1. The matrix group O(n) consists of real orthogonal $n \times n$ matrices (n is a positive integer), and SO(n) consists of the subgroup of O(n) matrices with determinant equal to one.

(a) Show that SO(n) is a normal subgroup of O(n).

A normal subgroup H of G has the property that for all $h \in H$ and $g \in G$, it follows that $ghg^{-1} \in H$. Applying this result to $G = \mathrm{O}(n)$ and $H = \mathrm{SO}(n)$, we note that

$$\det(ghg^{-1}) = \det h = 1\,,$$

after employing the well-known properties of the determinant, $\det(AB) = \det A \det B$ and $\det A^{-1} = (\det A)^{-1}$ [the latter being true for any invertible matrix A]. Since $\det(ghg^{-1}) = 1$, it follows that $ghg^{-1} \in \mathrm{SO}(n)$ for all $g \in \mathrm{O}(n)$ and $h \in \mathrm{SO}(n)$, as required if SO(n) is a normal subgroup of O(n).

<u>ALTERNATIVE PROOF</u>: Consider the map, $f : \mathrm{O}(n) \longrightarrow \{\boldsymbol{I}_n\,,\, -\boldsymbol{I}_n\}$, defined by

$$f(A) = \boldsymbol{I}_n \det A, \quad \text{for } A \in \mathrm{O}(n)\,,$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. Clearly f is an onto map. Moreover, we can identify $\mathbb{Z}_2 \cong \{\boldsymbol{I}_n\,,\, -\boldsymbol{I}_n\}$ since both groups possess the same group multiplication table. Finally, the kernel of the map is $\ker f = \mathrm{SO}(n)$. Using the theorem proved in class that states that if $f : G \to G'$ then ker f is a normal subgroup of G, it follows that SO(n) is a normal subgroup of O(n). Note that we can identify the image of the map as $\mathrm{im}\, f = \mathbb{Z}_2$. Thus, by the isomorphism theorem proved in class which states that $\mathrm{im}\, f \cong G/\ker f$, it follows that $\mathrm{O}(n)/\mathrm{SO}(n) \cong \mathbb{Z}_2$.

(b) If n is odd, show that $\mathbb{Z}_2 \cong \{\boldsymbol{I}_n\,,\, -\boldsymbol{I}_n\}$ is a normal subgroup of O(n), where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. Prove that O(n) can be written as an internal direct product, $\mathrm{O}(n) \cong \mathrm{SO}(n) \times \mathbb{Z}_2$.

Consider $g \in \mathrm{O}(2)$ and $k \in \{\boldsymbol{I}_n\,,\, -\boldsymbol{I}_n\}$. Since $\pm\boldsymbol{I}_n$ commute with all $n \times n$ matrices, it follows that $gkg^{-1} = k$ for $g \in \mathrm{O}(n)$ and $k \in \{\boldsymbol{I}_n\,,\, -\boldsymbol{I}_n\}$. By identifying $\mathbb{Z}_2 \cong \{\boldsymbol{I}_n\,,\, -\boldsymbol{I}_n\}$, it immediately follows that $\mathbb{Z}_2$ is a normal subgroup of O(n). This result is valid for both even and odd n.

However, an important distinction between the case of even and odd n is revealed by the observation that $\det(-\boldsymbol{I}_n) = -1$ if n is odd, whereas $\det(-\boldsymbol{I}_n) = +1$ if n is even. When n is odd, it is instructive to consider a function $g : \mathrm{O}(n) \longrightarrow \mathrm{SO}(n)$ defined by,

$$g(A) = A \det A, \quad \text{for } A \in \mathrm{O}(n).$$

Following the alternative proof given in part (a), we note that in this case, $\ker g = \{\boldsymbol{I}_n\,,\, -\boldsymbol{I}_n\}$, and it follows that $\ker g \cong \mathbb{Z}_2$ is a normal subgroup of O(n). Moreover, the image of the map is $\mathrm{im}\, g = \mathrm{SO}(n)$, which implies that $\mathrm{O}(n)/\mathbb{Z}_2 \cong \mathrm{SO}(n)$.

Having identified $\mathbb{Z}_2 \cong \{\boldsymbol{I}_n\,,\,-\boldsymbol{I}_n\}$, we have shown in parts (a) and (b) above that SO(n) and $\mathbb{Z}_2$ are normal subgroups of O(n) and that O(n)/SO(n) $\cong \mathbb{Z}_2$ and O(n)/$\mathbb{Z}_2 \cong$ SO(n). Moreover,

$$\mathrm{SO}(n) \cap \mathbb{Z}_2 = \{\boldsymbol{I}_n\}\,, \tag{1}$$

since $\det(-\boldsymbol{I}_n) = -1$ for odd n as previously noted. Finally, any O(n) matrix (with n odd) can be generated by $\{hk\,|\,h \in \mathrm{SO}(n)\,,\,k \in \mathbb{Z}_2\}$. Hence, by the results obtained in class, it follows that for odd n, $\mathrm{O}(n) \cong \mathrm{SO}(n) \times \mathbb{Z}_2$.

(c) Explain why the results of part (b) do not apply to the case of even n. Show that if n is even then O(n) can be written as a semidirect product, $\mathrm{O}(n) \cong \mathrm{SO}(n) \rtimes \mathbb{Z}_2$. Identify explicitly the subgroup of O(n) appearing in this semidirect product that is isomorphic to $\mathbb{Z}_2$.

When n is even, $\det(-\boldsymbol{I}_n) = +1$ so that eq. (1) is no longer valid, in which case one of the critical conditions for the direct product employed in part (b) does not hold. Thus, we need to identify a $\mathbb{Z}_2$ subgroup of O(n) for even n such that eq. (1) is satisfied. Let us introduce a matrix $B \in \mathrm{O}(n)$ with the properties that $B^2 = \boldsymbol{I}_n$ and $\det B = -1$. It follows that $\mathbb{Z}_2 \cong \{\boldsymbol{I}_n, B\}$, since both groups possess the same group multiplication table. Moreover, with this definition of $\mathbb{Z}_2$, we have $\mathrm{SO}(n) \cap \mathbb{Z}_2 = \{\boldsymbol{I}_n\}$, in light of the fact that $\det B = -1$.

However, $\mathbb{Z}_2 \cong \{\boldsymbol{I}_n, B\}$ is *not* a normal subgroup of O(n). In particular, it is *not* true that $gBg^{-1} \in \{\boldsymbol{I}_n, B\}$ for all $g \in \mathrm{O}(n)$. This result can be proven by using the properties of B stated above. Here, we shall demonstrate this result by adopting a particular choice for the matrix B,

$$B = \mathrm{diag}(-1, 1, 1, , \ldots, 1)\,. \tag{2}$$

That is B is a diagonal matrix whose diagonal elements are given by $B_{11} = -1$ and $B_{ii} = +1$ for $i = 2, 3, \ldots, n$. Consider the SO(n) matrix given in block diagonal form,

$$C = \begin{pmatrix} C_2 & 0 \\ 0 & \boldsymbol{I_{n-2}} \end{pmatrix}\,,$$

where $\boldsymbol{I_{n-2}}$ is the $(n-2) \times (n-2)$ identity matrix, and C_2 is the 2×2 matrix defined by,

$$C_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\,.$$

A simple computation shows that $CBC^{-1} \notin \{\boldsymbol{I}_n, B\}$ if $\theta \neq \frac{1}{2}m\pi$ (for integer m).

