The Lie algebra $\mathfrak{su}(n)$ consists of the set of traceless $n \times n$ anti-hermitian matrices. Following the physicist's convention, we shall multiply each matrix in this set by i to obtain the set of traceless $n \times n$ hermitian matrices. Any such matrix can be expressed as a linear combination of $n^2 - 1$ matrix generators that form the basis of the $\mathfrak{su}(n)$ Lie algebra.

It is convenient to define the following n^2 traceless $n \times n$ matrices,

$$(F^k_\ell)_{ij} = \delta_{\ell i}\delta_{kj} - \frac{1}{n}\delta_{k\ell}\delta_{ij}\,, \tag{1}$$

where ij indicates the row and column of the corresponding matrix (here i and j can take on the values 1,2,$\ldots, n$), and k and ℓ label the n^2 possible matrices F^k_ℓ (where $k, \ell = 1, 2, \ldots, n$). Note that

$$\sum_\ell F^\ell_\ell = 0\,, \tag{2}$$

which means that of the n^2 matrices, F^k_ℓ, only $n^2 - 1$ are independent. These $n^2 - 1$ generators will be employed to construct the basis for the $\mathfrak{su}(n)$ Lie algebra. The corresponding commutation relations are easily obtained,

$$\left[F^k_\ell\,,\, F^m_n\right] = \delta^k_n F^m_\ell - \delta^m_\ell F^k_n\,. \tag{3}$$

The matrices F^k_ℓ satisfy

$$(F^k_\ell)^\dagger = F^\ell_k\,. \tag{4}$$

Thus, we can use the F^k_ℓ to construct $n^2 - 1$ traceless $n \times n$ hermitian matrices by employing suitable linear combinations.

In these notes, we are interested in the $\mathfrak{su}(3)$ Lie algebra. Setting $n = 3$ in the equations above, we define the eight Gell-Mann matrices, which are related to the F^k_ℓ $(\ell, k = 1, 2, 3)$ defined in eq. (1) as follows:[1]

$$\begin{aligned}
\lambda_1 &= F^2_1 + F^1_2\,, & \lambda_2 &= -i(F^2_1 - F^1_2)\,,\\
\lambda_4 &= F^3_1 + F^1_3\,, & \lambda_5 &= -i(F^3_1 - F^1_3)\,,\\
\lambda_6 &= F^3_2 + F^2_3\,, & \lambda_7 &= -i(F^3_2 - F^2_3)\,,\\
\lambda_3 &= F^1_1 - F^2_2\,, & \lambda_8 &= -\sqrt{3}F^3_3 = \sqrt{3}(F^1_1 + F^2_2)\,,
\end{aligned} \tag{5}$$

where we have used eq. (2) to rewrite λ_8 in two different ways. In defining the Gell-Mann matrices above, we have chosen to normalize the $\mathfrak{su}(3)$ generators such that

$$\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}\,. \tag{6}$$

This explains the appearance of the $\sqrt{3}$ in the definition of λ_8 in eq. (5).

[1]Using eq. (4), one can easily check that the Gell-Mann matrices are hermitian as advertised.

The Gell-Mann matrices are the traceless hermitian generators of the $\mathfrak{su}(3)$ Lie algebra, analogous to the Pauli matrices of $\mathfrak{su}(2)$. Using eq. (1) with $n = 3$ and eq. (5), the eight Gell-Mann matrices are explicitly given by:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The Gell-Mann matrices satisfy commutation relation,

$$[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c\,, \qquad \text{where } a, b, c = 1, 2, 3, \ldots, 8\,,$$

where there is an implicit sum over c, and the structure constants f_{abc} are totally antisymmetric under the interchange of any pair of indices. The explicit form of the non-zero $\mathfrak{su}(3)$ structure constants are listed in Table 1.

Table 1: Non-zero structure constants[1] f_{abc} of $\mathfrak{su}(3)$.

abc	f_{abc}	abc	f_{abc}
123	1	345	$\frac{1}{2}$
147	$\frac{1}{2}$	367	$-\frac{1}{2}$
156	$-\frac{1}{2}$	458	$\frac{1}{2}\sqrt{3}$
246	$\frac{1}{2}$	678	$\frac{1}{2}\sqrt{3}$
257	$\frac{1}{2}$		

[1] The f_{abc} are antisymmetric under the permutation of any pair of indices.

The following properties of the Gell-Mann matrices are also useful:

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}\,, \qquad \{\lambda_a, \lambda_b\} = 2d_{abc}\lambda_c + \tfrac{4}{3}\delta_{ab}\mathbf{I}\,,$$

where $\mathbf{I}$ is the 3×3 identity matrix and $\{A, B\} \equiv AB + BA$ is the anticommutator of A and B. It follows that

$$f_{abc} = -\tfrac{1}{4} i\, \mathrm{Tr}\big(\lambda_a [\lambda_b, \lambda_c]\big)\,, \qquad d_{abc} = \tfrac{1}{4} \mathrm{Tr}\big(\lambda_a \{\lambda_b, \lambda_c\}\big)\,.$$

The d_{abc} are totally symmetric under the interchange of any pair of indices. The explicit form of the non-zero d_{abc} are listed in Table 2.

