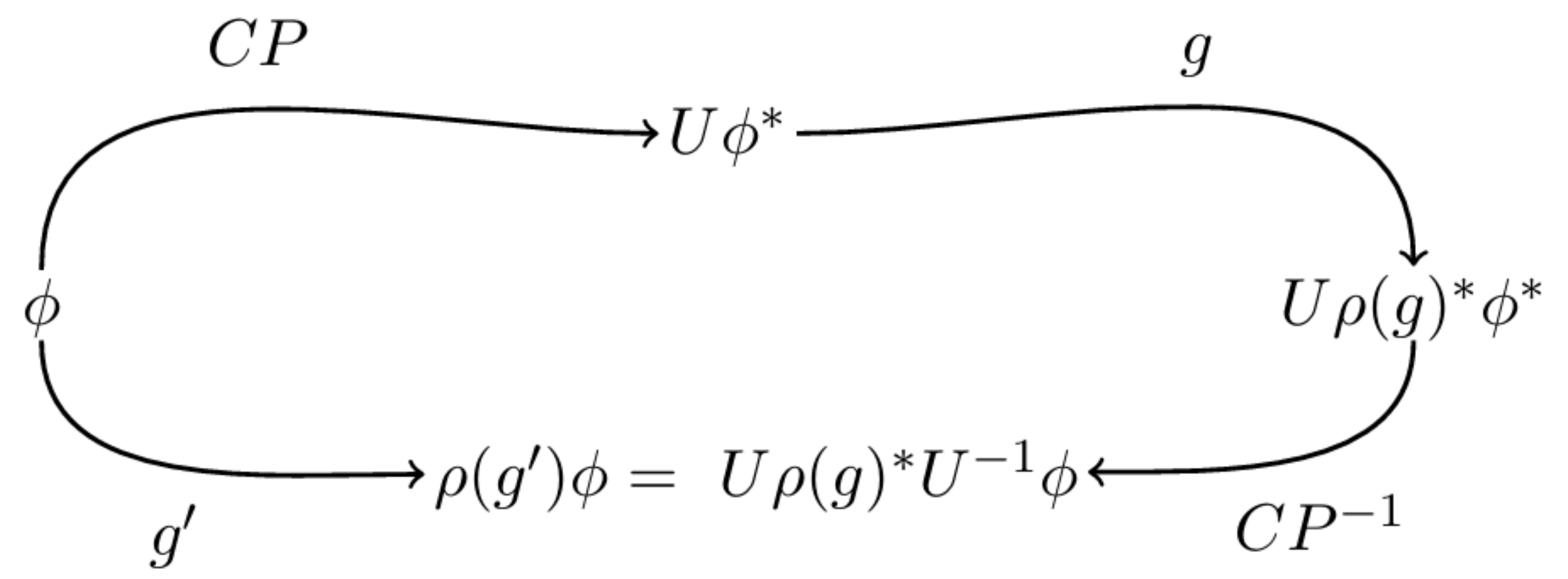

$$U\rho(g)^*U^{-1} = \rho\Big(u(g)\Big)$$

How to Use Outer Automorphisms in Physics

Discete Symmetry Compatability with Internal Symmetry Groups

$$U\rho(g)^*U^{-1} = \rho\Big(u(g)\Big)$$

$$\downarrow$$

$$\rho(h)U\rho(g)^*U^{-1}\rho(h^{-1}) = \rho(h)\rho\Big(u(g)\Big)\rho(h^{-1}) = \rho(hu(g)h^{-1})$$

Fundamental Result of Symmetry Application

$$\mathscr{L}(\phi) \overset{\equiv}{\to} \mathscr{L}(C\phi) \overset{\equiv}{\to} \mathscr{L}(\rho(g)C\phi)$$

$$u : G \mapsto G \qquad conj_h \circ u : G \mapsto G$$

The same Physics are described by Aut. u **and** $conj_h \circ u$

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Discrete Symmetries are implemented by Out

Action On the Full Multiplet

$$U\begin{pmatrix}N\\ \bar{N}\end{pmatrix} = \begin{pmatrix}0 & U_- \\ U_+ & 0\end{pmatrix}\begin{pmatrix}N\\ \bar{N}\end{pmatrix}$$

$$UU^\dagger = \begin{pmatrix}0 & U_- \\ U_+ & 0\end{pmatrix}\begin{pmatrix}0 & U_+ \\ U_- & 0\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$$

$$U^2 = \begin{pmatrix}0 & U_- \\ U_+ & 0\end{pmatrix}\begin{pmatrix}0 & U_- \\ U_+ & 0\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$$

$$\boxed{U_\pm^2 = 1,\, U_- = U_+^\dagger}$$

Action On the Full Multiplet

$$U\rho(g)U = \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} \begin{pmatrix} e^{iT^a\omega^a} & 0 \\ 0 & e^{-iT^{aT}\omega^a} \end{pmatrix} \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} = \begin{pmatrix} U^\dagger e^{-iT^{aT}\omega^a} I & 0 \\ 0 & U e^{iT^a\omega^a} U^\dagger \end{pmatrix}$$

$$U\rho(g)U^\dagger = \rho(u(g)) = \begin{pmatrix} U^\dagger e^{-iT^{aT}\omega^a} U & 0 \\ 0 & U e^{iT^a\omega^a} U^\dagger \end{pmatrix} = \begin{pmatrix} e^{iT^a\eta^a} & 0 \\ 0 & e^{-iT^{aT}\eta^a} \end{pmatrix}$$

Action On the Full Multiplet

$$U\rho(g)U^\dagger = \rho(u(g)) = \begin{pmatrix} U^\dagger e^{-iT^{aT}\omega^a}U & 0 \\ 0 & Ue^{iT^a\omega^a}U^\dagger \end{pmatrix} = \begin{pmatrix} e^{iT^a\eta^a} & 0 \\ 0 & e^{-iT^{aT}\eta^a} \end{pmatrix}$$

$$U^\dagger e^{-iT^{aT}\omega^a}U = e^{iT^a\eta^a} \Rightarrow U^T e^{iT^a\omega^a}U* = e^{-iT^{aT}\eta^a} = Ue^{iT^a\omega^a}U^\dagger$$

$$Ue^{iT^a\omega^a}U^\dagger = U^T e^{iT^a\omega^a}U* \Rightarrow U*Ue^{iT^a\omega^a}U^\dagger U^T = e^{iT^a\omega^a}$$

$$U*Ue^{iT^a\omega^a} = e^{iT^a\omega^a}U*U \Rightarrow U*U = a1$$

$$U^*Ue^{iT^a\omega^a} = e^{iT^a\omega^a}U^*U \Rightarrow U^*U = a1$$

$$U^*U = a1 \Rightarrow U = aU^T, U^T = aU$$

$$U = a^2U \Rightarrow U = \pm 1$$

THE CHARGE CONJUGATION HAS A SYMMETRIC AND ANTISYMMETRIC IMPLEMENTATION

$$detU = a^N detU^T \Rightarrow a^N = 1$$

THE CHARGE CONJUGATION FOR SU(2N) IS SYMMETRIC AND FOR SU(2N+1) THERE EXIST BOTH A SYMMETRIC AND ANTISYMMETRIC IMPLEMENTATIONS