Nevertheless, we can construct a map, $f : \mathrm{O}(n) \longrightarrow \{\boldsymbol{I}_n, B\}$, defined by

$$f(A) = \mathrm{diag}(\det A, 1, 1, \ldots, 1)\,, \quad \text{for } A \in \mathrm{O}(n).$$

The kernel of this map is $\ker f = \mathrm{SO}(n)$ and the image is $\mathrm{im}\, f = \mathbb{Z}_2 \cong \{\boldsymbol{I}_n, B\}$. Hence, it follows that SO(n) is a normal subgroup of O(n) and $\mathrm{O}(n)/\mathrm{SO}(n) \cong \mathbb{Z}_2$. Finally, any O($n$) matrix (with n even) can be generated by $\{hk\,|\,h \in \mathrm{SO}(n)\,,\,k \in \mathbb{Z}_2\}$. These results, taken together with $\mathrm{SO}(n) \cap \mathbb{Z}_2 = \{\boldsymbol{I}_n\}$ are sufficient to identify the semidirect product, $\mathrm{O}(n) \cong \mathrm{SO}(n) \rtimes \mathbb{Z}_2$.

2. Consider the dihedral group D_4 treated in Exercise 2.1 of Haber and Terning. The elements of this group are $D_4 = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$ with the group multiplication law determined by the relations $r^4 = e$, $f^2 = e$ and $fr = r^3f$, where e is the identity element.

(a) Write out an explicit irreducible two-dimensional representation of D_4. Check that the group multiplication table is preserved. Verify that this representation is irreducible.

In class, the following two-dimensional representation matrices for $r, f \in D_n$ were given,

$$r = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}, \qquad f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

For $n = 4$, eq. (3) yields:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad r^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad r^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad rf = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad r^2f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r^3f = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

One easily checks that the representation matrices exhibited in eq. (4) satisfy the D_4 group multiplication table.

To show that eq. (4) is an *irreducible* representation of D_4, we must prove that there is no basis in which the above matrices are reduced to block diagonal form. If such a basis existed, then we could simultaneously diagonalize the matrices that represent r and rf. But these elements do not commute and thus are not simultaneously diagonalizable.

For completeness, I now provide two other proofs that the above two-dimensional representation exhibited in eq. (4) is irreducible.

(i) In class, we proved that a necessary and sufficient condition for a representation $D(g)$ with characters $\chi(\mathcal{C}_k) \equiv \operatorname{Tr} D(g)$ [for $g \in \mathcal{C}_k$] to be irreducible is

$$\sum_{k=1}^{n_{\rm irr}} N_k |\chi(\mathcal{C}_k)|^2 = O(G)\,, \tag{5}$$

where N_k is the number of elements in conjugacy class $\mathcal{C}_k$, $n_{\rm irr}$ is the number of inequivalent irreducible representations of G, and $O(G)$ is the order of the group G. Using the results of problem 5(b) of Solution Set 1, the conjugacy classes of D_4 are

$$\mathcal{C}_1 = \{1\}\,, \quad \mathcal{C}_2 = \{r, r^3\}\,, \quad \mathcal{C}_3 = \{r^2\}\,, \quad \mathcal{C}_4 = \{f, r^2f\} \quad \text{and} \quad \mathcal{C}_5 = \{rf, r^3f\}\,. \tag{6}$$

Employing eqs. (4) and (6), we can immediately enumerate N_k and the $\chi(\mathcal{C}_k)$ for the two-dimensional irreducible representation of D_4,

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
N_k	1	2	1	2	2
$\chi(\mathcal{C}_k)$	2	0	-2	0	0

and check that eq. (5) is satisfied with $O(D_4) = 8$.

(ii) One can check explicitly that if $AD(g) = D(g)A$ for all $g \in D_4$, then A is a multiple of the identity. For this problem, it is enough to check that for an arbitrary 2×2 matrix A, if $Ar = rA$ and $Ad = fA$, with r and f given by the 2×2 matrices listed in eq. (4), then $A = c\mathbb{1}_{2\times2}$ for some complex number c (where $\mathbb{1}_{2\times2}$ is the 2×2 identity matrix). Hence, by Schur's second lemma, $D(g)$ is an irreducible representation of D_4.

(b) Construct the character table for the irreducible representations of D_4.

First, we need to specify all the irreducible representations of D_4. We already have identified a two-dimensional irreducible representation in part (c). Moreover, the trivial one-dimensional representation in which all elements of the group are represented by the 1×1 matrix (1) is always present. We now make use of the theorem obtained in class,

$$\sum_{i=1}^{n_{\rm irr}} n_i^2 = O(G)\,, \tag{7}$$

where n_i is the dimension of the ith inequivalent irreducible representation, and Part 2 which states that $n_{\rm irr} = n_c$, where $n_{\rm irr}$ the number of inequivalent irreducible representations and n_c is equal to the number of conjugacy classes. Applying eq. (7) to D_4, we have $O(D_4) = 8$, and it follows that D_4 must have four inequivalent one-dimensional representations.[1] The possible one-dimensional representations can be determined by inspection. In particular, we know that $f^2 = e$ and $fr = r^3 f$. Since f and r are represented by 1×1 matrices, it follows that as matrices, f and r commute. Then $fr = r^3 f$ implies that $r^2 = e$. Thus, the four one dimensional representations correspond to:

$$r = (1)\,,\ f = (1)\,, \qquad r = (1)\,,\ f = (-1)\,, \qquad r = (-1)\,,\ f = (1)\,, \quad r = (-1)\,,\ f = (-1)\,. \tag{8}$$

The character of a representation is equal to the trace of the corresponding representation matrix. Moreover, the character of all elements of a given class are equal. Thus, we can immediately write down the character table:

dimension	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
1	1	1	1	1	1
1	1	1	1	-1	-1
1	1	-1	1	1	-1
1	1	-1	1	-1	1
2	2	0	-2	0	0

The zeros in the last row of the character table are easy to understand. One can obtain equivalent two-dimensional irreducible representations by multiplying the irreducible representation exhibited in eq. (4) by either ± 1 according to the sign of r and f given by the four possible choices given in eq. (8). Since the characters of equivalent representations are equal, the characters of the two-dimensional irreducible representation corresponding to classes $\mathcal{C}_2$, $\mathcal{C}_4$ and $\mathcal{C}_5$ must be equal to their negatives, and hence must be zero. One can also check this character table by verifying that the class orthogonality relations are satisfied.

[1] By definition, all inequivalent one-dimensional representations of a group are irreducible.

3. A finite group G can be decomposed into conjugacy classes $\mathcal{C}_k$.

(a) Construct the set $\mathcal{C}'_k \equiv g\mathcal{C}_k g^{-1}$, which is obtained by replacing each element $x \in \mathcal{C}_k$ by gxg^{-1}. Prove that $\mathcal{C}'_k = \mathcal{C}_k$.

The elements of a finite group G (consisting of $n+1$ elements) will be denoted by $G = \{e, g_1, g_2, \ldots, g_n\}$ where e is the identity element. If $x \in \mathcal{C}_k$, then the elements of $\mathcal{C}_k$ are given by:

$$\mathcal{C}_k = \{x, g_1 x g_1^{-1}, g_2 x g_2^{-1}, \ldots, g_n x g_n^{-1}\}\,, \tag{9}$$

where we keep only distinct elements in the set $\mathcal{C}_k$ and discard any duplicate elements. The set of elements in $\mathcal{C}'_k \equiv g\mathcal{C}_k g^{-1}$, for a fixed element $g \in G$, is then given by:

$$\begin{aligned}
\mathcal{C}'_k &= \{gxg^{-1}, gg_1 x g_1^{-1} g^{-1}, gg_2 x g_2^{-1} g^{-1}, \ldots, gg_n x g_n^{-1} g^{-1}\} \\
&= \{gxg^{-1}, gg_1 x (gg_1)^{-1}, gg_2 x (gg_2)^{-1}, \ldots, gg_n x (gg_n)^{-1}\} \\
&= \{x, g_1 x g_1^{-1}, g_2 x g_2^{-1}, \ldots, g_n x g_n^{-1}\} \\
&= \mathcal{C}_k\,,
\end{aligned}$$

after using the rearrangement lemma in the penultimate step to reorder the elements of $\mathcal{C}'_k$.

