DUE: TUESDAY, JUNE 9, 2026

FINAL PROJECTS ALERT: The presentations of the final projects will take place on Tuesday June 9 from 12–3 pm in ISB 231. The slides from your presentations will be posted to the class website.

1. This problem concerns the Lie group SO(4) and its Lie algebra $\mathfrak{so}(4)$.

(a) Work out the Lie algebra $\mathfrak{so}(4)$ and verify that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

HINT: Show that there is a choice of basis for $\mathfrak{so}(4)$ consisting of 4×4 antisymmetric matrices that contain precisely two non-zero entries: 1 and -1. Evaluate the commutation relations of these $\mathfrak{so}(4)$ generators. Then, by choosing a new basis consisting of sums and differences of pairs of the original $\mathfrak{so}(4)$ generators, show that the resulting commutation relations are isomorphic to the commutation relations of the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

(b) What is the universal covering group of SO(4)? What is the center of SO(4)? Identify the adjoint group $\mathrm{Ad}\big(\mathrm{SO}(4)\big)$.

(c) Calculate the Killing form of $\mathfrak{so}(4)$ and verify that this Lie algebra is semisimple and compact.

(d) Consider the complexification of the corresponding Lie algebras of part (a). Show that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$. Do the conclusions of part (c) still hold? If not, explain how these conclusions are modified?

(e) Using the methods used in part (a), show that $\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, where $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is the realification of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. By complexification of this result, show that one recovers the result of part (d).

REMARK: Note the Lie algebra isomorphisms, $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ and $\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$. The latter is obtained from the former via complexification.

2. A Lie algebra $\mathfrak{g}$ is defined by the commutation relations of the generators,

$$[e_a, e_b] = f_{ab}^c e_c \,.$$

Consider an irreducible finite-dimensional matrix representation of $\mathfrak{g}$. The dimension of the representation R will be denoted by d_R, and the corresponding (matrix) generators will be denoted by R_a. In the special case of the adjoint representation (whose dimension d is equal to the dimension of the Lie algebra $\mathfrak{g}$), we shall denote the corresponding generators by F_a.

(a) Show that the Cartan-Killing metric g_{ab} can be written as $g_{ab} = \mathrm{Tr}(F_a F_b)$.

(b) If $\mathfrak{g}$ is a simple real compact Lie algebra, prove that

$$\mathrm{Tr}(R_a R_b) = c_R g_{ab} \,,$$

for any irreducible representation R, where c_R is called the *index* of R.

HINT: Choose a basis where g_{ab} is proportional to δ_{ab}. Then the f_{ab}^{c} are antisymmetric in all three indices. Show that $\mathrm{Tr}\,[R_a, R_b]R_c = \mathrm{Tr}\,R_a[R_b, R_c]$ and argue that this implies that $\mathrm{Tr}\,R_a R_b$, viewed as the ab element of a $d \times d$ matrix, commutes with all Lie algebra elements in the adjoint representation. Finally, invoke Schur's lemma.[1]

(c) The quadratic Casimir operator is defined as $C_2 \equiv g^{ab} e_a e_b$ where g^{ab} is the inverse of g_{ab}. Recall that C_2 commutes with all elements of the Lie algebra. Hence, by Schur's lemma, C_2 must be a multiple of the identity operator. Let us write $C_2 = C_2(R)\mathbf{I}$ where $\mathbf{I}$ is the $d_R \times d_R$ identity matrix and $C_2(R)$ is the eigenvalue of the Casimir operator in the irreducible representation R. Show that $C_2(R)$ is related to the index c_R by

$$C_2(R) = \frac{d c_R}{d_R} \,,$$

where d is the dimension of the Lie algebra $\mathfrak{g}$. Check the above formula in the case that R is the adjoint representation.

HINT: The matrix elements of the R_a are $(R_a)_{ij}$, where $i, j = 1, \ldots, d_R$. If you keep the matrix element indices explicit, then the derivation of the above result is straightforward.

(d) Compute the index of an arbitrary irreducible representation of $\mathfrak{su}(2)$.

(e) Compute the index of the fundamental (defining) representation of $\mathfrak{su}(3)$. Generalize this result to $\mathfrak{su}(n)$.

3. Various subalgebras of $\mathfrak{su}(3)$ may be identified with specific subsets of the $\mathfrak{su}(3)$ generators.

(a) Show that the Gell-Mann matrices λ_1, λ_2, and λ_3 generate an $\mathfrak{su}(2)$ subalgebra.

(b) Show that the Gell-Mann matrices λ_2, λ_5, and λ_7 generate an $\mathfrak{so}(3)$ subalgebra. (Why do you think I called this an $\mathfrak{so}(3)$ subalgebra rather than an $\mathfrak{su}(2)$ subalgebra?)

(c) Decompose (if necessary) the three-dimensional irreducible representation of $\mathfrak{su}(3)$ into representations that are irreducible under the subalgebras of parts (a) and (b).

