Local properties of a Lie group

1. The group manifold near the identity

Consider the element $A, B \in G$, where G is a real Lie group of dimension n. We shall parameterize the points on the group manifold such that the coordinates of A on the group manifold are $\vec{\boldsymbol{a}} = (a^1, a^2, \ldots, a^n)$ and the coordinates of B are $\vec{\boldsymbol{b}} = (b^1, b^2, \ldots, b^n)$. It is convenient to define the coordinates on the group manifold such that the identity element (henceforth denoted by E) lies at the origin; i.e., the coordinates of E are $\vec{\boldsymbol{0}} = (0, 0, \ldots, 0)$. The group multiplication law, $C = AB$, will be expressed in terns of the corresponding coordinates,

$$\vec{\boldsymbol{c}} = \vec{\boldsymbol{m}}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}})\,. \tag{1}$$

If A and B lie close to the identity element of G, then we can expand,

$$m^i(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}) = a^i + b^i + \alpha^i_{jk} a^j a^k + \beta^i_{jk} b^j b^k + c^i_{jk} a^j b^k + \ldots\,, \tag{2}$$

where we are employing the Einstein summation convention in which repeated indices of different heights are summed over. The ellipsis indicates that we have dropped terms cubic or higher in the coordinates. Note that $AE = A$ and $EB = B$ imply that $m^i(\vec{\boldsymbol{a}}, \vec{\boldsymbol{0}}) = a^i$ and $m^i(\vec{\boldsymbol{0}}, \vec{\boldsymbol{b}}) = b^i$. It then follows that $\alpha^i_{jk} = \beta^i_{jk} = 0$ so that

$$m^i(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}) = a^i + b^i + c^i_{jk} a^j b^k + \ldots \tag{3}$$

In particular, we can identify,

$$c^i_{jk} = \left.\frac{\partial^2 m^i(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}})}{\partial a^j \partial b^k}\right|_{\vec{\boldsymbol{a}} = \vec{\boldsymbol{b}} = 0}. \tag{4}$$

We next investigate the implications of an associative group multiplication law, $C(AB) = (CA)B$. In terms of the coordinates,

$$\vec{\boldsymbol{m}}\big(\vec{\boldsymbol{c}}, \vec{\boldsymbol{m}}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}})\big) = \vec{\boldsymbol{m}}\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{c}}, \vec{\boldsymbol{a}}), \vec{\boldsymbol{b}}\big)\,. \tag{5}$$

Differentiating eq. (5) with respect to c^k and employing the chain rule on the right hand side yields

$$\frac{\partial m^i\big(\vec{\boldsymbol{c}}, \vec{\boldsymbol{m}}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}})\big)}{\partial c^k} = \frac{\partial m^i\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{c}}, \vec{\boldsymbol{a}}), \vec{\boldsymbol{b}}\big)}{\partial m^j(\vec{\boldsymbol{c}}, \vec{\boldsymbol{a}})} \frac{\partial m^j(\vec{\boldsymbol{c}}, \vec{\boldsymbol{a}})}{\partial c^k}\,. \tag{6}$$

To examine the local behavior of the associative group multiplication law near the identity, we introduce the quantity,

$$\Theta^i_j(\vec{\boldsymbol{a}}) \equiv \left.\frac{\partial m^i(\vec{\boldsymbol{c}}, \vec{\boldsymbol{a}})}{\partial c^j}\right|_{\vec{\boldsymbol{c}} = 0}. \tag{7}$$

Taking the $c^k \to 0$ limit of eq. (6) and using $m^i(\vec{\boldsymbol{0}},\vec{\boldsymbol{a}}) = a^i$, we obtain,

$$\Theta^i_k(\vec{\boldsymbol{m}}) = \frac{\partial m^i(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})}{\partial a^j}\,\Theta^j_k(\vec{\boldsymbol{a}})\,. \tag{8}$$

where the argument of Θ^i_k in eq. (8) is $\vec{\boldsymbol{m}} \equiv \vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})$.

The matrix $\Theta^i_j(\vec{\boldsymbol{a}})$ is nonsingular. This can be easily understood by writing $b^i \equiv m^i(\vec{\boldsymbol{c}},\vec{\boldsymbol{a}})$. By holding $\vec{\boldsymbol{a}}$ fixed, we can view this latter relation as a change of coordinates on the group manifold from $\vec{\boldsymbol{c}}$ to $\vec{\boldsymbol{b}}$. That is, we are changing coordinates on the group manifold by right multiplication by a fixed element $A \in G$. The Jacobian matrix of this coordinate transformation is $\partial b^i/\partial c^j$. Since this coordinate change is nonsingular, the inverse Jacobian matrix, $\partial c^j/\partial b^i$ exists at all points in the group manifold. Hence, it follows that the inverse $\Theta^{-1\,j}_k(\vec{\boldsymbol{a}})$ exists and satisfies

$$\Theta^j_k(\vec{\boldsymbol{a}})\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}}) = \delta^j_\ell\,. \tag{9}$$

A more direct proof that Θ^j_k is invertible goes as follows (see, e.g. Ref. 4). Let us denote the coordinates of A^{-1} by $\vec{\boldsymbol{\imath}}(\vec{\boldsymbol{a}})$. The identity $ABB^{-1} = A$ in coordinates reads,

$$m^i\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}),\vec{\boldsymbol{\imath}}(\vec{\boldsymbol{b}})\big) = a^i\,. \tag{10}$$

Taking the derivative with respect to a^j and employing the chain rule yields,

$$\frac{\partial m^i\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}),\vec{\boldsymbol{\imath}}(\vec{\boldsymbol{b}})\big)}{\partial m^k(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})}\,\frac{\partial m^k(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})}{\partial a^j} = \delta^i_j\,. \tag{11}$$

Eq. (11) implies that $\partial m^k(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})/\partial a^j$ is nonsingular. A similar argument starting from $A^{-1}AB = B$ implies that $\partial m^k(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})/\partial b^j$ is nonsingular. Of course, these results are a consequence of the fact that the transformations $\vec{\boldsymbol{r}}(\vec{\boldsymbol{a}}) \equiv \vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})$ with $\vec{\boldsymbol{b}}$ fixed and $\vec{\boldsymbol{r}}(\vec{\boldsymbol{b}}) \equiv \vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})$ with $\vec{\boldsymbol{a}}$ fixed are necessarily invertible.

Having established that the inverse $\Theta^{-1\,j}_k(\vec{\boldsymbol{a}})$ exists, we can now multiply eq. (8) by $\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})$ to obtain,

$$\frac{\partial m^i(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})}{\partial a^\ell} = \Theta^i_k(\vec{\boldsymbol{m}})\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})\,. \tag{12}$$

As a check of eq. (12), let us take the limit of $\vec{\boldsymbol{a}} \to \vec{\boldsymbol{0}}$. Then, $\Theta^{-1\,k}_j(\vec{\boldsymbol{0}}) = \delta^k_j$ and $\vec{\boldsymbol{m}}(\vec{\boldsymbol{0}},\vec{\boldsymbol{b}}) = \vec{\boldsymbol{b}}$, in which case eq. (12) yields

$$\Theta^i_j(\vec{\boldsymbol{b}}) = \left.\frac{\partial m^i(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})}{\partial a^j}\right|_{\vec{\boldsymbol{a}}=0}\,. \tag{13}$$

which is consistent with the definition given in eq. (7).