(b) Suppose that $D^{(i)}(g)$ is the ith irreducible (finite-dimensional) matrix representation of the finite group G. For a fixed class $\mathcal{C}_k$, prove that

$$\sum_{g \in \mathcal{C}_k} D^{(i)}_{j\ell}(g) = \frac{N_k}{n_i} \chi^{(i)}(\mathcal{C}_k)\delta_{j\ell}\,, \tag{10}$$

where n_i is the dimension of the ith irreducible representation of G, N_k is the number of elements in the kth conjugacy class and $\chi^{(i)}(\mathcal{C}_k)$ is the irreducible character corresponding to the kth conjugacy class.

Define the matrix

$$A^{(i)}_k \equiv \sum_{\tilde{g} \in \mathcal{C}_k} D^{(i)}(\tilde{g})\,. \tag{11}$$

Then,

$$D^{(i)}(g) A^{(i)}_k = \sum_{\tilde{g} \in \mathcal{C}_k} D^{(i)}(g) D^{(i)}(\tilde{g}) = \sum_{\tilde{g} \in \mathcal{C}_k} D^{(i)}(g\tilde{g}) = \sum_{\tilde{g} \in \mathcal{C}_k} D^{(i)}(g\tilde{g}g^{-1}) D^{(i)}(g)\,,$$

where we have used the fact that the matrix representation $D^{(i)}(g)$ must obey the group multiplication law, $D^{(i)}(g_1 g_2) = D^{(i)}(g_1) D^{(i)}(g_2)$. Using the result of part (a), it follows that

$$\sum_{\tilde{g} \in \mathcal{C}_k} D^{(i)}(g\tilde{g}g^{-1}) D^{(i)}(g) = \sum_{g' \in \mathcal{C}'_k} D^{(i)}(g') D^{(i)}(g) = A^{(i)}_k D^{(i)}(g)\,,$$

after noting that $\mathcal{C}'_k = \mathcal{C}_k$ and g' is a dummy summation variable.

Thus, we have established that for any i and k,

$$D^{(i)}(g)A_k^{(i)} = A_k^{(i)}D^{(i)}(g)\,, \qquad \text{for } all \ g \in G.$$

Hence, Schur's second lemma applies, and it follows that

$$A_k^{(i)} = \lambda_k^{(i)}\mathbf{I}\,, \tag{12}$$

for some complex constant $\lambda_k^{(i)}$ (which can depend on i and k), where $\mathbf{I}$ is the identity matrix. Using eq. (11), we can rewrite eq. (12) as

$$\sum_{g\in\mathcal{C}_k} D_{j\ell}^{(i)}(g) = \lambda_k^{(i)}\delta_{j\ell}\,. \tag{13}$$

To determine the constant $\lambda_k^{(i)}$, we set $j=\ell$ in eq. (13) and sum over j, which yields:

$$N_k\chi^{(i)}(\mathcal{C}_k) = n_i\lambda_k^{(i)}\,,$$

where N_k, n_i and $\chi^{(i)}(\mathcal{C}_k)$ are defined in the statement of the problem. Solving for $\lambda_k^{(i)}$ then completes the derivation of eq. (10).

(c) Starting from the completeness result that is satisfied by the matrix elements of the irreducible matrix representations of G and using the result of part (b), derive the completeness relation for the irreducible characters,

$$\frac{N_k}{O(G)}\sum_i \chi^{(i)}(\mathcal{C}_k)[\chi^{(i)}(\mathcal{C}_\ell)]^* = \delta_{k\ell}\,, \tag{14}$$

where $O(G)$ is the order of the group G (i.e. the number of elements of G), and the sum is taken over all inequivalent (finite-dimensional) irreducible representations.

The completeness relation is given by:

$$\frac{1}{O(G)}\sum_i\sum_{m,n} n_i D_{mn}^{(i)}(g)D_{mn}^{(i)}(g')^* = \delta_{gg'}\,, \tag{15}$$

where $O(G)$ is the order of the group G (i.e., the number of elements of G). The sum over i runs over all inequivalent unitary irreducible representations of G. We now sum eq. (15) over $g\in\mathcal{C}_k$ and $g'\in\mathcal{C}_\ell$ and make use of eq. (10), which yields:

$$\frac{1}{O(G)}\sum_i\sum_{m,n} n_i\frac{N_k}{n_i}\chi^{(i)}(\mathcal{C}_k)\frac{N_\ell}{n_i}\chi^{(i)}(\mathcal{C}_\ell)^*\delta_{mn}\delta_{mn} = N_k\delta_{k\ell}\,, \tag{16}$$

since there are N_k elements in the class $\mathcal{C}_k$. Noting that $\delta_{mn}\delta_{mn} = \delta_{mm} = n_i$, it follows that the factors of n_i cancel in eq. (16). Thus, after summing over m and n, we obtain the completeness relation for the irreducible characters given in eq. (14).

(d) Using the orthogonality and the completeness relations satisfied by the irreducible characters, prove that the number of inequivalent irreducible representations of G is equal to the number of conjugacy classes.

The orthogonality relation satisfied by the irreducible characters was derived in class,

$$\frac{1}{O(G)}\sum_{k=1}^{n_c} N_k[\chi^{(i)}(\mathcal{C}_k)]^*\chi^{(j)}(\mathcal{C}_k) = \delta_{ij}\,. \tag{17}$$

The completeness relation derived in part (c) is given by

$$\frac{N_k}{O(G)}\sum_{i=1}^{n_{\rm irr}} \chi^{(i)}(\mathcal{C}_k)[\chi^{(i)}(\mathcal{C}_\ell)]^* = \delta_{k\ell}\,. \tag{18}$$

If we set $i = j$ in eq. (17) and sum over i, we obtain

$$n_{\rm irr} = \sum_{i=1}^{n_{\rm irr}} 1 = \sum_{k=1}^{n_c}\frac{N_k}{O(G)}\sum_{i=1}^{n_{\rm irr}}[\chi^{(i)}(\mathcal{C}_k)]^*\chi^{(i)}(\mathcal{C}_k) = \sum_{k=1}^{n_c} 1 = n_c\,,$$

after interchanging the order of the summation and using eq. (18). Hence, the number of inequivalent irreducible representations of G is equal to the number of conjugacy classes.

The same conclusion can be obtained by regarding the characters as vectors in "class" space, whose dimension is given by the number of distinct classes, n_c. Different vectors are labeled by i. The orthogonality relation then implies that there are $n_{\rm irr}$ mutually orthogonal vectors living in the n_c-dimensional class vector space, where $n_{\rm irr}$ are the number of inequivalent irreducible representations. But, the completeness relation implies that any vector in the class space can be expressed as a linear combination of the $n_{\rm irr}$ class vectors. That is, the $n_{\rm irr}$ class vectors form a complete set of mutually orthogonal vectors, which span the class space. It then immediately follows that $n_{\rm irr} = n_c$, since the number of vectors in a complete set of mutually orthogonal vectors is equal to the dimension of the vector space.

4. Consider the transformations of the triangle that make up the dihedral group D_3. The elements of this group are $D_3 = \{e, r, r^2, f, rf, r^2f\}$, with the group multiplication law determined by the relations $r^3 = e$, $f^2 = e$ and $fr = r^2f$, where e is the identity element. In class, the following two-dimensional representation matrices for $r, f \in D_n$ were given,

$$r = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n)\end{pmatrix}\,, \qquad f = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\,. \tag{19}$$

Setting $n = 3$, one can construct a two-dimensional matrix representation of D_3.

(a) Consider the six-dimensional function space W consisting of polynomials of degree 2 in two real variables (x, y):

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + h\,, \tag{20}$$

where $a, b, \ldots, h$ are complex constants. We can view $(a, b, \ldots, h)$ as a six-dimensional vector that lives in a vector space which is isomorphic to W. If we perform a transformation of (x, y) under D_3 according to the two-dimensional representation obtained from eq (19) with $n = 3$, then the polynomial $f(x, y)$ given by eq. (20) transforms into another polynomial. That is, the vector $(a, b, \ldots, h)$ transforms under D_3 according to a six-dimensional representation. Compute the 6×6 matrices that represent the elements of D_3. Determine which irreducible representations of D_3 are contained in this six-dimensional representation and their corresponding multiplicities.

