DUE: TUESDAY, May 16, 2023

1. (a) A homomorphism from the vector space $\mathbb{R}^3$ to the set of traceless Hermitian 2×2 matrices is defined by $\vec{\boldsymbol{x}} \to \vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{\sigma}}$, where $\vec{\boldsymbol{\sigma}} = (\sigma_1\,,\,\sigma_2\,,\,\sigma_3)$ are the Pauli matrices. First, show that $\det(\vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{\sigma}}) = -|\vec{\boldsymbol{x}}|^2$. Second, prove the identity:

$$x_i = \tfrac{1}{2}\mathrm{Tr}\,(\vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{\sigma}}\,\sigma_i)\,.$$

This identity provides the inverse transformation from the set of traceless 2×2 Hermitian matrices to the vector space $\mathbb{R}^3$.

(b) Let $U \in \mathrm{SU}(2)$. Show that $U\,\vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{\sigma}}\,U^{-1} = \vec{\boldsymbol{y}} \boldsymbol{\cdot} \vec{\boldsymbol{\sigma}}$ for some vector $\vec{\boldsymbol{y}}$. Using the results of part (a), prove that an element of the rotation group exists such that $\vec{\boldsymbol{y}} = R\vec{\boldsymbol{x}}$ and determine an explicit form for $R \in \mathrm{SO}(3)$. Display a homomorphism from SU(2) onto SO(3) and prove that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$.

(c) The Lie group $\mathrm{SU}(1\,,1)$ is defined as the group of 2×2 matrices V that satisfy $V\sigma_3 V^\dagger = \sigma_3$ and $\det\,V = 1$. (Note that V is *not* a unitary matrix.) The Lie group $\mathrm{SO}(2\,,1)$ is the group of transformations on vectors $\vec{\boldsymbol{x}} \in \mathbb{R}^3$ (with determinant equal to one) that preserves $x_1^2 + x_2^2 - x_3^2$. Display the homomorphism from $\mathrm{SU}(1\,,1)$ onto $\mathrm{SO}(2\,,1)$ and compare with part (b).

2. The Möbius group is defined as the set of linear fractional transformations:

$$M = \left\{ m(z) = \frac{az+b}{cz+d}\,, \quad ad - bc = 1 \right\}\,,$$

where a, b, c, d and z are complex numbers.

(a) Show that the mapping $f : \mathrm{SL}(2, \mathbb{C}) \to M$ defined by:

$$f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto m(z)$$

is a group homomorphism. [*HINT:* the multiplication law on M is defined by the composition of functions.]

(b) Prove that M is not simply connected and identify its universal covering group.

3. In class, we showed that the invariant measure on a Lie group manifold is given by

$$d\mu(g) = |\det c(\vec{\boldsymbol{\xi}})|\, d\xi_1 d\xi_2 \cdots d\xi_n\,, \tag{1}$$

where the the matrix elements $c_{jk}(\vec{\boldsymbol{\xi}})$ are the coefficients of the Lie algebra element $A^{-1}\partial A/\partial\xi_k$ with respect to some basis, and $A(\vec{\boldsymbol{\xi}})$ are elements of the corresponding Lie group that is parameterized by the coordinates $\vec{\boldsymbol{\xi}}$. That is, given an n-dimensional Lie group G, the corresponding real Lie algebra $\mathfrak{g}$ consists of real linear combinations of basis vectors $\mathcal{A}_j \in \mathfrak{g}$. Since $A^{-1}\partial A/\partial\xi_k \in \mathfrak{g}$ for any $A \in G$, one can therefore write,

$$A^{-1}\frac{\partial A}{\partial \xi_k} = \sum_{j=1}^{n} c_{jk}(\vec{\boldsymbol{\xi}})\mathcal{A}_j\,, \tag{2}$$

which defines the coefficients $c_{jk}(\vec{\boldsymbol{\xi}})$ needed in the determination of the invariant measure.

(a) An element of SO(3) can be parameterized by $\vec{\boldsymbol{\xi}} = (\alpha, \beta, \gamma)$, where α, β and γ are the three Euler angles defined in Appendix E of the class handout entitled *Properties of Proper and Improper Rotation Matrices*. Using the Euler angle parameterization of the SO(3) group manifold, compute the invariant integration measure $d\mu(g)$ for SO(3).