Starting with eq. (12), we shall impose the integrability condition,

$$\frac{\partial m^i(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})}{\partial a^\ell \partial a^j} = \frac{\partial m^i(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})}{\partial a^j \partial a^\ell}\,. \tag{14}$$

It follows that

$$\frac{\partial}{\partial a^\ell}\left[\Theta^i_k(\vec{\boldsymbol{m}})\Theta^{-1\,k}_j(\vec{\boldsymbol{a}})\right] = \frac{\partial}{\partial a^j}\left[\Theta^i_k(\vec{\boldsymbol{m}})\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})\right]\,, \tag{15}$$

where $\vec{\boldsymbol{m}} \equiv \vec{\boldsymbol{m}}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}})$. Expanding out eq. (15) yields,

$$\Theta^i_k(\vec{\boldsymbol{m}}) \left(\frac{\partial}{\partial a^\ell} \Theta^{-1\,k}_j(\vec{\boldsymbol{a}}) - \frac{\partial}{\partial a^j} \Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}}) \right) = \Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}}) \frac{\partial}{\partial a^j} \Theta^i_k(\vec{\boldsymbol{m}}) - \Theta^{-1\,k}_j(\vec{\boldsymbol{a}}) \frac{\partial}{\partial a^\ell} \Theta^i_k(\vec{\boldsymbol{m}}) \,. \tag{16}$$

The derivatives on the right hand side above are evaluated with the help of the chain rule. For example,

$$\frac{\partial}{\partial a^j} \Theta^i_k(\vec{\boldsymbol{m}}) = \frac{\partial m^n}{\partial a^j} \frac{\partial}{\partial m^n} \Theta^i_k(\vec{\boldsymbol{m}}) = \Theta^n_p(\vec{\boldsymbol{m}}) \Theta^{-1\,p}_j(\vec{\boldsymbol{a}}) \frac{\partial}{\partial m^n} \Theta^i_k(\vec{\boldsymbol{m}}) \,, \tag{17}$$

where we have used eq. (12) in the final step to evaluate $\partial m^n / \partial a^j$. We can now insert eq. (17) (and a similar result with the index j replaced by ℓ) back into eq. (16). Our strategy is to manipulate the resulting equation so that variables depending on $\vec{\boldsymbol{a}}$ appear on the left hand side whereas variables depending on $\vec{\boldsymbol{m}}$ appear on the right hand side. To accomplish this, we multiply eq. (16) by $\Theta^{-1\,r}_i(\vec{\boldsymbol{m}}) \Theta^\ell_s(\vec{\boldsymbol{a}}) \Theta^j_t(\vec{\boldsymbol{a}})$ on both sides of the equation. The end result is,

$$\begin{aligned} &\left(\frac{\partial}{\partial a^\ell} \Theta^{-1\,k}_j(\vec{\boldsymbol{a}}) - \frac{\partial}{\partial a^j} \Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}}) \right) \Theta^\ell_s(\vec{\boldsymbol{a}}) \Theta^j_t(\vec{\boldsymbol{a}}) \\ &\qquad = \left[\Theta^n_t(\vec{\boldsymbol{m}}) \frac{\partial}{\partial m^n} \Theta^i_s(\vec{\boldsymbol{m}}) - \Theta^n_s(\vec{\boldsymbol{m}}) \frac{\partial}{\partial m^n} \Theta^i_t(\vec{\boldsymbol{m}}) \right] \Theta^{-1\,k}_i(\vec{\boldsymbol{m}}) \,. \end{aligned} \tag{18}$$

We have succeeded in the separation of variables technique. Thus, we can conclude that both sides of eq. (18) must be equal to some constant, which we henceforth denote by f^k_{st}. These are the *structure constants* of the Lie group. In particular,

$$f^k_{st} = \left(\frac{\partial}{\partial a^\ell} \Theta^{-1\,k}_j(\vec{\boldsymbol{a}}) - \frac{\partial}{\partial a^j} \Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}}) \right) \Theta^\ell_s(\vec{\boldsymbol{a}}) \Theta^j_t(\vec{\boldsymbol{a}}) \,. \tag{19}$$

Due to the separation of variables in eq. (18), f^k_{st} is a constant, independent of $\vec{\boldsymbol{a}}$. Thus, we are free to take the limit of $\vec{\boldsymbol{a}} \to \vec{\boldsymbol{0}}$ on the right hand side of eq. (19) without changing the values of f^k_{st}. That is, we can assume that the group elements A and B are close to the identity E and employ eqs. (3) and (7) to evaluate,

$$\Theta^\ell_s(\vec{\boldsymbol{a}}) \simeq \delta^\ell_s + c^\ell_{sn} a^n \,, \qquad\qquad \Theta^{-1\,k}_j(\vec{\boldsymbol{a}}) \simeq \delta^k_j - c^k_{jn} a^n \,, \tag{20}$$

where we have dropped terms quadratic in the coordinates. Plugging these results back into the right hand side of eq. (19) and taking $\vec{\boldsymbol{a}} \to \vec{\boldsymbol{0}}$ at the end of the computation yields,

$$f^k_{st} = (c^k_{\ell j} - c^k_{j\ell}) \delta^\ell_s \delta^j_t = c^k_{st} - c^k_{ts} \,. \tag{21}$$

Note that eqs. (18) and (19) imply that

$$\Theta^n_t(\vec{\boldsymbol{a}}) \frac{\partial \Theta^i_s(\vec{\boldsymbol{a}})}{\partial a^n} - \Theta^n_s(\vec{\boldsymbol{a}}) \frac{\partial \Theta^i_t(\vec{\boldsymbol{a}})}{\partial a^n} = f^k_{st} \Theta^i_k(\vec{\boldsymbol{a}}) \,. \tag{22}$$

In obtaining eq. (22), we have replaced the dummy variable $\vec{\boldsymbol{m}}$ by $\vec{\boldsymbol{a}}$.

2. Properties of the structure constants

The structure constants of a Lie group are given in eq. (21). One immediate consequence is that f^k_{st} is antisymmetric under the interchange of its two lower indices,

$$f^k_{st} = -f^k_{ts}\,. \tag{23}$$

In particular, for a one-dimensional Lie group (i.e., $n = 1$), it follows that $f = 0$.

A second important property of the f^k_{st} can be obtained as follows. We first rewrite eq. (19) as,

$$f^k_{st}\,\Theta^{-1\,s}_{\ell}(\vec{\boldsymbol{a}})\,\Theta^{-1\,t}_{j}(\vec{\boldsymbol{a}}) = \left(\frac{\partial}{\partial a^\ell}\Theta^{-1\,k}_{j}(\vec{\boldsymbol{a}}) - \frac{\partial}{\partial a^j}\Theta^{-1\,k}_{\ell}(\vec{\boldsymbol{a}})\right)\,. \tag{24}$$

Taking a derivative of eq. (24) with respect to a^p and multiplying the resulting expression by $\Theta^p_x\Theta^j_w\Theta^\ell_v$ yields,

$$\Theta^p_x\Theta^j_w\Theta^\ell_v\left[\frac{\partial^2}{\partial a^\ell\partial a^p}\,\Theta^{-1\,k}_{j} - \frac{\partial^2}{\partial a^j\partial a^p}\,\Theta^{-1\,k}_{\ell}\right] = f^k_{sw}\,\Theta^p_x\,\Theta^\ell_v\frac{\partial}{\partial a^p}\,\Theta^{-1\,s}_{\ell} + f^k_{vs}\,\Theta^p_x\,\Theta^j_w\frac{\partial}{\partial a^p}\,\Theta^{-1\,s}_{j}\,. \tag{25}$$