One can rewrite eq. (20) as

$$f(x, y) = z^{\mathsf{T}} A z + z^{\mathsf{T}} B + h \,,$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}, \qquad B = \begin{pmatrix} d \\ e \end{pmatrix}.$$

Under a D_3 transformation, $z \to D(g)z$, where $D(g)$ is the corresponding 2×2 matrix obtained by using the two dimensional matrix representation given in eq. (19) with $n = 3$. It follows that under a D_3 transformation,

$$f(x, y) \longrightarrow z^{\mathsf{T}} A' z + z^{\mathsf{T}} B' + h \,,$$

where

$$A' = D(g)^{\mathsf{T}} A D(g) \,, \qquad B' = D(g)^{\mathsf{T}} B \,.$$

In particular, if

$$D(g) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

then we find after some algebra,

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ h \end{pmatrix} = \begin{pmatrix} d_{11}^2 & d_{11}d_{21} & d_{21}^2 & 0 & 0 & 0 \\ 2d_{11}d_{12} & d_{11}d_{22} + d_{12}d_{21} & 2d_{21}d_{22} & 0 & 0 & 0 \\ d_{12}^2 & d_{12}d_{22} & d_{22}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{11} & d_{21} & 0 \\ 0 & 0 & 0 & d_{12} & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ h \end{pmatrix},$$

under the action of D_3. Using the specific matrices given in eq. (19), we obtain the following six-dimensional representation of D_3,

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r^2 = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$rf = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r^2 f = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We know that D_3 possesses one two-dimensional and two inequivalent one-dimensional irreducible representations. To compute the number of times a given irreducible representation appears in the six-dimensional representation obtained above, we make use of the formula derived in class,

$$n_i = \sum_{k=1}^{n_c} \chi^{(i)\,*}(\mathcal{C}_k)\chi(\mathcal{C}_k)\frac{N_k}{O(G)}\,, \tag{21}$$

where the sum is taken over the n_c possible classes of D_3, N_k is the number of elements in the kth class, and $O(G)$ is the number of elements of G (the order of the group). We can read off the irreducible characters, $\chi^{(i)}(\mathcal{C}_k)$ from the character table of $D_3 \cong S_3$ obtained in class:

i	dimension	$\mathcal{C}_1 = \{e\}$	$\mathcal{C}_2 = \{f, rf, r^2 f\}$	$\mathcal{C}_3 = \{r, r^2\}$
1	1	1	1	1
2	1	1	-1	1
3	2	2	0	-1

where i labels one of the three possible irreducible representations of D_3. For the six-dimensional matrix representation obtained above,

$$\chi(\mathcal{C}_1) = 6\,, \qquad \chi(\mathcal{C}_2) = 2\,, \qquad \chi(\mathcal{C}_3) = 0\,.$$

Applying eq. (21), it follows that:

$$n_1 = 2\,, \qquad n_2 = 0\,, \qquad n_3 = 2\,.$$

That is, the six-dimensional representation of D_3 obtained above contains the trivial representation twice and the two-dimensional irreducible representation twice.

(b) Identify the irreducible invariant subspaces of W under D_3. Check that your result is consistent with the results of part (a).

The irreducible invariant subspaces are easily identified. The space spanned by (0 0 0 0 0 1) clearly corresponds to the trivial representation.[2] By inspection of the six-dimensional representation matrices given above, it follows that the space spanned by (0 0 0 1 0 0) and (0 0 0 0 1 0) corresponds to the irreducible two-dimensional representation given in eq. (19).

The final task is to decompose the remaining three-dimensional subspace into its irreducible components. Recall that

$$A' = D(g)^{\mathsf{T}} A D(g)\,, \qquad \text{where} \quad A = \begin{pmatrix} a & \tfrac{1}{2}b \\ \tfrac{1}{2}b & c \end{pmatrix}\,.$$

Then, if we choose $a = c$ and $b = 0$, it follows that $A' = A$ since in this case $A = a\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix.[3] Thus, it follows that (1 0 1 0 0 0) spans an invariant space, which is thus one-dimensional. We can now construct two linearly independent vectors that are orthogonal to the four vectors already identified. By inspection, these vectors can be chosen to be $(1\ 0\ -1\ 0\ 0\ 0)$ and (0 1 0 0 0 0). In light of the result of part (b), these two vectors must also span an invariant subspace under the action of D_3. This is indeed the case, as one can easily check by applying the six 6×6 matrices that represent the elements of D_3 to an arbitrary linear combination of $(1\ 0\ -1\ 0\ 0\ 0)$ and (0 1 0 0 0 0).

Thus, we have explicitly identified two 1-dimensional and two 2-dimensional irreducible subspaces of the vector space $\mathbb{C}^6$ under the action of the six-dimensional representation of D_3 obtained above.

5. Suppose that D is an irreducible n-dimensional representation of a finite group G, and $D^{(1)}$ is a (nontrivial) one-dimensional representation of G. Prove that the direct product $D \otimes D^{(1)}$ is an irreducible representation of G.

I will provide two different proofs.

Proof 1: Suppose that $D^{(i)}$ is an irreducible representation and $D^{(j)}$ is a one-dimensional irreducible representation. Then, no invertible matrix S exists such that

$$SD^{(i)}(g)S^{-1} = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix}\,.$$

Consider the matrix representation, $D^{(i\otimes j)} \equiv D^{(i)}(g)D^{(j)}(g)$. Since $D^{(j)}(g)$ is one-dimensional, we see that $D^{(j)}(g)$ is simply a (complex) number. Recalling that all matrix representations of a finite group are equivalent to unitary representations, it follows that $D^{(j)}(g)$ must be a complex number of unit modulus (since all complex numbers commute). That is,

$$|D^{(j)}(g)| = 1\,. \tag{22}$$

[2]To save space, all vectors will henceforth be specified by row vectors rather than the usual column vectors

[3]Since $D^{\mathsf{T}} = D^{-1}$ for the representation given in eq. (19) and $A = a\mathbf{I}$, we have $A' = D^{-1}a\mathbf{I}\,D = a\mathbf{I} = A$.

Thus, we conclude that no invertible matrix S exists such that:

$$SD^{(i\otimes j)}(g)S^{-1} = D^{(j)}(g)SD^{(i)}(g)S^{-1} = \begin{pmatrix} D^{(j)}(g)A(g) & D^{(j)}(g)B(g) \\ 0 & D^{(j)}(g)C(g) \end{pmatrix} .$$

It immediately follows that $D^{(i\otimes j)}(g)$ is irreducible.

<u>**Proof 2:**</u> Using eq. (22), it follows that any one-dimensional representation $D^{(j)}$ of a finite group must satisfy:

$$|\chi^{(j)}(g)| = |D^{(j)}(g)| = 1 ,$$

where $\chi^{(i)}(g) \equiv \operatorname{Tr} D^{(i)}(g)$ is the character of $g \in G$ for the ith irreducible representation. We also showed in class that for a direct product representation,

$$\chi^{(i\otimes j)}(g) = \chi^{(i)}(g)\chi^{(j)}(g) .$$

Hence, if $D^{(j)}$ is a one-dimensional representation,

$$|\chi^{(i\otimes j)}(g)| = |\chi^{(i)}(g)| . \tag{23}$$

The necessary and sufficient condition for a representation of a finite group to be irreducible is that

$$\sum_{k=1}^{n_{\rm irr}} N_k |\chi(\mathcal{C}_k)|^2 = O(G) , \tag{24}$$

where N_k is the number of elements in class $\mathcal{C}_k$ and $O(G)$ is the order of the group. If the $\chi^{(i)}$ satisfy eq. (24), then the $\chi^{(i\otimes j)}$ must also satisfy eq. (24) in light of eq. (23). Hence $D^{(i\otimes j)}$ must be an irreducible representation of G.