4. Consider the simple Lie algebra $\mathfrak{g}$ generated by the ten 4×4 matrices: $\sigma_a \otimes \mathbf{I}$, $\sigma_a \otimes \tau_1$, $\sigma_a \otimes \tau_3$ and $\mathbf{I} \otimes \tau_2$, where $(\mathbf{I}, \sigma_a)$ and $(\mathbf{I}, \tau_a)$ are the 2×2 identity and Pauli matrices in orthogonal spaces. For example, since $\tau_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we obtain in block matrix form:

$$\sigma_a \otimes \tau_3 = \left(\begin{array}{c|c} \sigma_a & \mathbf{0} \\ \hline \mathbf{0} & -\sigma_a \end{array}\right) , \qquad (a = 1, 2, 3) \,,$$

[1]Note that by complexifying the simple real compact Lie algebra, one can easily show that the above result also holds for any simple complex Lie algebra.

where $\mathbf{0}$ is the 2×2 zero matrix. The remaining seven matrices can be likewise obtained. Take $H_1 = \sigma_3 \otimes \mathbf{I}$ and $H_2 = \sigma_3 \otimes \tau_3$ as the generators of the Cartan subalgebra. Note that if A, B, C, and D are 2×2 matrices, then $(A \otimes B)(C \otimes D) = AC \otimes BD$.

(a) Find the roots of $\mathfrak{g}$. Normalize the roots such that the shortest root vector has length 1. What is the rank of $\mathfrak{g}$?

(b) Determine the simple roots and evaluate the corresponding Cartan matrix. Deduce the Dynkin diagram for this Lie algebra and identify it by name.

(c) Define the fundamental weights $\boldsymbol{m}_i$ in terms of the simple roots $\boldsymbol{\alpha}_j \in \Pi$ such that

$$\frac{2(\boldsymbol{m}_i, \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j)} = \delta_{ij}\,, \qquad \text{for } i, j = 1, 2, \ldots, r\,,$$

where $r \equiv \text{rank}\ \mathfrak{g}$. Using the results of part (b), determine all the fundamental weights of $\mathfrak{g}$.

HINT: Expand the $\boldsymbol{m}_i$ as linear combinations of the simple roots and solve for the coefficients.

(d) [*EXTRA CREDIT*] Each of the r fundamental weights is the highest weight for an irreducible representation of $\mathfrak{g}$. Collectively, these are called the fundamental (or basic) representations of $\mathfrak{g}$. For each fundamental representation of $\mathfrak{g}$, compute the complete set of weights and draw the corresponding weight diagrams.[2] What are the corresponding dimensions of the fundamental representations of $\mathfrak{g}$.

HINT: In this example, all weights of the fundamental representations of $\mathfrak{g}$ appear with multiplicity equal to one. The complete set of weights for the irreducible representations of $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$ corresponding to the highest weights $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, respectively, can be obtained by the method of block weight diagrams given in Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Dover Publications, Inc., Mineola, NY, 2006).[3] Note that the Cartan matrix employed by Cahn is the transpose of Cartan matrix defined in class.

5. A basis for a three-dimensional complex Lie algebra $\mathfrak{g}$ satisfies the following commutation relations (where k is a nonzero real number):

$$[e_3, e_1] = e_1\,, \qquad [e_3, e_2] = ike_2\,, \qquad [e_1, e_2] = 0\,.$$

(a) Obtain the corresponding Cartan-Killing metric. Is $\mathfrak{g}$ semisimple? solvable? nilpotent?

(b) Suppose that $k = 1$. Find the explicit form of the basis transformation that yields commutation relations of the generators such that all structure constants (with respect to the transformed basis) are real.

(c) [*EXTRA CREDIT*] Suppose instead that $k = 2$. Show that in this case, no change of basis exists in which all structure constants (with respect to the transformed basis) are real.

[2]The weight diagrams should be plotted on a two dimensional plane, where the axes correspond to the diagonalized generators normalized such that the shortest root vector has length 1.

[3]A link to an electronic copy of this book can be found on the Physics 251 course webpage.

HINT: It is convenient to define a 3×3 matrix A, whose matrix elements are given by $A^{\ell k} = \frac{1}{2}\epsilon^{\ell ij} f_{ij}^{k}$, or equivalently, $f_{ij}^{k} = \epsilon_{\ell ij} A^{\ell k}$, where the f_{ij}^{k} are the structure constants of $\mathfrak{g}$. How do the matrix elements of A change under a basis transformation? To show that no real basis exists, one must show that no change of basis can be performed such that the transformed A is a real matrix. In this problem, you should find that the third row and column of A contain only zero entries. If the upper 2×2 block of A is denoted by A_2, define a new matrix $B = A_2^{\mathsf{T}} A_2^{-1}$. How do the matrix elements of B change under a basis transformation? Show that if $k = 2$ then no basis transformation exists such that the transformed B is a real matrix. Conclude that in this case, no basis exists in which the structure constants of $\mathfrak{g}$ are all real.