(b) The SO(3) group manifold can be also be described as a ball of radius π with antipodal points identified. A point in the SO(3) group manifold is specified by a vector $\vec{\boldsymbol{\xi}}$ with $|\vec{\boldsymbol{\xi}}| \leq \pi$. Thus, the SO(3) manifold is parameterized by $\vec{\boldsymbol{\xi}} = (\xi, \theta, \phi)$, where (θ, ϕ) are the spherical angles (such that $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$) and ξ is the magnitude of the vector $\vec{\boldsymbol{\xi}}$. [*NOTE:* This is equivalent to the angle-and-axis parameterization where the rotation angle is ξ and the rotation axis, $\hat{\boldsymbol{\xi}}$, is specified by a polar angle θ and an azimuthal angle ϕ.]

Show that the the matrix elements of $c(\vec{\boldsymbol{\xi}})$ defined in eq. (2) are given by,

$$c(\vec{\boldsymbol{\xi}})_{nk} = \tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k}\,, \tag{3}$$

and $R_{ij} \equiv R_{ij}(\vec{\boldsymbol{\xi}})$ is the SO(3) matrix given in problem 7(b) of problem set 2.

(c) [*EXTRA CREDIT*] Using eqs. (1) and (3), evaluate the invariant integration measure $d\mu(g)$ for the angle-and-axis parameterization of SO(3) and show that

$$d\mu(\vec{\boldsymbol{\xi}}) = 2(1-\cos\xi)\sin\theta\, d\theta\, d\phi\, d\xi\,.$$

HINT: First evaluate $d\mu(\vec{\boldsymbol{\xi}})$ in terms of Cartesian coordinates ξ_1, ξ_2 and ξ_3. Convert to spherical coordinate (ξ, θ, ϕ) at the very end of the calculation.

4. Consider a Lie group of transformations G acting on a manifold M. That is, for every $g \in G$, we have $gx = y$ for some $x, y \in M$.

(a) Let H be the set of all transformations in G that map a given point $x \in M$ into itself. Show that H is a subgroup. H has at least three names in the mathematical literature: the little group, the isotropy group, or the stability group of the point x.

(b) Consider the submanifold of M defined by $\{gx \,|\, g \in G\}$, for fixed $x \in M$. This is called the *orbit* through x with respect to G. Show that there is a one-to-one correspondence between the points of the orbit and the set of left cosets of H. Explain why we may conclude that $\{gx \,|\, g \in G\} = G/H$. Show that the coset space G/H is a homogeneous space.

(c) Prove that $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ by considering the action of the rotation group on the point $(1, 0, 0, \ldots, 0) \in \mathbb{R}^n$.

(d) Prove that $S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1)$ by considering the action of the $\mathrm{U}(n)$ matrices on the point $(1, 0, 0, \ldots, 0) \in \mathbb{C}^n$.

(e) Complex projective space $\mathbb{CP}^n$ is defined as the space of complex lines in $\mathbb{C}^{n+1}$ through the origin. That is, $\mathbb{CP}^n$ consists of the set of nonzero vectors in $\mathbb{C}^{n+1}$ where we identify $(z_0, z_1, \ldots, z_n) \sim \lambda(z_0, z_1, \ldots, z_n)$, for any nonzero complex number λ. Without loss of generality, we can restrict our considerations to the vectors $\vec{\boldsymbol{v}} \in \mathbb{C}^{n+1}$ such that $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}^* = 1$. Show that $\mathrm{U}(1)\otimes\mathrm{U}(n)$ is the little group of the point $z = (1, 0, 0, \ldots, 0) \in \mathbb{CP}^n$, and that $\mathbb{CP}^n$ is the orbit through z with respect to $\mathrm{U}(n+1)$. Conclude that $\mathbb{CP}^n = \mathrm{U}(n+1)/\mathrm{U}(1)\otimes\mathrm{U}(n)$.

(f) Real projective space $\mathbb{RP}^n$ can be defined analogously to $\mathbb{CP}^n$ of part (e) by replacing the field of complex numbers with the field of real numbers. What coset space can be identified with $\mathbb{RP}^n$?

(g) In parts (c)–(f), check that $\dim(G/H) = \dim G - \dim H$.