6. (a) Display all the standard Young tableaux of the permutation group S_4. From this result, enumerate the inequivalent irreducible representations of S_4 and specify their dimensions.

All the standard Young tableaux for S_4 are listed below, where each irreducible representation corresponds to a different Young diagram.

There are five possible Young diagrams involving four boxes, corresponding to five possible partitions of 4:

$$4 \,,\; 3+1 \,,\; 2+1+1 \,,\; 2+2 \text{ and } 1+1+1+1 \,.$$

Hence there are five inequivalent irreducible representations of S_4. The number of standard Young tableaux corresponding to a given Young diagram is equal to the dimension of the corresponding irreducible representation. As a check, we can use eq. (7) to verify that

$$1^2 + 3^2 + 3^2 + 2^2 + 1^2 = 24 = 4! \,,$$

which is equal to the order of the group S_4 as required.

<table>
<tr><th colspan="3">standard Young tableaux</th><th>dimension</th></tr>
<tr><td>$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$</td><td></td><td></td><td>1</td></tr>
<tr><td>$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$</td><td>$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$</td><td>$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$</td><td>3</td></tr>
<tr><td>$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$</td><td>$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$</td><td>$\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$</td><td>3</td></tr>
<tr><td>$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$</td><td>$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$</td><td></td><td>2</td></tr>
<tr><td>$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$</td><td></td><td></td><td>1</td></tr>
</table>

(b) Show that the normal subgroup $\{e, (12)(34), (13)(24), (14)(23)\}$ of S_4 is isomorphic to D_2. Using this result, prove that $D_3 \cong S_4/D_2$.

The multiplication table for $\{e, (12)(34), (13)(24), (14)(23)\}$ is given by:

	e	$(12)(34)$	$(13)(24)$	$(14)(23)$
e	e	$(12)(34)$	$(13)(24)$	$(14)(23)$
$(12)(34)$	$(12)(34)$	e	$(14)(23)$	$(13)(24)$
$(13)(24)$	$(13)(24)$	$(14)(23)$	e	$(12)(34)$
$(14)(23)$	$(14)(24)$	$(13)(24)$	$(12)(34)$	e

We recognize this as the multiplication table of the group $K_4 \equiv D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.[4] One can check that $D_2 \cong \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4. That is, for any element $h \in D_2$ and $g \in S_4$, one can verify that $ghg^{-1} \in D_2$.

Next, we examine the cosets of the form gD_2, for $g \in S_4$. First, we have

$$eD_2 = D_2 = \{e\,,\ (12)(34)\,,\ (13)(24)\,,\ (14)(23)\}\,. \tag{25}$$

[4]There are only two possible finite groups of four elements: $\mathbb{Z}_4$ and $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $\mathbb{Z}_4$ is a cyclic group with one generator. However, the multiplication table for $\{e, (12)(34), (13)(24), (14)(23)\}$ is clearly not a cyclic group with one generator. By the process of elimination, the only possible conclusion is that the multiplication table corresponds to the group $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Using the multiplication table of D_3, it then follows that:

$$(123)D_2 = \{(123)\,, (134)\,, (243)\,, (142)\}\,, \tag{26}$$
$$(132)D_2 = \{(132)\,, (234)\,, (124)\,, (143)\}\,, \tag{27}$$
$$(12)D_2 = \{(12)\,, (34)\,, (1324)\,, (1423)\}\,, \tag{28}$$
$$(13)D_2 = \{(13)\,, (24)\,, (1234)\,, (1432)\}\,, \tag{29}$$
$$(23)D_2 = \{(23)\,, (14)\,, (1243)\,, (1342)\}\,. \tag{30}$$

All other cosets coincide with one of the six specified above.

Since D_2 is a normal subgroup of S_4, it follows that S_4/D_2 is a group of six elements. There are only two possible groups of six elements: $\mathbb{Z}_6$ which is an abelian cyclic group and $D_3 \cong S_3$ which is non-abelian. It is straightforward to check that the group multiplication law of the cosets given above is non-abelian. Since the multiplication rule of the cosets is given by $(g_1D_2)(g_2D_2) = g_1g_2D_2$, it follows that

$$[(12)D_2][(13)D_2] = (12)(13)D_2 = (132)D_2\,, \tag{31}$$
$$[(13)D_2][(12)D_2] = (13)(12)D_2 = (123)D_2\,, \tag{32}$$

so that $[(12)D_2][(13)D_2] \neq [(13)D_2][(12)D_2]$. Hence, S_4/D_2 is non-abelian, and we conclude that $D_3 \cong S_4/D_2$

One can also verify directly that $D_3 \cong S_4/D_2$ by checking that there exists a one-to-one correspondence of the multiplication table of the cosets,

$$S_4/D_2 = \{eD_2\,, (123)D_2\,, (132)D_2\,, (12)D_3\,, (13)D_2\,, (23)D_2\}\,,$$

and the multiplication table of $D_3 \cong S_3 = \{e, (12), (13), (23), (123), (132)\}$. Indeed, one can identify the elements of D_3 with those of S_4/D_2 as follows. Recall that D_3 can be formally defined as:

$$D_3 = \{f^k r^\ell \mid f^2 = e\,,\ r^3 = e\,,\ fr = r^2 f\}\,.$$

Thus, we can identify $f = (12)D_2$ and $r = (123)D_2$. In particular,

$$(12)(12) = e\,, \qquad (123)(123)(123) = e\,, \qquad (12)(123) = (123)(123)(12)\,,$$

where we have chosen the simplest representative elements from the cosets $(12)D_2$ and $(123)D_2$. It then follows that

$$r^2 = (132)D_2\,, \qquad rf = (13)D_2\,, \qquad r^2 f = (23)D_2\,,$$

and of course $e = eD_2$. This completes the explicit identification of the elements of D_3 with the cosets gD_2.

(c) Using the two-dimensional irreducible representation of D_3 given in class and the result of part (b), construct a two-dimensional representation of S_4 and determine its characters. Is the latter an *irreducible* representation of S_4?

HINT: Show that given a normal subgroup N of a group G and a representation $D^{G/N}$ of the factor group G/N, one can construct a representation D^G of the group G by defining $D^G(g) \equiv D^{G/N}(gN)$ for all $g \in G$.

In light of the hint provided above, we first prove that $D^G(g)$ is a representation of G. That is, we must show that

$$D^G(g_1)D^G(g_2) = D^G(g_1g_2)\,. \tag{33}$$

This result follows from the definition, $D^G(g) \equiv D^{G/N}(gN)$ In particular, recall that the group multiplication law for G/N is:

$$(g_1N)(g_2N) = g_1g_2N\,.$$

Thus, if $D^{G/N}$ is a representation of G/N, it follows that:

$$D^{G/N}(g_1N)D^{G/N}(g_2N) = D^{G/N}(g_1g_2N)\,. \tag{34}$$

Hence, it follows that:

$$D^G(g_1)D^G(g_2) = D^{G/N}(g_1N)D^{G/N}(g_2N) = D^{G/N}(g_1g_2N) = D^G(g_1g_2)\,,$$

which confirms eq. (33).[5]

In part (b), we showed that $D_3 \cong S_4/D_2$. In light of the hint, we can employ a representation of D_3 to obtain an unfaithful representation of S_4. In particular, the two-dimensional representation of D_3 is given by eq. (19) with $n = 3$,

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad r^2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad rf = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \qquad r^2f = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

Note that r corresponds to an active counterclockwise rotation by $2\pi/3$ radians about an axis perpendicular to the rotation plane.

