

The Lie algebra $\mathfrak{su}(n)$ consists of the set of traceless $n \times n$ anti-hermitian matrices. Following the physicist's convention, we shall multiply each matrix in this set by i to obtain the set of traceless $n \times n$ hermitian matrices. Any such matrix can be expressed as a linear combination of $n^2 - 1$ matrix generators that form the basis of the $\mathfrak{su}(n)$ Lie algebra.

It is convenient to define the following n^2 traceless $n \times n$ matrices,

$$(F_\ell^k)_{ij} = \delta_{li}\delta_{kj} - \frac{1}{n}\delta_{k\ell}\delta_{ij}, \quad (1)$$

where ij indicates the row and column of the corresponding matrix (here i and j can take on the values $1, 2, \dots, n$), and k and ℓ label the n^2 possible matrices F_ℓ^k (where $k, \ell = 1, 2, \dots, n$). Note that

$$\sum_\ell F_\ell^\ell = 0, \quad (2)$$

which means that of the n^2 matrices, F_ℓ^k , only $n^2 - 1$ are independent. These $n^2 - 1$ generators will be employed to construct the basis for the $\mathfrak{su}(n)$ Lie algebra. The corresponding commutation relations are easily obtained,

$$[F_\ell^k, F_n^m] = \delta_n^k F_\ell^m - \delta_\ell^m F_n^k. \quad (3)$$

The matrices F_ℓ^k satisfy

$$(F_\ell^k)^\dagger = F_k^\ell. \quad (4)$$

Thus, we can use the F_ℓ^k to construct $n^2 - 1$ traceless $n \times n$ hermitian matrices by employing suitable linear combinations.

In these notes, we are interested in the $\mathfrak{su}(3)$ Lie algebra. Setting $n = 3$ in the equations above, we define the eight Gell-Mann matrices, which are related to the F_ℓ^k ($\ell, k = 1, 2, 3$) defined in eq. (1) as follows:¹

$$\begin{aligned} \lambda_1 &= F_1^2 + F_2^1, & \lambda_2 &= -i(F_1^2 - F_2^1), \\ \lambda_4 &= F_1^3 + F_3^1, & \lambda_5 &= -i(F_1^3 - F_3^1), \\ \lambda_6 &= F_2^3 + F_3^2, & \lambda_7 &= -i(F_2^3 - F_3^2), \\ \lambda_3 &= F_1^1 - F_2^2, & \lambda_8 &= -\sqrt{3}F_3^3 = \sqrt{3}(F_1^1 + F_2^2), \end{aligned} \quad (5)$$

where we have used eq. (2) to rewrite λ_8 in two different ways. In defining the Gell-Mann matrices above, we have chosen to normalize the $\mathfrak{su}(3)$ generators such that

$$\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}. \quad (6)$$

This explains the appearance of the $\sqrt{3}$ in the definition of λ_8 in eq. (5).

¹Using eq. (4), one can easily check that the Gell-Mann matrices are hermitian as advertised.

The Gell-Mann matrices are the traceless hermitian generators of the $\mathfrak{su}(3)$ Lie algebra, analogous to the Pauli matrices of $\mathfrak{su}(2)$. Using eq. (1) with $n = 3$ and eq. (5), the eight Gell-Mann matrices are explicitly given by:

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

The Gell-Mann matrices satisfy commutation relation,

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c, \quad \text{where } a, b, c = 1, 2, 3, \dots, 8,$$

where there is an implicit sum over c , and the structure constants f_{abc} are totally antisymmetric under the interchange of any pair of indices. The explicit form of the non-zero $\mathfrak{su}(3)$ structure constants are listed in Table 1.

Table 1: Non-zero structure constants¹ f_{abc} of $\mathfrak{su}(3)$.

abc	f_{abc}	abc	f_{abc}
123	1	345	$\frac{1}{2}$
147	$\frac{1}{2}$	367	$-\frac{1}{2}$
156	$-\frac{1}{2}$	458	$\frac{1}{2}\sqrt{3}$
246	$\frac{1}{2}$	678	$\frac{1}{2}\sqrt{3}$
257	$\frac{1}{2}$		

¹The f_{abc} are antisymmetric under the permutation of any pair of indices.