Note that we have replaced the dummy index t with s in the last term on the right hand side of eq. (25). Next, we rewrite eq. (25) by making the following index interchanges, $x \leftrightarrow v$ and $p \leftrightarrow \ell$, and then subtracting the resulting equation from eq. (25). The end result is,

$$\begin{aligned}\Theta^p_x\Theta^j_w\Theta^\ell_v\left[\frac{\partial^2}{\partial a^j\partial a^\ell}\,\Theta^{-1\,k}_{p} - \frac{\partial^2}{\partial a^j\partial a^p}\,\Theta^{-1\,k}_{\ell}\right] &= f^k_{sw}\,\Theta^p_x\,\Theta^\ell_v\left(\frac{\partial}{\partial a^p}\,\Theta^{-1\,s}_{\ell} - \frac{\partial}{\partial a^\ell}\,\Theta^{-1\,s}_{p}\right)\\ &\quad + f^k_{vs}\,\Theta^p_x\,\Theta^j_w\frac{\partial}{\partial a^p}\,\Theta^{-1\,s}_{j} - f^k_{xs}\,\Theta^\ell_v\,\Theta^j_w\frac{\partial}{\partial a^\ell}\,\Theta^{-1\,s}_{j}\,,\end{aligned} \tag{26}$$

after noting that the mixed partial derivatives satisfy $\partial^2/\partial a^\ell\partial a^p = \partial^2/\partial a^p\partial a^\ell$. Likewise, we can rewrite eq. (25) by making the following index interchanges, $x \leftrightarrow w$ and $p \leftrightarrow j$, and then subtracting the resulting equation from eq. (25). The end result is,

$$\begin{aligned}\Theta^p_x\Theta^j_w\Theta^\ell_v\left[\frac{\partial^2}{\partial a^\ell\partial a^p}\,\Theta^{-1\,k}_{j} - \frac{\partial^2}{\partial a^\ell\partial a^j}\,\Theta^{-1\,k}_{p}\right] &= f^k_{vs}\,\Theta^p_x\,\Theta^j_w\left(\frac{\partial}{\partial a^p}\,\Theta^{-1\,s}_{j} - \frac{\partial}{\partial a^j}\,\Theta^{-1\,s}_{p}\right)\\ &\quad + f^k_{sw}\,\Theta^p_x\,\Theta^\ell_v\frac{\partial}{\partial a^p}\,\Theta^{-1\,s}_{\ell} - f^k_{sx}\,\Theta^j_w\,\Theta^\ell_v\frac{\partial}{\partial a^j}\,\Theta^{-1\,s}_{\ell}\,.\end{aligned} \tag{27}$$

Adding eqs. (26) and (27) and employing eqs. (19) and (23) yields,

$$\begin{aligned}\Theta^p_x\Theta^j_w\Theta^\ell_v\left[\frac{\partial^2}{\partial a^\ell\partial a^p}\,\Theta^{-1\,k}_{j} - \frac{\partial^2}{\partial a^j\partial a^p}\,\Theta^{-1\,k}_{\ell}\right] &= f^k_{sw}f^s_{xv} + f^k_{vs}f^s_{xw} + f^k_{xs}f^s_{wv}\\ &\quad + f^k_{vs}\,\Theta^p_x\,\Theta^j_w\frac{\partial}{\partial a^p}\,\Theta^{-1\,s}_{j} + f^k_{sw}\,\Theta^p_x\,\Theta^\ell_v\frac{\partial}{\partial a^p}\,\Theta^{-1\,s}_{\ell}\,.\end{aligned} \tag{28}$$

Finally, after subtracting eq. (25) from eq. (28), we end up with

$$f^k_{sw}f^s_{xv} + f^k_{vs}f^s_{xw} + f^k_{xs}f^s_{wv} = 0\,. \tag{29}$$

That is, the structure constants, f^k_{sw}, satisfy the Jacobi identity.

3. Basis vectors of the Lie algebra and their commutators

We now demonstrate that the f^{k}_{sw} are the structure constants of the Lie algebra $\mathfrak{g}$ corresponding to the Lie group G. Consider an analytic curve, $A(t)$, that lies in the group manifold and passes through the identity, $A(t=0)=E$. The analytic curve $A(t)$ is parameterized by $t\in\mathbb{R}$, where $\vec{\boldsymbol{a}}(t)=\bigl(a^1(t),a^2(t),\ldots,a^n(t)\bigr)$ are the coordinates of A, such that $\vec{\boldsymbol{a}}(t=0)=(0,0,\ldots,0)$ are the coordinates of E. The elements of a Lie algebra are tangent vectors at the identity element of the group manifold. In particular, a tangent vector $\mathcal{A}\in\mathfrak{g}$ is identified by the value of the slope of an analytic curve $A(t)\equiv A\bigl(\vec{\boldsymbol{a}}(t)\bigr)$ at $t=0$. For any element $\mathcal{A}\in\mathfrak{g}$, we can employ the chain rule to write,

$$\mathcal{A}=\frac{d}{dt}A\bigl(\vec{\boldsymbol{a}}(t)\bigr)\bigg|_{t=0}=\sum_{i=1}^{n}\frac{\partial A(\vec{\boldsymbol{a}})}{\partial a^i}\,\frac{\partial a^i}{\partial t}\bigg|_{t=0}=\sum_{i=1}^{n}v^i\mathcal{A}_i\,, \tag{30}$$

where $v^i\equiv(\partial a^i/\partial t)_{t=0}$ are the components of $\mathcal{A}$, and the basis vectors of the Lie algebra are given by,

$$\mathcal{A}_i=\left(\frac{\partial A(\vec{\boldsymbol{a}})}{\partial a^i}\right)_{\vec{\boldsymbol{a}}=0}\,. \tag{31}$$

Associativity of the group multiplication in G implies that

$$A(\vec{\boldsymbol{c}})A\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\bigr)=A\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{c}},\vec{\boldsymbol{a}})\bigr)A(\vec{\boldsymbol{b}})\,. \tag{32}$$

We now differentiate eq. (32) with respect to c^k and then $\vec{\boldsymbol{c}}=0$. In light of eq. (31),

$$\mathcal{A}_k\,A\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\bigr)=\left(\frac{\partial A\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{c}},\vec{\boldsymbol{a}})\bigr)}{\partial m^i}\,\frac{\partial m^i(\vec{\boldsymbol{c}},\vec{\boldsymbol{a}})}{\partial c^k}\right)_{\vec{\boldsymbol{c}}=0}A(\vec{\boldsymbol{b}})\,. \tag{33}$$

after employing the chain rule of differentiation. Using eq. (7) and $\vec{\boldsymbol{m}}(\vec{\boldsymbol{0}},\vec{\boldsymbol{a}})=\vec{\boldsymbol{a}}$, it follows that

$$\mathcal{A}_k\,A\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\bigr)=\frac{\partial A(\vec{\boldsymbol{a}})}{\partial a^i}\,\Theta^i_{\ k}(\vec{\boldsymbol{a}})\,A(\vec{\boldsymbol{b}})\,. \tag{34}$$

Hence,

$$\frac{\partial A(\vec{\boldsymbol{a}})}{\partial a^i}\,A(\vec{\boldsymbol{b}})=\mathcal{A}_k\,\Theta^{-1\,k}_{\ i}(\vec{\boldsymbol{a}})\,A\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\bigr)\,. \tag{35}$$

Our strategy going forward is similar to the analysis that yielded eq. (19). Namely impose an integrability condition and then look to separate variables. The integrability condition is this case can be expressed as an equality of mixed partial derivatives,

$$\frac{\partial^2 A(\vec{\boldsymbol{a}})}{\partial a^i\partial a^j}\,A(\vec{\boldsymbol{b}})=\frac{\partial^2 A(\vec{\boldsymbol{a}})}{\partial a^j\partial a^i}\,A(\vec{\boldsymbol{b}})\,. \tag{36}$$