The representation of S_4 obtained from the representation of D_3 (following the hint) is unfaithful, since four elements of S_4 are mapped into the same matrix. In particular,

[5]One can also check that $D(e) = \mathbf{I}$ and $D^G(g^{-1}) = [D^G(g)]^{-1}$. For example, since $D^{G/N}$ is a representation of G/N and $eN = N$ is the identity of the subgroup G/N, it follows that $D^G(e) = \mathbf{I}$. Likewise, using the fact that the inverse of gN is $g^{-1}N$, it follows that $D^G(g^{-1}) = D^{G/N}(g^{-1}N) = [D^{G/N}(gN)]^{-1} = [D^G(g)]^{-1}$.

eqs. (25)–(30) yield,

$$e\,,\,(12)(34)\,,\,(13)(24)\,,\,(14)(23)\quad\longrightarrow\quad\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,,\tag{35}$$

$$(123)\,,\,(134)\,,\,(243)\,,\,(142)\quad\longrightarrow\quad\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\,,\tag{36}$$

$$(132)\,,\,(234)\,,\,(124)\,,\,(143))\quad\longrightarrow\quad\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\,,\tag{37}$$

$$(12)\,,\,(34)\,,\,(1324)\,,\,(1423)\quad\longrightarrow\quad\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\,,\tag{38}$$

$$(13)\,,\,(24)\,,\,(1234)\,,\,(1432)\quad\longrightarrow\quad\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\,,\tag{39}$$

$$(23)\,,\,(14)\,,\,(1243)\,,\,(1342)\quad\longrightarrow\quad\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\,.\tag{40}$$

The characters are obtained by taking the trace of the corresponding representation matrices.

Since elements of the same conjugacy class possess the same characters, we first determine the conjugacy classes of S_4. Recall that all elements of S_n of a given conjugacy class possess the same cycle structure (which are in one-to-one correspondence with the possible Young diagrams). Hence,

$$\begin{aligned}
&\mathcal{C}_1=\{e\}\,,\\
&\mathcal{C}_2=\{(12)\,,\,(13)\,,\,(23)\,,\,(14)\,,\,(24)\,,\,(34)\}\,,\\
&\mathcal{C}_3=\{(123)\,,\,(132)\,,\,(124)\,,\,(142)\,,\,(134)\,,\,(143)\,,\,(234)\,,\,(243)\}\,,\\
&\mathcal{C}_4=\{(12)(34)\,,\,(13)(24)\,,\,(14)(23)\}\,,\\
&\mathcal{C}_5=\{(1234)\,,\,(1243)\,,\,(1324)\,,\,(1342)\,,\,(1423)\,,\,(1432)\}
\end{aligned}$$

Comparing the elements of the conjugacy classes with the cosets given in eqs. (25)–(30), it follows that:

$$\begin{aligned}
&\text{elements of } eD_2 \text{ belong to classes } \mathcal{C}_1 \text{ and } \mathcal{C}_4\,,\\
&\text{elements of } (123)D_2 \text{ belong to class } \mathcal{C}_3\,,\\
&\text{elements of } (132)D_2 \text{ belong to classes } \mathcal{C}_3\,,\\
&\text{elements of } (12)D_2 \text{ belong to classes } \mathcal{C}_2 \text{ and } \mathcal{C}_5\,,\\
&\text{elements of } (13)D_2 \text{ belong to classes } \mathcal{C}_2 \text{ and } \mathcal{C}_5\,,\\
&\text{elements of } (23)D_2 \text{ belong to classes } \mathcal{C}_2 \text{ and } \mathcal{C}_5\,.
\end{aligned}$$

Thus, using the two-dimensional representation of S_4 obtained in eqs. (35)–(40), it follows that the corresponding characters are given by:

$$\chi(\mathcal{C}_1) = \chi(\mathcal{C}_4) = 2\,, \qquad \chi(\mathcal{C}_3) = -1\,, \qquad \chi(\mathcal{C}_2) = \chi(\mathcal{C}_5) = 0\,. \tag{41}$$

We can check whether the two-dimensional representation of S_4 obtained in eqs. (35)–(40) is irreducible by employing the theorem that states that a representation is irreducible if and only if

$$\sum_{k=1}^{n_{\rm irr}} N_k |\chi(\mathcal{C}_k)|^2 = O(G)\,,$$

where N_k is the number of elements in class $\mathcal{C}_k$ and $O(G)$ is the order of the group. For S_4, we have $O(G) = 4! = 24$ and $N_1 = 1\,,\ N_2 = 6\,,\ N_3 = 8\,,\ N_4 = 3\,,\ N_5 = 6$. The representation given by the matrices in eqs. (35)–(40) is irreducible if:

$$1 \cdot 2^2 + 6 \cdot 0 + 8 \cdot (-1)^2 + 3 \cdot 2^2 + 6 \cdot 0 \stackrel{?}{=} 24\,.$$

Indeed, $4 + 8 + 12 = 24$, so that we conclude that the irreducible (simple) characters corresponding to the two-dimensional irreducible representation of S_4 are given by eq. (41).

(d) Using the known one-dimensional representations of S_4 and the results of parts (a) and (c), construct the character table for the group S_4. Determine any unknown entries in the character table by using the orthonormality and completeness relations for the irreducible characters. Using this technique, all entries of the character table can be uniquely determined up to a sign ambiguity in some of the entries.

Using all known information (up to this point), the character table for S_4 is given by:

irreps	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
$\yng(4)$	1	1	1	1	1
$\yng(1,1,1,1)$	1	-1	1	1	-1
$\yng(2,2)$	2	0	-1	2	0
$\yng(3,1)$	3	a	b	c	d
$\yng(2,1,1)$	3	e	f	g	h

Note that the dimension d of the irreducible representation (irrep) is equal to the character of the identity element, which corresponds to conjugacy class $\mathcal{C}_1$, since the matrix representation of the identity is the $d \times d$ identity matrix. Moreover, the totally symmetric one-dimensional representation corresponds to representing each permutation by 1, and the totally antisymmetric one-dimensional representation corresponds to representing each permutation by $(-1)^p$, which is $+1$ for even permutations (i.e., classes $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_4$), and -1 for odd permutations (i.e., classes $\mathcal{C}_2$ and $\mathcal{C}_5$). This immediately yields the first two lines of the character table above. The third line of the character table was obtained in eq. (41).

The numbers a, b,...,h represent the presently unknown entries of the character table, which we shall determine by employing the orthogonality and completeness relations for the simple characters,

$$\sum_{k=1}^{n_c} N_k \chi^{(i)\,*}(\mathcal{C}_k)\chi^{(j)}(\mathcal{C}_k) = O(G)\delta_{ij}\,, \qquad\qquad \sum_{i=1}^{n_{\rm irr}} \chi^{(i)\,*}(\mathcal{C}_k)\chi^{(i)}(\mathcal{C}_\ell) = \frac{O(G)}{N_k}\delta_{k\ell}\,,$$

where $k = 1, 2, \ldots, n_c$ labels the conjugacy classes, N_k is equal to the number of group elements in conjugacy class $\mathcal{C}_k$, $i = 1, 2, \ldots n_{\rm irr}$ labels the irreps, and $O(G)$ is equal to the number of elements in the group G.