The following properties of the Gell-Mann matrices are also useful:

$$\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}, \quad \{\lambda_a, \lambda_b\} = 2d_{abc}\lambda_c + \frac{4}{3}\delta_{ab}\mathbf{I},$$

where \mathbf{I} is the 3×3 identity matrix and $\{A, B\} \equiv AB + BA$ is the anticommutator of A and B . It follows that

$$f_{abc} = -\frac{1}{4}i \text{Tr}(\lambda_a[\lambda_b, \lambda_c]), \quad d_{abc} = \frac{1}{4}\text{Tr}(\lambda_a\{\lambda_b, \lambda_c\}).$$

The d_{abc} are totally symmetric under the interchange of any pair of indices. The explicit form of the non-zero d_{abc} are listed in Table 2.

Table 2: Non-zero independent elements of the tensor² d_{abc} of $\mathfrak{su}(3)$.

abc	d_{abc}	abc	d_{abc}
118	$\frac{1}{\sqrt{3}}$	355	$\frac{1}{2}$
146	$\frac{1}{2}$	366	$-\frac{1}{2}$
157	$\frac{1}{2}$	377	$-\frac{1}{2}$
228	$\frac{1}{\sqrt{3}}$	448	$-\frac{1}{2\sqrt{3}}$
247	$-\frac{1}{2}$	558	$-\frac{1}{2\sqrt{3}}$
256	$\frac{1}{2}$	668	$-\frac{1}{2\sqrt{3}}$
338	$\frac{1}{\sqrt{3}}$	778	$-\frac{1}{2\sqrt{3}}$
344	$\frac{1}{2}$	888	$-\frac{1}{\sqrt{3}}$

²The d_{abc} are symmetric under the permutation of any pair of indices.

Using the explicit form for the structure constants f_{abc} , one can construct the Cartan-Killing metric tensor,²

$$g_{ab} = f_{acd}f_{bcd} = 3\delta_{ab},$$

and the inverse metric tensor is $g^{ab} = \frac{1}{3}\delta^{ab}$. The latter can be used to construct the quadratic Casimir operator in the defining representation of $\mathfrak{su}(3)$,

$$C_2 = \frac{3}{4}g^{ab}\lambda_a\lambda_b = \frac{1}{4}\sum_a(\lambda_a)^2 = \frac{4}{3}\mathbf{I},$$

where \mathbf{I} is the 3×3 identity matrix and the overall factor of $\frac{3}{4}$ is conventional.

One can define C_2 for any d -dimensional irreducible representation R of $\mathfrak{su}(3)$. We shall denote the the corresponding traceless hermitian generators in representation R by R_a . The normalization of the matrix generators in the defining representation of $\mathfrak{su}(3)$ will be fixed by $\text{Tr}(R_a R_b) = \frac{1}{2}\delta_{ab}$. Thus, in the defining representation of $\mathfrak{su}(3)$, we identify $R_a = \frac{1}{2}\lambda_a$ [cf. eq. (6)]. In the adjoint representation of $\mathfrak{su}(3)$, we may identify $(R_a)_{bc} = -if_{abc}$.

²Since we are employing the physicist's convention in which the $\mathfrak{su}(3)$ generators $\frac{1}{2}\lambda_a$ are hermitian, the Cartan-Killing metric tensor is positive definite. This is in contrast with the mathematician's convention of anti-hermitian generators, where the corresponding Cartan-Killing metric tensor of $\mathfrak{su}(3)$ is negative definite.

For any irreducible representation R of $\mathfrak{su}(3)$, the Casimir operator is defined by

$$C_2(R) = 3g^{ab}R_aR_b = \sum_a (R_a)^2 = c_{2R}\mathbf{I}_d. \quad (7)$$

where \mathbf{I}_d is the $d \times d$ identity matrix. Indeed, by using $[R_a, R_b] = if_{abc}R_c$, it is straightforward to prove that,

$$[R_a, C_2] = 0, \quad \text{for } a = 1, 2, 3, \dots, 8.$$

Since C_2 commutes with all the $\mathfrak{su}(3)$ generators of the irreducible representation R , it follows from Schur's lemma that C_2 is a multiple of the identity, as indicated in eq. (7). As an example, in the adjoint representation A where $(R_a)_{bc} = -if_{abc}$, it follows that

$$C_2(A)_{cd} = f_{abc}f_{abd} = g_{cd} = 3\delta_{cd},$$

which yields $c_{2A} = 3$.