Table 2: Non-zero independent elements of the tensor[2] d_{abc} of $\mathfrak{su}(3)$.

abc	d_{abc}	abc	d_{abc}
118	$\frac{1}{\sqrt{3}}$	355	$\frac{1}{2}$
146	$\frac{1}{2}$	366	$-\frac{1}{2}$
157	$\frac{1}{2}$	377	$-\frac{1}{2}$
228	$\frac{1}{\sqrt{3}}$	448	$-\frac{1}{2\sqrt{3}}$
247	$-\frac{1}{2}$	558	$-\frac{1}{2\sqrt{3}}$
256	$\frac{1}{2}$	668	$-\frac{1}{2\sqrt{3}}$
338	$\frac{1}{\sqrt{3}}$	778	$-\frac{1}{2\sqrt{3}}$
344	$\frac{1}{2}$	888	$-\frac{1}{\sqrt{3}}$

[2]The d_{abc} are symmetric under the permutation of any pair of indices.

Using the explicit form for the structure constants f_{abc}, one can construct the Cartan-Killing metric tensor,[2]

$$g_{ab} = f_{acd} f_{bcd} = 3\delta_{ab}\,,$$

and the inverse metric tensor is $g^{ab} = \frac{1}{3}\delta^{ab}$. The latter can be used to construct the quadratic Casimir operator in the defining representation of $\mathfrak{su}(3)$,

$$C_2 = \tfrac{3}{4} g^{ab}\lambda_a\lambda_b = \tfrac{1}{4}\sum_a (\lambda_a)^2 = \tfrac{4}{3}\mathbf{I}\,,$$

where $\mathbf{I}$ is the 3×3 identity matrix and the overall factor of $\frac{3}{4}$ is conventional.

One can define C_2 for any d-dimensional irreducible representation R of $\mathfrak{su}(3)$. We shall denote the the corresponding traceless hermitian generators in representation R by R_a. The normalization of the matrix generators in the defining representation of $\mathfrak{su}(3)$ will be fixed by $\mathrm{Tr}(R_aR_b) = \frac{1}{2}\delta_{ab}$. Thus, in the defining representation of $\mathfrak{su}(3)$, we identify $R_a = \frac{1}{2}\lambda_a$ [cf. eq. (6)]. In the adjoint representation of $\mathfrak{su}(3)$, we may identify $(R_a)_{bc} = -if_{abc}$.

[2]Since we are employing the physicisit's convention in which the $\mathfrak{su}(3)$ generators $\frac{1}{2}\lambda_a$ are hermitian, the Cartan-Killing metric tensor is positive definite. This is in contrast with the mathematician's convention of anti-hermitian generators, where the corresponding Cartan-Killing metric tensor of $\mathfrak{su}(3)$ is negative definite.

For any irreducible representation R of $\mathfrak{su}(3)$, the Casimir operator is defined by

$$C_2(R) = 3g^{ab}R_aR_b = \sum_a (R_a)^2 = c_{2R}\mathbf{I_d}\,. \tag{7}$$

where $\mathbf{I_d}$ is the $d \times d$ identity matrix. Indeed, by using $[R_a, R_b] = if_{abc}R_c$, it is straightforward to prove that,

$$[R_a, C_2] = 0\,, \qquad \text{for } a = 1, 2, 3, \ldots, 8\,.$$

Since C_2 commutes with all the $\mathfrak{su}(3)$ generators of the irreducible representation R, it follows from Schur's lemma that C_2 is a multiple of the identity, as indicated in eq. (7). As an example, in the adjoint representation A where $(R_a)_{bc} = -if_{abc}$, it follows that

$$C_2(A)_{cd} = f_{abc}f_{abd} = g_{cd} = 3\delta_{cd}\,,$$

which yields $c_{2A} = 3$.

For an irreducible representation of $\mathfrak{su}(3)$ denoted by (n, m), corresponding to a Young diagram with $n+m$ boxes in the first row and n boxes in the second row,[3] the eigenvalue of the quadratic Casimir operator is given by,

$$c_2 = \tfrac{1}{3}(m^2 + n^2 + mn) + m + n\,.$$

The d_{abc} can be employed to construct a cubic Casimir operator in the defining representation of $\mathfrak{su}(3)$,

$$C_3 \equiv \tfrac{1}{8}d_{abc}\lambda_a\lambda_b\lambda_c = \tfrac{10}{9}\mathbf{I}\,,$$

where all repeated indices are summed over. The overall factor of $\frac{1}{8}$ is conventional.

For any d-dimensional irreducible representation R of $\mathfrak{su}(3)$, the cubic Casimir operator is defined by

$$C_3(R) \equiv d_{abc}R_aR_bR_c = c_{3R}\mathbf{I_d}\,. \tag{8}$$

As before, it is straightforward to prove that,

$$[R_a, C_3] = 0\,, \qquad \text{for } a = 1, 2, 3, \ldots, 8\,.$$

Since C_3 commutes with all the $\mathfrak{su}(3)$ generators of the irreducible representation R, it follows from Schur's lemma that C_3 is a multiple of the identity, as indicated in eq. (8).