(h) [*EXTRA CREDIT*] $\mathbb{CP}^n$ is a manifold of n complex (or $2n$ real) dimensions. $\mathbb{CP}^1$ is homeomorphic to which well-known two-dimensional real manifold? Justify your answer.

5. Let A be an even-dimensional complex antisymmetric $2n\times 2n$ matrix, where n is a positive integer. We define the *pfaffian* of A, denoted by $\mathrm{pf}\,A$, by:

$$\mathrm{pf}\,A = \frac{1}{2^n n!} \sum_{p\in S_{2n}} (-1)^p \, A_{i_1 i_2} A_{i_3 i_4} \cdots A_{i_{2n-1} i_{2n}} \,, \tag{4}$$

where the sum is taken over all permutations

$$p = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ i_1 & i_2 & \cdots & i_{2n} \end{pmatrix}$$

and $(-1)^p$ is the sign of the permutation $p \in S_{2n}$. If A is an odd-dimensional complex antisymmetric matrix, the corresponding pfaffian is defined to be zero.

(a) By explicit calculation, show that[1]

$$\det A = (\mathrm{pf}\,A)^2 \,, \tag{5}$$

for any 2×2 and 4×4 complex antisymmetric matrix A.

(b) Prove that the determinant of any odd-dimensional complex antisymmetric matrix vanishes. As a result, the definition of the pfaffian in the odd-dimensional case is consistent with the result of eq. (5).

(c) Given an arbitrary $2n \times 2n$ complex matrix B and complex antisymmetric $2n \times 2n$ matrix A, use the definition of the pfaffian given in eq. (4) to prove the following identity:

$$\mathrm{pf}\,(BAB^T) = \mathrm{pf}\,A \;\det B \,.$$

[1] In fact, eq. (5) holds for all complex antisymmetric $2n \times 2n$ matrices, where n is any positive number. A general proof will be provided in a class handout.

(d) A complex $2n \times 2n$ matrix S is called *symplectic* if $S^{\mathsf{T}} J S = J$, where S^{T} is the transpose of S and J is a $2n \times 2n$ matrix which is given in block matrix form by

$$J \equiv \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ -\mathbb{1} & \mathbb{O} \end{pmatrix},$$

where $\mathbb{1}$ is the $n \times n$ identity matrix and $\mathbb{O}$ is the $n \times n$ zero matrix. Prove that the set of $2n \times 2n$ complex symplectic matrices, denoted by $\mathrm{Sp}(n, \mathbb{C})$, is a matrix Lie group[2] [*i.e.*, it is a topologically closed subgroup of $\mathrm{GL}(2n, \mathbb{C})$].

(e) Prove that if S is a symplectic matrix, then $\det S = 1$.

HINT: It is very easy to prove that $\det S = \pm 1$ by taking the determinant of the equation $S^{\mathsf{T}} J S = J$. To prove that there are no symplectic matrices with $\det S = -1$, use the result of part (c).

(f) Using the results of parts (d) and (e), prove that the matrix Lie groups $\mathrm{Sp}(1, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C})$ are isomorphic.

6. The two-dimensional Poincaré group P(2) is the group consisting of two-dimensional Lorentz transformations [i.e., transformations on 2-vectors $\left(\begin{smallmatrix} ct \\ x \end{smallmatrix}\right)$ that preserve $x^2 - c^2 t^2$] and translations in time and space. P(2) can be represented by 3×3 matrices acting homogeneously on the column vector, $\left(\begin{smallmatrix} ct \\ x \\ 1 \end{smallmatrix}\right)$, in analogy with the two-dimensional Euclidean group, E(2), worked out in class.

(a) Find the infinitesimal generators (i.e., differential operators) of the corresponding Lie algebra, $\mathfrak{p}(2)$. Work out the commutation relations of $\mathfrak{p}(2)$.

(b) Compute the Cartan-Killing form. Show that P(2) is noncompact and non-semisimple.

(c) Express the Lie algebra $\mathfrak{p}(2)$ as a semidirect sum of two abelian subalgebras.

[2] Warning: many authors denote the group of $2n \times 2n$ complex symplectic matrices by $\mathrm{Sp}(2n, \mathbb{C})$.