Then, eq. (35) yields,

$$\mathcal{A}_k\,\frac{\partial}{\partial a^j}\Bigl(\Theta^{-1\,k}_{\ i}(\vec{\boldsymbol{a}})\,A\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\bigr)\Bigr)=\mathcal{A}_k\,\frac{\partial}{\partial a^i}\Bigl(\Theta^{-1\,k}_{\ j}(\vec{\boldsymbol{a}})\,A\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\bigr)\Bigr)\,. \tag{37}$$

Using the product rule for derivatives,

$$\mathcal{A}_k\left(\frac{\partial}{\partial a^j}\Theta_i^{-1\,k}(\vec{\boldsymbol{a}})-\frac{\partial}{\partial a^i}\Theta_j^{-1\,k}(\vec{\boldsymbol{a}})\right)A\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\big)$$
$$=\mathcal{A}_k\left(\Theta_j^{-1\,k}(\vec{\boldsymbol{a}})\,\frac{\partial}{\partial a^i}A\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\big)-\Theta_i^{-1\,k}(\vec{\boldsymbol{a}})\,\frac{\partial}{\partial a^j}A\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\big)\right)\,. \tag{38}$$

Multiplying the above result on the right by $A^{-1}\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\big)\,\Theta_\ell^j(\vec{\boldsymbol{a}})\,\Theta_n^i(\vec{\boldsymbol{a}})$, we end up with,

$$\mathcal{A}_k\left(\frac{\partial}{\partial a^j}\Theta_i^{-1\,k}(\vec{\boldsymbol{a}})-\frac{\partial}{\partial a^i}\Theta_j^{-1\,k}(\vec{\boldsymbol{a}})\right)\Theta_\ell^j(\vec{\boldsymbol{a}})\,\Theta_n^i(\vec{\boldsymbol{a}})$$
$$=\left(\mathcal{A}_\ell\,\Theta_n^i(\vec{\boldsymbol{a}})\,\frac{\partial}{\partial a^i}A\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\big)-\mathcal{A}_n\,\Theta_\ell^j(\vec{\boldsymbol{a}})\,\frac{\partial}{\partial a^j}A\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\big)\right)A^{-1}\big(\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\big)\,. \tag{39}$$

Using eq. (19), we recognize that the right hand side of eq. (39) is given by $f_{\ell n}^{k}\mathcal{A}_k$. Hence, it follows that

$$\left(\mathcal{A}_\ell\,\Theta_n^i(\vec{\boldsymbol{a}})\,\frac{\partial}{\partial a^i}A(\vec{\boldsymbol{m}})-\mathcal{A}_n\,\Theta_\ell^j(\vec{\boldsymbol{a}})\,\frac{\partial}{\partial a^j}A(\vec{\boldsymbol{m}})\right)A^{-1}(\vec{\boldsymbol{m}})=f_{\ell n}^{k}\mathcal{A}_k\,, \tag{40}$$

where $\vec{\boldsymbol{m}}\equiv\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})$. We can simplify the right hand side of eq. (40) with the help of the chain rule. In particular, note that

$$\Theta_n^i(\vec{\boldsymbol{a}})\,\frac{\partial}{\partial a^i}A(\vec{\boldsymbol{m}})=\Theta_n^i(\vec{\boldsymbol{a}})\,\frac{\partial A(\vec{\boldsymbol{m}})}{\partial m^k}\,\frac{\partial m^k(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})}{\partial a^i}=\Theta_n^i(\vec{\boldsymbol{a}})\,\frac{\partial A(\vec{\boldsymbol{m}})}{\partial m^k}\,\Theta_p^k(\vec{\boldsymbol{m}})\,\Theta_i^{-1\,p}(\vec{\boldsymbol{a}})$$
$$=\frac{\partial A(\vec{\boldsymbol{m}})}{\partial m^k}\,\Theta_n^k(\vec{\boldsymbol{m}})\,, \tag{41}$$

where we have used eq. (12) in the penultimate step above. Consequently, we can rewrite eq. (40) as follows:

$$\left(\mathcal{A}_\ell\,\Theta_n^k(\vec{\boldsymbol{m}})\,\frac{\partial A(\vec{\boldsymbol{m}})}{\partial m^k}-\mathcal{A}_n\,\Theta_\ell^k(\vec{\boldsymbol{m}})\,\frac{\partial A(\vec{\boldsymbol{m}})}{\partial m^k}\right)A^{-1}(\vec{\boldsymbol{m}})=f_{\ell n}^{k}\mathcal{A}_k\,. \tag{42}$$

The separation of variables is complete. The left hand side of eq. (42) depends on $\vec{\boldsymbol{m}}$ whereas the right hand side is independent of $\vec{\boldsymbol{m}}$. This implies that eq. (42) is consistent only if the left-hand side of eq. (42) is actually independent of $\vec{\boldsymbol{m}}$. Hence, to evaluate the left hand side of eq. (42), we are free to take the limit $\vec{\boldsymbol{m}}\to 0$. Using eq. (31) and noting that $A(\vec{\boldsymbol{0}})=A^{-1}(\vec{\boldsymbol{0}})=E$ and $\Theta_n^k(\vec{\boldsymbol{0}})=\delta_n^k$ [cf. eq. (20)], we see that eq. (42) reduces to

$$\mathcal{A}_\ell\mathcal{A}_n-\mathcal{A}_n\mathcal{A}_\ell=f_{\ell n}^{k}\mathcal{A}_k\,. \tag{43}$$

That is,

$$\big[\mathcal{A}_\ell\,,\,\mathcal{A}_n\big]=f_{\ell n}^{k}\mathcal{A}_k\,. \tag{44}$$

Indeed, the $f_{\ell n}^{k}$ are the structure constants of the Lie algebra $\mathfrak{g}$.

4. Lie group action on a manifold

Lie groups can act on manifolds. We define a Lie transformation group by the action of a Lie group G on a manifold M. It is convenient to consider a left action in which the group G acts from the left on M. In this section, we assume that the group action is effective. This means that the only group element whose action on M has no effect on any of the points of M is the identity element of G. The left group action is given by

$$x'^{\,i} = \Phi^i(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})\,, \tag{45}$$

where A, with coordinates $\vec{\boldsymbol{a}} = (a^1, a^2, \ldots, a^n) \in G$ acts on $\vec{\boldsymbol{x}} \in M$ from the left and sends the latter to $\vec{\boldsymbol{x}}' \in M$. The index i takes on m possible values, where m is the dimension of the manifold M. Note that m need not be the same as $n = \dim G$.

The function Φ which represents the group action has the following properties:

$$1. \quad \Phi^i(\vec{\boldsymbol{0}}\,;\,\vec{\boldsymbol{x}}) = x^i\,, \tag{46}$$

$$2. \quad \Phi^i\bigl(\vec{\boldsymbol{b}}\,;\,\vec{\boldsymbol{\Phi}}(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})\bigr) = \Phi^i\bigl(\vec{\boldsymbol{m}}(\vec{\boldsymbol{b}},\vec{\boldsymbol{a}})\,;\,\vec{\boldsymbol{x}}\bigr)\,. \tag{47}$$

One often applies a shorthand notation by writing $\vec{\boldsymbol{x}}' = A\vec{\boldsymbol{x}}$ for the (left) group action. The group G acts effectively on the manifold M if $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}$ for all $\vec{\boldsymbol{x}} \in M$ implies that $A = E$ (where E is the identity element of G). The two properties above correspond to $E\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}$ and $A(B\vec{\boldsymbol{x}}) = (AB)\vec{\boldsymbol{x}}$, respectively, for $\vec{\boldsymbol{x}} \in M$ and $A, B \in G$.