According to a theorem proved in class, the simple characters of S_n are all real. Thus, we can ignore the complex conjugation in the orthogonality and completeness relations. First, we make use of the orthogonality of row 4 with rows 1, 2 and 3, respectively. Using $N_1 = 1$, $N_2 = 6$, $N_3 = 8$, $N_4 = 3$ and $N_5 = 6$, we obtain three relations,

$$\begin{aligned} 3 + 6a + 8b + 3c + 6d &= 0\,, \\ 3 - 6a + 8b + 3c - 6d &= 0\,, \\ 6 - 8b + 6c &= 0\,. \end{aligned}$$

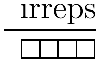
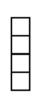
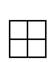
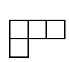
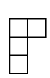
Solving these three equations yields $d = -a$, $c = -1$ and $b = 0$. Next, we make use of the orthogonality of row 5 with rows 1, 2 and 3, respectively. By an identical analysis, we obtain $h = -e$, $g = -1$ and $f = 0$. Next, we make use of the orthogonality of rows 4 and 5 to obtain

$$9 + 6ae + 8bf + 3cg + 6dh = 0\,.$$

Using the relations previously obtained, it follows that $ae = -1$. Finally, we make use of the completeness relation for columns 1 and 2, which yields:[6]

$$1 - 1 + 0 + 3a + 3e = 0\,,$$

which implies that $a = -e$. Combining this with $ae = -1$ yields $a^2 = e^2 = 1$. Hence, $a = -e = \pm 1$, where the sign ambiguity is not yet resolved.[7] Thus, the character table of S_4 is given at the top of the next page.

irreps	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
	1	1	1	1	1
	1	-1	1	1	-1
	2	0	-1	2	0
	3	± 1	0	-1	∓ 1
	3	∓ 1	0	-1	± 1

The sign ambiguity above will be resolved in part (e) of this problem.

[6] We could also make use of the orthogonality relation for row 4 (or 5) by itself. This yields $a^2 = e^2 = 1$, which when combined with $ae = -1$ yields $a = -e = \pm 1$.

[7] Note that the completeness relation for column 2 (or 5) alone yields, $1 + 1 + 0 + a^2 + e^2 = \frac{24}{6}$. Combining this with $a = -e$, it again follows that $a^2 = e^2 = 1$.

(e) Resolve the sign ambiguity of part (d). One possible approach is to construct the matrix representative of the transposition (1 2) corresponding to the three-dimensional irreducible representation of S_4. By taking the trace of this matrix, complete the character table of S_4.

Using the method discussed in class, I shall determine an explicit irreducible three-dimensional matrix representation for the element $(1\ 2) \in \mathcal{C}_2$. The Young element corresponding to

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{42}$$

is given by

$$\begin{aligned}\mathcal{Y} &= \big[e + (1\ 2)\big]\big[e + (1\ 3\ 4) + (1\ 4\ 3) - (1\ 3) - (1\ 4) - (3\ 4)\big] \\ &= e + (1\ 3\ 4) + (1\ 4\ 3) - (1\ 3) - (1\ 4) - (3\ 4) \\ &\qquad +(1\ 2) + (1\ 3\ 4\ 2) + (1\ 4\ 3\ 2) - (1\ 3\ 2) - (1\ 4\ 2) - (1\ 2)(3\ 4)\,.\end{aligned} \tag{43}$$

Since eq. (42) corresponds to a three-dimensional irreducible representation of S_4, we need to find two additional elements of the group algebra of S_4 that span the irreducible three-dimensional subspace of the regular representation. Since (2 3) and (2 4) do not appear in eq. (43), two other possible elements of the group algebra that span the irreducible three-dimensional subspace of the regular representation are:

$$\begin{aligned}(2\ 3)\mathcal{Y} &= (2\ 3) + (1\ 2\ 3\ 4) + (1\ 4\ 2\ 3) - (1\ 2\ 3) - (1\ 4)(2\ 3) - (2\ 3\ 4) \\ &\qquad +(1\,3\ 2) + (1\ 2)(3\ 4) + (1\ 4\ 2) - (1\ 2) - (1\ 4\ 3\ 2) - (1\ 3\ 4\ 2)\,,\end{aligned} \tag{44}$$

and

$$\begin{aligned}(2\ 4)\mathcal{Y} &= (2\ 4) + (1\ 3\ 2\ 4) + (1\ 2\ 4\ 3) - (1\ 2\ 4) - (1\ 3)(2\ 4) - (2\ 4\ 3) \\ &\qquad +(1\,4\ 2) + (1\ 2)(3\ 4) + (1\ 3\ 2) - (1\ 2) - (1\ 3\ 4\ 2) - (1\ 4\ 3\ 2)\,.\end{aligned} \tag{45}$$

Thus, we choose the basis, $\big\{\mathcal{Y}\,,\,(2\ 3)\mathcal{Y}\,,\,(2\ 4)\mathcal{Y}\big\}$, for the irreducible three-dimensional subspace of the regular representation corresponding to eq. (42). We represent these basis vectors by

$$\mathcal{Y} = \begin{pmatrix}1\\0\\0\end{pmatrix}\,,\qquad (2\ 3)\mathcal{Y} = \begin{pmatrix}0\\1\\0\end{pmatrix}\,,\qquad (2\ 4)\mathcal{Y} = \begin{pmatrix}0\\0\\1\end{pmatrix}\,.$$

With respect to this basis choice, any element $g \in S_4$ can be represented by the matrix

$$D[g] = \left(\begin{array}{c|c|c} g\mathcal{Y} & g(2\ 3)\mathcal{Y} & g(2\ 4)\mathcal{Y}\end{array}\right)\,, \tag{46}$$

where the three respective columns of the matrix representation of g are indicated above.

We now compute the three-dimensional matrix representation of $g = (1\ 2)$. Using

$$(1\ 2)(2\ 3) = (1\ 2\ 3)\,, \qquad\qquad (1\ 2)(2\ 4) = (1\ 2\ 4)\,,$$

all we need to compute is:

$$\begin{aligned}
(1\ 2)\mathcal{Y} &= (1\ 2) + (1\ 3\ 4\ 2) + (1\ 4\ 3\ 2) - (1\ 3\ 2) - (1\ 4\ 2) - (1\ 2)(3\ 4) \\
&\qquad +e + (1\ 3\ 4) + (1\ 4\ 3) - (1\ 3) - (1\ 4) - (1\ 3\ 4) \\
&= \mathcal{Y}\,, \\
(1\ 2)(2\ 3)\mathcal{Y} = (1\ 2\ 3)\mathcal{Y} &= (1\ 2\ 3) + (2\ 3\ 4) + (1\ 4)(2\ 3) - (2\ 3) - (1\ 4\ 2\ 3) - (1\ 2\ 3\ 4) \\
&\qquad +(1\ 3) + (3\ 4) + (1\ 4) - e - (1\ 4\ 3) - (1\ 3\ 4) \\
&= -\mathcal{Y} - (2\ 3)\mathcal{Y}\,, \\
(1\ 2)(2\ 4)\mathcal{Y} = (1\ 2\ 4)\mathcal{Y} &= (1\ 2\ 4) + (2\ 4\ 3) + (1\ 3)(2\ 4) - (2\ 4) - (1\ 3\ 2\ 4) - (1\ 2\ 4\ 3) \\
&\qquad +(1\ 4) + (3\ 4) + (1\ 3) - e - (1\ 3\ 4) - (1\ 4\ 3) \\
&= -\mathcal{Y} - (2\ 4)\mathcal{Y}\,.
\end{aligned}$$

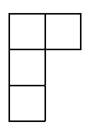
That is,

$$(1\ 2)\mathcal{Y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad (1\ 2)(2\ 3)\mathcal{Y} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \qquad (1\ 2)(2\ 4)\mathcal{Y} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}.$$

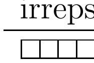
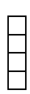
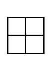
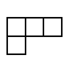
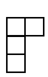
Thus, eq. (46) yields:

$$D[(1\ 2)] = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The trace of $D[(1\ 2)]$ yields the character, $\chi(\mathcal{C}_2) = \mathrm{Tr}\ D[(1\ 2)] = -1$, for the three-dimensional irreducible representation corresponding to the Young diagram,



That is, in the notation of part (e), we have $e = -1$ and therefore $a = 1$. The character table for S_4 is now complete:

irreps	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
	1	1	1	1	1
	1	-1	1	1	-1
	2	0	-1	2	0
	3	1	0	-1	-1
	3	-1	0	-1	1

7. (a) Derive the following properties of the Pauli matrices $\vec{\boldsymbol{\sigma}} \equiv (\sigma_1\,,\,\sigma_2\,,\,\sigma_3)$:

$$\begin{aligned}
&\text{(i)}\ \sigma_i\sigma_j = \mathbf{I}\delta_{ij} + i\epsilon_{ijk}\sigma_k\,,\\
&\text{(ii)}\ \sigma_2\,\vec{\boldsymbol{\sigma}}\,\sigma_2 = -\vec{\boldsymbol{\sigma}}^{\,*}\,,\\
&\text{(iii)}\ \exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2) = \mathbf{I}\cos(\theta/2) - i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)\,,
\end{aligned}$$

where I is the 2×2 identity matrix.