For an irreducible representation of $\mathfrak{su}(3)$ denoted by (n, m), corresponding to a Young diagram with $n+m$ boxes in the first row and n boxes in the second row, the eigenvalue of the cubic Casimir operator is given by:

$$c_3 = \tfrac{1}{2}(m-n)\left[\tfrac{2}{9}(m+n)^2 + \tfrac{1}{9}mn + m + n + 1\right]\,.$$

In particular, the eigenvalue of cubic Casimir operator in the adjoint representation vanishes.

[3] In particular, $(1, 0)$ is the defining representation and $(1, 1)$ is the adjoint representation of $\mathfrak{su}(3)$.

It is convenient to rewrite the commutation relations of the generators of the $\mathfrak{su}(3)$ Lie algebra in the Cartan-Weyl form. Defining $T_a \equiv \frac{1}{2}\lambda_a$, and using the F^k_ℓ of eq. (1) [with $n=3$], it follows from eq. (3) that,

$$
\begin{aligned}
&\bigl[T_3\,,\,F^2_1\bigr] = F^2_1\,, && \bigl[T_3\,,\,F^1_2\bigr] = -F^1_2\,,\\
&\bigl[T_8\,,\,F^2_1\bigr] = 0\,, && \bigl[T_8\,,\,F^1_2\bigr] = 0\,,\\
&\bigl[T_3\,,\,F^3_1\bigr] = \tfrac{1}{2}F^3_1\,, && \bigl[T_3\,,\,F^1_3\bigr] = -\tfrac{1}{2}F^1_3\,,\\
&\bigl[T_8\,,\,F^3_1\bigr] = \tfrac{1}{2}\sqrt{3}\,F^3_1\,, && \bigl[T_8\,,\,F^1_3\bigr] = -\tfrac{1}{2}\sqrt{3}\,F^1_3\,,\\
&\bigl[T_3\,,\,F^3_2\bigr] = -\tfrac{1}{2}F^3_2\,, && \bigl[T_3\,,\,F^2_3\bigr] = \tfrac{1}{2}F^2_3\,,\\
&\bigl[T_8\,,\,F^3_2\bigr] = \tfrac{1}{2}\sqrt{3}\,F^3_2\,, && \bigl[T_8\,,\,F^2_3\bigr] = -\tfrac{1}{2}\sqrt{3}\,F^2_3\,.
\end{aligned}
$$

These commutation relations can be rewritten in the following notation,

$$\bigl[T_i\,,\,F_{\boldsymbol{\alpha}}\bigr] = \alpha_i F_{\boldsymbol{\alpha}}\,,$$

where $i=3,8$ and $F_{\boldsymbol{\alpha}} = \{F^2_1\,,\,F^1_2\,,\,F^3_1\,,\,F^1_3\,,\,F^3_2\,,\,F^2_3\}$. Using the explicit form of the commutation relations given above, we can read off the six root vectors corresponding to the six generators $F_{\boldsymbol{\alpha}}$,

$$(1\,,\,0)\;,\quad(-1\,,\,0)\;,\quad\left(\frac{1}{2}\,,\,\frac{\sqrt{3}}{2}\right)\,,\quad\left(-\frac{1}{2}\,,\,-\frac{\sqrt{3}}{2}\right)\,,\quad\left(-\frac{1}{2}\,,\,\frac{\sqrt{3}}{2}\right)\,,\quad\left(\frac{1}{2}\,,\,-\frac{\sqrt{3}}{2}\right)\,.$$

Thus, the root diagram of the complexified $\mathfrak{su}(3)$ Lie algebra [that is, $\mathfrak{sl}(3,\mathbb{C})$] is

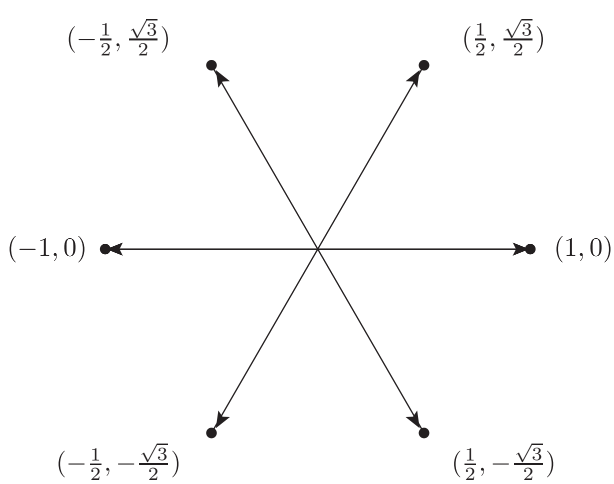

Figure 1: The root diagram for $\mathfrak{sl}(3,\mathbb{C})$.