If we differentiate the second property above with respect to b^k and employ the chain rule, we obtain,

$$\frac{\partial \Phi^i\bigl(\vec{\boldsymbol{b}}\,;\,\vec{\boldsymbol{\Phi}}(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})\bigr)}{\partial b^k} = \frac{\partial \Phi^i(\vec{\boldsymbol{m}}(\vec{\boldsymbol{b}},\vec{\boldsymbol{a}})\,;\,\vec{\boldsymbol{x}})}{\partial m^j}\,\frac{\partial m^j}{\partial b^k}\,, \tag{48}$$

where $m^j \equiv\ m^j(\vec{\boldsymbol{b}},\vec{\boldsymbol{a}})$ and there is an implicit sum over the repeated index j. It is convenient to introduce the matrix,

$$u^i_k(\vec{\boldsymbol{x}}) = \left.\frac{\partial \Phi^i(\vec{\boldsymbol{b}}\,;\,\vec{\boldsymbol{x}})}{\partial b^k}\right|_{\vec{\boldsymbol{b}}=0}\,. \tag{49}$$

Note the similarity between this definition and that of eq. (7). Indeed, if in the special case where the manifold M is taken to be the group manifold G, then $u^i_k = \Theta^i_k$. However, in contrast to Θ^i_k, one cannot assume that the matrix $u^i_k(\vec{\boldsymbol{x}})$ is nonsingular for the general case where M is an m-dimensional manifold.[1] If we now set $\vec{\boldsymbol{b}} = 0$ in eq. (48) and employ $\vec{\boldsymbol{m}}(\vec{\boldsymbol{0}},\vec{\boldsymbol{a}}) = \vec{\boldsymbol{a}}$, it follows that

$$u^i_k(\vec{\boldsymbol{\Phi}}) = \frac{\partial \Phi^i(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})}{\partial a^j}\,\Theta^j_k(\vec{\boldsymbol{a}})\,, \tag{50}$$

after using eq. (7), where $\vec{\boldsymbol{\Phi}} \equiv \vec{\boldsymbol{\Phi}}(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})$. Since $\Theta^j_k(\vec{\boldsymbol{a}})$ is nonsingular as discussed below eq. (7), we end up with[2]

$$\frac{\partial \Phi^i(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})}{\partial a^\ell} = u^i_k(\vec{\boldsymbol{\Phi}})\Theta^{-1\,k}_{\ \ell}(\vec{\boldsymbol{a}})\,, \tag{51}$$

[1] Indeed, the proof that Θ^i_k is invertible given below eq. (9) does not apply to u^i_k in the case of a Lie group action on a generic manifold.

[2] Eq. (51) corresponds to Lie's first theorem in Ref. 6. The converse to this theorem is also valid and is proven in Ref. 3.

We now follow the same analysis employed following eq. (12). Starting with eq. (51), we shall impose the integrability condition,

$$\frac{\partial\Phi^i(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})}{\partial a^\ell\partial a^j} = \frac{\partial\Phi^i(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})}{\partial a^j\partial a^\ell}\,. \tag{52}$$

It follows that

$$\frac{\partial}{\partial a^\ell}\left[u^i_k(\vec{\boldsymbol{\Phi}})\Theta^{-1\,k}_j(\vec{\boldsymbol{a}})\right] = \frac{\partial}{\partial a^j}\left[u^i_k(\vec{\boldsymbol{\Phi}})\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})\right]\,, \tag{53}$$

where $\Phi \equiv \Phi(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})$. Expanding out eq. (53) yields,

$$u^i_k(\vec{\boldsymbol{\Phi}})\left(\frac{\partial}{\partial a^\ell}\Theta^{-1\,k}_j(\vec{\boldsymbol{a}}) - \frac{\partial}{\partial a^j}\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})\right) = \Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})\frac{\partial}{\partial a^j}u^i_k(\vec{\boldsymbol{\Phi}}) - \Theta^{-1\,k}_j(\vec{\boldsymbol{a}})\frac{\partial}{\partial a^\ell}u^i_k(\vec{\boldsymbol{\Phi}})\,. \tag{54}$$

The derivatives on the right hand side above are evaluated with the help of the chain rule. For example,

$$\frac{\partial}{\partial a^j}u^i_k(\vec{\boldsymbol{\Phi}}) = \frac{\partial\Phi^n}{\partial a^j}\frac{\partial u^i_k(\vec{\boldsymbol{\Phi}})}{\partial\Phi^n} = u^n_p(\vec{\boldsymbol{\Phi}})\Theta^{-1\,p}_j(\vec{\boldsymbol{a}})\frac{\partial u^i_k(\vec{\boldsymbol{\Phi}})}{\partial\Phi^n}\,, \tag{55}$$

where we have used eq. (51) in the final step to evaluate $\partial\Phi^n/\partial a^j$. We can now insert eq. (55) (and a similar result with the index j replaced by ℓ) back into eq. (54). Hence, eq. (54) yields,

$$\begin{aligned} u^i_k(\vec{\boldsymbol{\Phi}})&\left(\frac{\partial}{\partial a^\ell}\Theta^{-1\,k}_j(\vec{\boldsymbol{a}}) - \frac{\partial}{\partial a^j}\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})\right) \\ &= u^n_p(\vec{\boldsymbol{\Phi}})\Theta^{-1\,p}_j(\vec{\boldsymbol{a}})\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})\frac{\partial u^i_k(\vec{\boldsymbol{\Phi}})}{\partial\Phi^n} - u^n_p(\vec{\boldsymbol{\Phi}})\Theta^{-1\,p}_\ell(\vec{\boldsymbol{a}})\Theta^{-1\,k}_j(\vec{\boldsymbol{a}})\frac{\partial u^i_k(\vec{\boldsymbol{\Phi}})}{\partial\Phi^n}\,. \end{aligned} \tag{56}$$

Multiplying both sides of the above equation by $\Theta^\ell_s(\vec{\boldsymbol{a}})\Theta^j_t(\vec{\boldsymbol{a}})$, we end up with

$$u^i_k(\vec{\boldsymbol{\Phi}})\left(\frac{\partial}{\partial a^\ell}\Theta^{-1\,k}_j(\vec{\boldsymbol{a}}) - \frac{\partial}{\partial a^j}\Theta^{-1\,k}_\ell(\vec{\boldsymbol{a}})\right)\Theta^\ell_s(\vec{\boldsymbol{a}})\Theta^j_t(\vec{\boldsymbol{a}}) = u^n_t(\vec{\boldsymbol{\Phi}})\frac{\partial u^i_s(\vec{\boldsymbol{\Phi}})}{\partial\Phi^n} - u^n_s(\vec{\boldsymbol{\Phi}})\frac{\partial u^i_t(\vec{\boldsymbol{\Phi}})}{\partial\Phi^n}\,. \tag{57}$$

Employing the definition of f^k_{st} given by eq. (19) in eq. (57) yields

$$u^n_t(\vec{\boldsymbol{\Phi}})\frac{\partial u^i_s(\vec{\boldsymbol{\Phi}})}{\partial\Phi^n} - u^n_s(\vec{\boldsymbol{\Phi}})\frac{\partial u^i_t(\vec{\boldsymbol{\Phi}})}{\partial\Phi^n} = f^k_{st}u^i_k(\vec{\boldsymbol{\Phi}})\,. \tag{58}$$

Since $\vec{\boldsymbol{\Phi}}$ is a dummy in the above equation, one can simply change variables to $\vec{\boldsymbol{x}}$ to obtain

$$u^n_t(\vec{\boldsymbol{x}})\frac{\partial u^i_s(\vec{\boldsymbol{x}})}{\partial x^n} - u^n_s(\vec{\boldsymbol{x}})\frac{\partial u^i_t(\vec{\boldsymbol{x}})}{\partial x^n} = f^k_{st}u^i_k(\vec{\boldsymbol{x}})\,. \tag{59}$$

Note that one can consider the action of the Lie group G on its group manifold consisting of group multiplication from the left. In this case, the manifold $M = G$ and we can identify $u^i_k = \Theta^i_k$. For example, in this special case, eq. (59) reduces to that of eq. (22).