For an irreducible representation of $\mathfrak{su}(3)$ denoted by (n, m) , corresponding to a Young diagram with $n + m$ boxes in the first row and n boxes in the second row,³ the eigenvalue of the quadratic Casimir operator is given by,

$$c_2 = \frac{1}{3}(m^2 + n^2 + mn) + m + n.$$

The d_{abc} can be employed to construct a cubic Casimir operator in the defining representation of $\mathfrak{su}(3)$,

$$C_3 \equiv \frac{1}{8}d_{abc}\lambda_a\lambda_b\lambda_c = \frac{10}{9}\mathbf{I},$$

where all repeated indices are summed over. The overall factor of $\frac{1}{8}$ is conventional.

For any d -dimensional irreducible representation R of $\mathfrak{su}(3)$, the cubic Casimir operator is defined by

$$C_3(R) \equiv d_{abc}R_aR_bR_c = c_{3R}\mathbf{I}_d. \quad (8)$$

As before, it is straightforward to prove that,

$$[R_a, C_3] = 0, \quad \text{for } a = 1, 2, 3, \dots, 8.$$

Since C_3 commutes with all the $\mathfrak{su}(3)$ generators of the irreducible representation R , it follows from Schur's lemma that C_3 is a multiple of the identity, as indicated in eq. (8).

For an irreducible representation of $\mathfrak{su}(3)$ denoted by (n, m) , corresponding to a Young diagram with $n + m$ boxes in the first row and n boxes in the second row, the eigenvalue of the cubic Casimir operator is given by:

$$c_3 = \frac{1}{2}(m - n) \left[\frac{2}{9}(m + n)^2 + \frac{1}{9}mn + m + n + 1 \right].$$

In particular, the eigenvalue of cubic Casimir operator in the adjoint representation vanishes.

³In particular, $(1, 0)$ is the defining representation and $(1, 1)$ is the adjoint representation of $\mathfrak{su}(3)$.

It is convenient to rewrite the commutation relations of the generators of the $\mathfrak{su}(3)$ Lie algebra in the Cartan-Weyl form. Defining $T_a \equiv \frac{1}{2}\lambda_a$, and using the F_ℓ^k of eq. (1) [with $n = 3$], it follows from eq. (3) that,

$$\begin{aligned} [T_3, F_1^2] &= F_1^2, & [T_3, F_2^1] &= -F_2^1, \\ [T_8, F_1^2] &= 0, & [T_8, F_2^1] &= 0, \\ [T_3, F_1^3] &= \frac{1}{2}F_1^3, & [T_3, F_3^1] &= -\frac{1}{2}F_3^1, \\ [T_8, F_1^3] &= \frac{1}{2}\sqrt{3}F_1^3, & [T_8, F_3^1] &= -\frac{1}{2}\sqrt{3}F_3^1, \\ [T_3, F_2^3] &= -\frac{1}{2}F_2^3, & [T_3, F_3^2] &= \frac{1}{2}F_3^2, \\ [T_8, F_2^3] &= \frac{1}{2}\sqrt{3}F_2^3, & [T_8, F_3^2] &= -\frac{1}{2}\sqrt{3}F_3^2. \end{aligned}$$

These commutation relations can be rewritten in the following notation,

$$[T_i, F_\alpha] = \alpha_i F_\alpha,$$

where $i = 3, 8$ and $F_\alpha = \{F_1^2, F_2^1, F_1^3, F_3^1, F_2^3, F_3^2\}$. Using the explicit form of the commutation relations given above, we can read off the six root vectors corresponding to the six generators F_α ,

$$(1, 0), \quad (-1, 0), \quad \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \quad \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Thus, the root diagram of the complexified $\mathfrak{su}(3)$ Lie algebra [that is, $\mathfrak{sl}(3, \mathbb{C})$] is

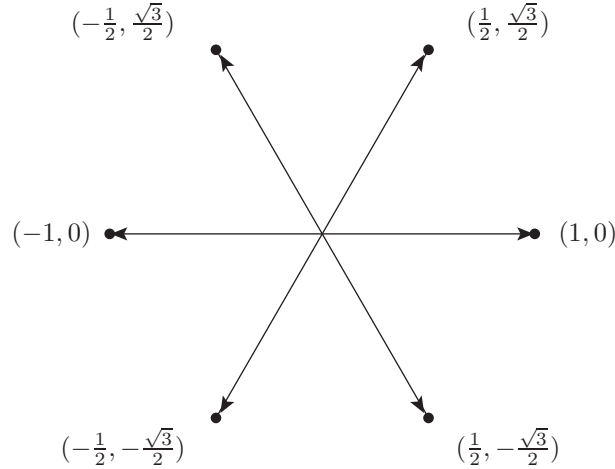


Figure 1: The root diagram for $\mathfrak{sl}(3, \mathbb{C})$.