5. Generators of infinitesimal Lie group transformations

Consider a function $f(\vec{\boldsymbol{x}})$, where $\vec{\boldsymbol{x}} \in M$. The action of G on $\vec{\boldsymbol{x}} \in M$ is given by $\vec{\boldsymbol{x}}' = \vec{\boldsymbol{\Phi}}(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})$ [cf. eq. (45)]. It then follows that under the action of G, the function is transformed,

$$f'(\vec{\boldsymbol{x}}') = f(\vec{\boldsymbol{x}})\,. \tag{60}$$

This simply means that the action of the Lie transformation group is to redefine the coordinate $\vec{\boldsymbol{x}}$, such that $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}}'$ represent the same physical point. In this way of thinking, the action of the Lie transformation group is a passive transformation in which the physical points do not move, but the coordinates of a given point change because the "axes" that define the coordinates of a given point have been transformed.

In the physics literature, one typically writes $\vec{\boldsymbol{x}}' = A\vec{\boldsymbol{x}}$ and

$$f'(A\vec{\boldsymbol{x}}) = f(\vec{\boldsymbol{x}})\,, \quad \text{for } A \in G \text{ and } \vec{\boldsymbol{x}} \in M. \tag{61}$$

One can rewrite eq. (61) equivalently as

$$f'(\vec{\boldsymbol{x}}') = f(A^{-1}\vec{\boldsymbol{x}}')\,. \tag{62}$$

Since $\vec{\boldsymbol{x}}'$ is a dummy variable, we can simply drop the primes and write

$$f'(\vec{\boldsymbol{x}}) = f(A^{-1}\vec{\boldsymbol{x}})\,. \tag{63}$$

This equation indicates how the function f must change under the action of the Lie transformation group.

Consider the case in which $A \in G$ is close to the identity E. We shall write $A = E + \delta A$, where the coordinates of A are given by $\delta\vec{\boldsymbol{a}} = (\delta a^1, \delta a^2, \ldots, \delta a^n)$, and the δa_i are small quantities. To first order in the small quantities, it follows that $A^{-1} \simeq E - \delta A$, or equivalently, the coordinates of A^{-1} are given by $-\delta\vec{\boldsymbol{a}} = (-\delta a^1, -\delta a^2, \ldots, -\delta a^n)$. Working to first order, it follows that

$$(A^{-1}\vec{\boldsymbol{x}})^i = \Phi^i(-\delta\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}}) \simeq \Phi^i(\vec{\boldsymbol{0}}\,;\,\vec{\boldsymbol{x}}) + (-\delta a^k)\left(\frac{\partial \Phi^i(\vec{\boldsymbol{b}}\,;\,\vec{\boldsymbol{x}})}{\partial b^k}\right)_{\vec{\boldsymbol{b}}=0} = x^i - \delta a^k\, u^i_k(\vec{\boldsymbol{x}})\,, \tag{64}$$

after employing eqs. (46) and (49). Plugging the above result into eq. (63) and expanding to first order,

$$f'(\vec{\boldsymbol{x}}) = f\big(x^i - \delta a^k\, u^i_k(\vec{\boldsymbol{x}})\big) \simeq f(\vec{\boldsymbol{x}}) - \delta a^k\, u^i_k(\vec{\boldsymbol{x}})\frac{\partial f}{\partial x^i}\,. \tag{65}$$

Hence, we can write,

$$f'(\vec{\boldsymbol{x}}) - f(\vec{\boldsymbol{x}}) = \delta a^k\, X_k(\vec{\boldsymbol{x}}) f(\vec{\boldsymbol{x}})\,, \tag{66}$$

where

$$X_k(\vec{\boldsymbol{x}}) = -u^i_k(\vec{\boldsymbol{x}})\frac{\partial}{\partial x^i}\,, \tag{67}$$

is called the generator of infinitesimal Lie group transformations. The name derives from the fact that the differential operator X_k determines how the function $f(\vec{\boldsymbol{x}})$ changes due to the action of the Lie group G on the manifold M.

The commutator of two Lie group generators is notable. We calculate,

$$
\begin{aligned}
\big[X_j(\vec{\boldsymbol{x}})\,,\,X_k(\vec{\boldsymbol{x}})\big]f &= u_j^\ell(\vec{\boldsymbol{x}})\frac{\partial}{\partial x^\ell}\left(u_k^i(\vec{\boldsymbol{x}})\frac{\partial f}{\partial x^i}\right) - u_k^\ell(\vec{\boldsymbol{x}})\frac{\partial}{\partial x^\ell}\left(u_j^i(\vec{\boldsymbol{x}})\frac{\partial f}{\partial x^i}\right) \\
&= u_j^\ell(\vec{\boldsymbol{x}})u_k^i(\vec{\boldsymbol{x}})\frac{\partial^2 f}{\partial x^\ell \partial x^i} - u_k^\ell(\vec{\boldsymbol{x}})u_j^i(\vec{\boldsymbol{x}})\frac{\partial^2 f}{\partial x^\ell \partial x^i} + u_j^\ell(\vec{\boldsymbol{x}})\frac{\partial u_k^i(\vec{\boldsymbol{x}})}{\partial x^\ell}\,\frac{\partial f}{\partial x^i} - u_k^\ell(\vec{\boldsymbol{x}})\frac{\partial u_j^i(\vec{\boldsymbol{x}})}{\partial x^\ell}\,\frac{\partial f}{\partial x^i}\,.
\end{aligned}
\tag{68}
$$

Since the mixed second order partial derivatives commute, we see that after an index relabeling,

$$
u_j^\ell(\vec{\boldsymbol{x}})u_k^i(\vec{\boldsymbol{x}})\frac{\partial^2 f}{\partial x^\ell \partial x^i} - u_k^\ell(\vec{\boldsymbol{x}})u_j^i(\vec{\boldsymbol{x}})\frac{\partial^2 f}{\partial x^\ell \partial x^i} = u_j^\ell(\vec{\boldsymbol{x}})u_k^i(\vec{\boldsymbol{x}})\left(\frac{\partial^2 f}{\partial x^\ell \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^\ell}\right) = 0\,. \tag{69}
$$

It then follows that

$$
\big[X_j(\vec{\boldsymbol{x}})\,,\,X_k(\vec{\boldsymbol{x}})\big]f = \left(u_j^\ell(\vec{\boldsymbol{x}})\frac{\partial u_k^i(\vec{\boldsymbol{x}})}{\partial x^\ell} - u_k^\ell(\vec{\boldsymbol{x}})\frac{\partial u_j^i(\vec{\boldsymbol{x}})}{\partial x^\ell}\right)\frac{\partial f}{\partial x^i}\,. \tag{70}
$$

Employing eqs. (59) and (67), it follows that

$$
\big[X_j(\vec{\boldsymbol{x}})\,,\,X_k(\vec{\boldsymbol{x}})\big]f = -f_{jk}^{\,n}u_n^i(\vec{\boldsymbol{x}})\,\frac{\partial f}{\partial x^i} = f_{jk}^{\,n}X_n(\vec{\boldsymbol{x}})f\,. \tag{71}
$$

This result is true for any function f. Thus, the following operator equation is satisfied,[3]

$$
\big[X_j\,,\,X_k\big] = f_{jk}^{\,n}X_n\,, \tag{72}
$$

where we have suppressed the arguments of the generators. Comparing with eq. (44), we see that the commutation relations satisfied by the generators of the infinitesimal Lie group transformations are identical to the commutation relations satisfied by the basis vectors of the Lie algebra. One can view the generators X_k defined in eq. (67) as a differential operator representation of the Lie algebra basis vectors. Colloquially, it is common to refer to the basis vectors of the Lie algebra as the generators of the Lie algebra, although one should keep in mind the distinction between these two objects as described above.

One can also introduce the generators of infinitesimal Lie group transformations consisting of group multiplication from the left on the group manifold. In this case, we can use all the results obtained in this section by replacing u_k^i with Θ_k^i. The corresponding generators are

$$
X_k(\vec{\boldsymbol{b}}) = -\Theta_k^i(\vec{\boldsymbol{b}})\frac{\partial}{\partial b^i}\,. \tag{73}
$$

These generators also satisfy eq. (72), as the derivation is identical to the one given above after replacing u_k^i with Θ_k^i.

[3]Eq. (72) is called Lie's Second Theorem in Ref. 6.

6. Example: the Lie algebra $\mathfrak{so}(3)$

It is instructive to study the Lie group SO(3) to illustrate the results presented in these notes. The group SO(3) acts on the manifold $\mathbb{R}^3$ by rotations. That is, $\vec{\boldsymbol{x}}' = R\vec{\boldsymbol{x}}$, where[4]

$$R_{ij} = n_i n_j + (\delta_{ij} - n_i n_j)\cos\theta - \epsilon_{ijk} n_k \sin\theta \,. \tag{74}$$

R_{ij} describes a rotation by an angle θ about an axis pointing along the unit vector $\hat{\boldsymbol{n}}$. For an infinitesimal rotation,

$$R_{ij} \simeq \delta_{ij} - \epsilon_{ijk}\theta^k \,, \tag{75}$$

where we have chosen the coordinates $\theta^k \equiv \theta n_k$ to parameterize the infinitesimal rotation. That is, the SO(3) group manifold is parameterized in a neighborhood of the identity by coordinates $\vec{\boldsymbol{\theta}} = (\theta^1, \theta^2, \theta^3)$.

To compute the basis for the $\mathfrak{so}(3)$ Lie algebra, we note that an infinitesimal rotation given in eq. (75) can be written in matrix form,

$$R(\vec{\boldsymbol{\theta}}) \simeq \begin{pmatrix} 1 & -\theta^3 & \theta^2 \\ \theta^3 & 1 & -\theta^1 \\ -\theta^2 & \theta^1 & 1 \end{pmatrix} . \tag{76}$$

Using eq. (31), we identify the basis vectors of $\mathfrak{so}(3)$ by

$$\mathcal{A}_i = \left(\frac{\partial R(\vec{\boldsymbol{\theta}})}{\partial \theta^i}\right)_{\vec{\boldsymbol{\theta}}=0} . \tag{77}$$

Inserting eq. (75) into the above equations yields the three generators of $\mathfrak{so}(3)$,

$$(\mathcal{A}_i)_{jk} = -\epsilon_{ijk} \,. \tag{78}$$

Equivalently, one can insert eq. (76) into eq. (77) to obtain,

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \qquad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \qquad \mathcal{A}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{79}$$

One can check that

$$\left[\mathcal{A}_i \,,\, \mathcal{A}_j\right] = \epsilon_{ijk}\mathcal{A}_k \,. \tag{80}$$

Next, we compute the generators of infinitesimal rotations. In the notation of eq. (45),[5]

$$x'_i = \Phi_i(\vec{\boldsymbol{\theta}}\,;\,\vec{\boldsymbol{x}}) \simeq (\delta_{ij} - \epsilon_{ijk}\theta^k)x^j = x_i - \epsilon_{ijk}\theta^k x^j \,. \tag{81}$$

Using eq. (49),

$$u^i_k(\vec{\boldsymbol{x}}) = \left(\frac{\partial \Phi_i}{\partial \theta^k}\right)_{\vec{\boldsymbol{\theta}}=0} = -\epsilon_{ijk}x^j \,. \tag{82}$$

---

[4]See eq. (13) of the class notes entitled, *Properties of Proper and Improper Rotation Matrices.*

[5]In Euclidean space, there is no distinction between raised and lowered indices.

Hence, eq. (67) yields,

$$X_k(\vec{\boldsymbol{x}}) = \epsilon_{ijk} x^j \frac{\partial}{\partial x^i}\,. \tag{83}$$

Explicitly, the generators of infinitesimal rotations are given by,

$$X_1 = x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3}\,, \qquad X_2 = x^1\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^1}\,, \qquad X_3 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}\,. \tag{84}$$

One can check that the generators satisfy the commutation relations of the $\mathfrak{so}(3)$ Lie algebra,[6]

$$\big[X_i\,,\,X_j\big] = \epsilon_{ijk} X_k\,. \tag{85}$$

As anticipated, the commutation relations satisfied by the generators of infinitesimal rotations and the basis vectors of $\mathfrak{so}(3)$ are identical.

Finally, let us consider the action of SO(3) on its group manifold via left group multiplication. In order to determine the generators of infinitesimal transformations on the group manifold, we need to determine the explicit form of Θ^i_k defined in eq. (7). However, this requires an explicit form for the SO(3) group multiplication law. Employing the angle-and-axis parameterization of SO(3), one must determine the dependence of $(\widehat{\boldsymbol{n}}_3, \theta_3)$ in terms of $(\widehat{\boldsymbol{n}}_1, \theta_1)$ and $(\widehat{\boldsymbol{n}}_2, \theta_2)$ from the equation $R(\widehat{\boldsymbol{n}}_3, \theta_3) = R(\widehat{\boldsymbol{n}}_1, \theta_1)R(\widehat{\boldsymbol{n}}_2, \theta_2)$, under the assumption that $|\theta_1| \ll 1$. Given the complexity of the corresponding expressions, this is not a practical strategy for evaluating Θ^i_k.

Instead, I will take an approach inspired by Ref. 4 in which two different coordinate systems are employed for the SO(3) group manifold. In the neighborhood of the identity, we parameterize the SO(3) group manifold by $(\theta_1, \theta_2, \theta_3)$ employed above, where $R(\vec{\boldsymbol{\theta}})$ is given by eq. (76). For a generic point of the SO(3) group manifold, we will use the Euler angle representation of the rotation matrix,

$$R(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}\,, \tag{86}$$

where $0 \leq \alpha, \gamma < 2\pi$ and $0 \leq \beta \leq \pi$, as described in Appendix E of the class handout entitled, *Properties of Proper and Improper Rotation Matrices*. If the matrix elements of R_{ij} are known, then the Euler angles can be determined from the following relations,

$$\tan\alpha = \frac{R_{23}}{R_{13}}\,, \qquad \cos\beta = R_{33}\,, \qquad \tan\gamma = -\frac{R_{32}}{R_{31}}\,. \tag{87}$$

Eq. (86) leaves the quadrants of the angles α and γ ambiguous, but these can be fixed from the signs of R_{23} and R_{32}, respectively, which determine the respective signs of $\sin\alpha$ and $\sin\gamma$ (in light of the fact that $0 \leq \sin\beta \leq 1$).

[6]Note that in the physicist's conventions, the generators of three dimensional infinitesimal rotations are chosen to be $L_i \equiv iX_k$, which satisfy $[L_i, L_j] = i\epsilon_{ijk}L_k$. Indeed, we recognize the L_k as the orbital angular momentum operators of quantum mechanics (in units where $\hbar = 1$).

Thus, we take $\vec{\boldsymbol{b}} = (b^1, b^2, b^3) = (\alpha, \beta, \gamma)$ and $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\theta}} = (\theta^1, \theta^2, \theta^3)$ in the calculation of

$$\Theta^i_j(\vec{\boldsymbol{b}}) = \frac{\partial m^i(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}})}{\partial a^j}\bigg|_{\vec{\boldsymbol{a}}=0}, \tag{88}$$

and $\vec{\boldsymbol{m}}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}) = (\alpha', \beta', \gamma')$. Consider the product, $R(\alpha', \beta', \gamma') = R(\vec{\boldsymbol{\theta}})R(\alpha, \beta, \gamma)$. To compute $\Theta^i_1(\alpha, \beta, \gamma)$, we can immediately set $\theta_2 = \theta_3 = 0$. Hence, we examine,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta^1 \\ 0 & \theta^1 & 1 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} - \theta^1 R_{31} & R_{22} - \theta^1 R_{32} & R_{23} - \theta^1 R_{33} \\ \theta^1 R_{21} + R_{31} & \theta^1 R_{22} + R_{32} & \theta^1 R_{23} + R_{33} \end{pmatrix}. \tag{89}$$

In light of eq. (87),

$$\tan\alpha' = \frac{R_{23} - \theta^1 R_{33}}{R_{13}}, \qquad \cos\beta' = \theta^1 R_{23} + R_{33}, \qquad \tan\gamma' = -\frac{\theta^1 R_{22} + R_{32}}{\theta^1 R_{21} + R_{31}}. \tag{90}$$

Hence, it follows that

$$(\sec^2\alpha')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^1_1(\alpha, \beta, \gamma) = -\frac{R_{33}}{R_{13}} = -\frac{\cos\beta}{\cos\alpha\sin\beta}, \tag{91}$$

$$-(\sin\beta')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^2_1(\alpha, \beta, \gamma) = R_{23} = \sin\alpha\sin\beta, \tag{92}$$

$$(\sec^2\gamma')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^3_1(\alpha, \beta, \gamma) = \frac{R_{32}R_{21} - R_{31}R_{22}}{R_{31}^2} = \frac{\cos\alpha}{\sin\beta\cos^2\gamma}. \tag{93}$$

Since we set $\vec{\boldsymbol{\theta}} = 0$ when evaluating $\Theta^i_1(\alpha, \beta, \gamma)$, we can drop the primes on the left hand side of eqs. (91)–(93). That is, $(\alpha', \beta', \gamma')_{\vec{\boldsymbol{\theta}}=0} = (\alpha, \beta, \gamma)$. Thus, we end up with

$$\Theta^1_1(\alpha, \beta, \gamma) = -\frac{\cos\alpha\cos\beta}{\sin\beta}, \qquad \Theta^2_1(\alpha, \beta, \gamma) = -\sin\alpha \qquad \Theta^3_1 = \frac{\cos\alpha}{\sin\beta}. \tag{94}$$

Likewise, to compute $\Theta^i_2(\alpha, \beta, \gamma)$, we can immediately set $\theta_1 = \theta_3 = 0$. Consider

$$\begin{pmatrix} 1 & 0 & \theta^2 \\ 0 & 1 & 0 \\ -\theta^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} R_{11} + \theta^2 R_{31} & R_{12} + \theta^2 R_{32} & R_{13} + \theta^2 R_{33} \\ R_{21} & R_{22} & R_{23} \\ -\theta^2 R_{11} + R_{31} & -\theta^2 R_{12} + R_{32} & -\theta^2 R_{13} + R_{33} \end{pmatrix}. \tag{95}$$

In light of eq. (87),

$$\tan\alpha' = \frac{R_{23}}{R_{13} + \theta^2 R_{33}}, \qquad \cos\beta' = -\theta^2 R_{13} + R_{33}, \qquad \tan\gamma' = \frac{\theta^2 R_{12} - R_{32}}{-\theta^2 R_{11} + R_{31}}. \tag{96}$$

Hence, it follows that

$$(\sec^2\alpha')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^1_2(\alpha, \beta, \gamma) = -\frac{R_{23}R_{33}}{R_{13}^2} = -\frac{\sin\alpha\cos\beta}{\cos^2\alpha\sin\beta} \tag{97}$$

$$-(\sin\beta')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^2_2(\alpha, \beta, \gamma) = -R_{13} = -\cos\alpha\sin\beta \tag{98}$$

$$(\sec^2\gamma')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^3_2(\alpha, \beta, \gamma) = \frac{R_{31}R_{12} - R_{32}R_{11}}{R_{31}^2} = \frac{\sin\alpha}{\sin\beta\cos^2\gamma}. \tag{99}$$

Thus, we end up with

$$\Theta^1_2(\alpha,\beta,\gamma) = -\frac{\sin\alpha\cos\beta}{\sin\beta}\,, \qquad \Theta^2_2(\alpha,\beta,\gamma) = \cos\alpha \qquad \Theta^3_2 = \frac{\sin\alpha}{\sin\beta}\,. \tag{100}$$

Finally, to compute $\Theta^i_3(\alpha,\beta,\gamma)$, we can immediately set $\theta_1 = \theta_2 = 0$. Consider

$$\begin{pmatrix} 1 & -\theta^3 & 0 \\ \theta^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} R_{11} - \theta^3 R_{21} & R_{12} - \theta^3 R_{22} & R_{13} - \theta^3 R_{23} \\ \theta^3 R_{11} + R_{21} & \theta^3 R_{12} + R_{22} & \theta^3 R_{13} + R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}. \tag{101}$$

In light of eq. (87),

$$\tan\alpha' = \frac{\theta^3 R_{13} + R_{23}}{R_{13} - \theta^3 R_{23}}\,, \qquad \cos\beta' = R_{33}\,, \qquad \tan\gamma' = -\frac{R_{32}}{R_{31}}\,. \tag{102}$$

Hence, it follows that

$$(\sec^2\alpha')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^1_3(\alpha,\beta,\gamma) = \frac{R^2_{13} + R^2_{23}}{R^2_{13}} = \frac{1}{\cos^2\alpha} \tag{103}$$

$$-(\sin\beta')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^2_3(\alpha,\beta,\gamma) = 0\,, \tag{104}$$

$$(\sec^2\gamma')_{\vec{\boldsymbol{\theta}}=0}\,\Theta^3_2(\alpha,\beta,\gamma) = 0\,. \tag{105}$$

Thus, we end up with

$$\Theta^1_3(\alpha,\beta,\gamma) = 1\,, \qquad \Theta^2_3(\alpha,\beta,\gamma) = 0 \qquad \Theta^3_3 = 0,. \tag{106}$$

Collecting the above results, we have computed the matrix elements $\Theta^i_j(\alpha,\beta,\gamma)$. Regarding j as indexing the rows and i as indexing the columns of a 3×3 matrix,

$$\Theta^i_j(\alpha,\beta,\gamma) = \begin{pmatrix} -\cos\alpha\cot\beta & -\sin\alpha\cot\beta & 1 \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\alpha\csc\beta & \sin\alpha\csc\beta & 0 \end{pmatrix}. \tag{107}$$

The generators of infinitesimal SO(3) transformations on the group manifold are given by

$$X_1(\alpha,\beta,\gamma) = \cos\alpha\cot\beta\frac{\partial}{\partial\alpha} + \sin\alpha\frac{\partial}{\partial\beta} - \cos\alpha\csc\beta\frac{\partial}{\partial\gamma}\,, \tag{108}$$

$$X_2(\alpha,\beta,\gamma) = \sin\alpha\cot\beta\frac{\partial}{\partial\alpha} - \cos\alpha\frac{\partial}{\partial\beta} - \sin\alpha\csc\beta\frac{\partial}{\partial\gamma}\,, \tag{109}$$

$$X_3(\alpha,\beta,\gamma) = -\frac{\partial}{\partial\alpha}\,. \tag{110}$$

It is straightforward to check that $\big[X_i\,,\,X_j\big] = \epsilon_{ijk}X_k$, as expected.

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