By direct calculation, $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbf{I}$, where $\mathbf{I} \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Moreover,

$$\sigma_1\sigma_2 = i\sigma_3\,, \qquad \sigma_2\sigma_3 = i\sigma_1\,, \qquad \sigma_3\sigma_1 = i\sigma_2\,.$$

These results are summarized by one equation,

$$\sigma_i\sigma_j = \mathbf{I}\delta_{ij} + i\epsilon_{ijk}\sigma_k\,, \tag{47}$$

where there is an implicit sum over the repeated index k. Next, it is a simple exercise of matrix multiplication to show that $\sigma_2\,\vec{\boldsymbol{\sigma}}\,\sigma_2 = -\vec{\boldsymbol{\sigma}}^{\,*}$ by verifying that $\sigma_2\sigma_i\sigma_2 = -\sigma_i^*$ for $i = 1, 2, 3$.

Finally, we compute

$$e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}}{2}\right)^k\,.$$

Using eq. (47), it follows that $(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})^2 = \hat{n}_i\hat{n}_j\sigma_i\sigma_j = \mathbf{I}$, since $\hat{\boldsymbol{n}}$ is a unit vector. Thus, for any positive integer k, $(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})^{2k} = \mathbf{I}$ and $(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})^{2k+1} = \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}$. Hence,

$$\begin{aligned}
e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2} &= \mathbf{I}\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\frac{i\theta}{2}\right)^{2k} - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{i\theta}{2}\right)^{2k+1}\\
&= \mathbf{I}\cos(\theta/2) - i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)\,.
\end{aligned}$$

(b) In the angle-and-axis parameterization of SO(3), a rotation by an angle θ about an axis that points along the unit vector $\hat{\boldsymbol{n}}$ is represented by an SO(3) matrix given by $R_{ij}(\hat{\boldsymbol{n}},\theta) = \exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})_{ij}$, with $(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})_{ij} \equiv -i\epsilon_{ijk}n_k$. By convention, we assume that $0 \leq \theta \leq \pi$, and the axis $\hat{\boldsymbol{n}}$ can point in any direction. Evaluate R_{ij} explicitly and show that

$$R_{ij}(\hat{\boldsymbol{n}},\theta) = n_in_j + (\delta_{ij} - n_in_j)\cos\theta - \epsilon_{ijk}n_k\sin\theta\,. \tag{48}$$

To evaluate the 3×3 matrix $R(\hat{\boldsymbol{n}},\theta)$, we compute

$$R(\hat{\boldsymbol{n}},\theta) = e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}} = \sum_{k=0}^{\infty}\frac{1}{k!}(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})^k\,, \tag{49}$$

where $(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})_{ij} \equiv -i\epsilon_{ijk}n_k$. Note that

$$(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})^2_{ij} = -\epsilon_{i\ell k}n_k\epsilon_{\ell jm}n_m = (\delta_{ij}\delta_{km} - \delta_{im}\delta_{jk})n_kn_m = \delta_{ij} - n_in_j\,,$$

after employing the well-known ϵ-tensor identity and noting that $\delta_{km}n_k n_m = 1$ for the unit vector $\hat{\boldsymbol{n}}$. Next, we compute:

$$(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})^3_{ij} = -i(\delta_{i\ell} - n_i n_\ell)\epsilon_{\ell jk}n_k = -i\epsilon_{ijk}n_k = (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})_{ij}\,.$$

Thus, for any positive integers k,

$$(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})^{2k-1} = \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}\,, \qquad\qquad (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})^{2k} = (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})^2\,.$$

Inserting these results in eq. (49), we obtain:

$$\begin{aligned}
R_{ij}(\hat{\boldsymbol{n}},\theta) &= \delta_{ij} - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})_{ij}\sum_{k=0}^\infty \frac{1}{(2k+1)!}(i\theta)^{2k+1} + (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})^2_{ij}\sum_{k=1}^\infty \frac{1}{(2k)!}(i\theta)^{2k} \\
&= \delta_{ij} - i(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})_{ij}\sin\theta + (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})^2_{ij}(\cos\theta - 1) \\
&= \delta_{ij} - \epsilon_{ijk}n_k\sin\theta + (\delta_{ij} - n_i n_j)(\cos\theta - 1) \\
&= n_i n_j + (\delta_{ij} - n_i n_j)\cos\theta - \epsilon_{ijk}n_k\sin\theta\,.
\end{aligned}$$

(c) Verify the formula:

$$e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2} = R_{ij}(\hat{\boldsymbol{n}},\theta)\,\sigma_i\,.$$

To evaluate $e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}$, we first note that this quantity is a traceless hermitian 2×2 matrix. The traceless condition follows from:

$$\mathrm{Tr}(e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}) = \mathrm{Tr}\,\sigma_j = 0\,,$$

after cyclically permuting the terms inside the parenthesis. Hermiticity is also easily demonstrated given that $\sigma_j^\dagger = \sigma_j$. Since an arbitrary traceless hermitian matrix can always be written as a real linear combination of σ_1, σ_2 and σ_3, it follows that

$$e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2} = R_{ij}(\hat{\boldsymbol{n}},\theta)\,\sigma_i\,, \tag{50}$$

for some real coefficients R_{ij}. To determine these coefficients, we multiply eq. (50) by σ_k and take the trace of the resulting equation. Using $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$ [which immediately follows after taking the trace of eq. (47)], it follows that:

$$R_{ij} = \tfrac{1}{2}\mathrm{Tr}\left(e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\sigma_i\right)\,. \tag{51}$$

This is easily evaluated using the results of part (a).

$$\begin{aligned}
R_{ij} &= \tfrac{1}{2}\mathrm{Tr}\Big\{[\cos(\theta/2) - i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)]\,\sigma_j\,[\cos(\theta/2) - i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)]\,\sigma_i\Big\} \\
&= \tfrac{1}{2}\cos^2(\theta/2)\mathrm{Tr}\,\sigma_i\sigma_j + \tfrac{1}{2}\sin^2(\theta/2)\mathrm{Tr}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_j) \\
&\qquad + \tfrac{1}{2}i\cos(\theta/2)\sin(\theta/2)\mathrm{Tr}(\sigma_j\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_i - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_j\sigma_i)\,.
\end{aligned} \tag{52}$$

We proceed to work out the three traces. The first trace has already been obtained, $\text{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$. The second trace is given by:

$$
\begin{aligned}
\text{Tr}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_j) &= n_k n_\ell \text{Tr}(\sigma_k\sigma_i\sigma_\ell\sigma_j) \\
&= n_k n_\ell \text{Tr}\big(\big[(\mathbf{I}\delta_{ki} + i\epsilon_{kim}\sigma_m)(\mathbf{I}\delta_{\ell j} + i\epsilon_{\ell jn}\sigma_n)\big] \\
&= 2n_k n_\ell \delta_{ki}\delta_{\ell j} - 2n_k n_\ell \epsilon_{kim}\epsilon_{\ell jn}\delta_{mn} \\
&= 2n_i n_j - 2n_k n_\ell(\delta_{k\ell}\delta_{ij} - \delta_{kj}\delta_{i\ell}) \\
&= 4n_i n_j - 2\delta_{ij}\,,
\end{aligned}
$$

after using $\text{Tr}\,\mathbf{I} = 2$, $\text{Tr}\,\sigma_i = 0$, and $\text{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$. Finally, the third trace is given by:

$$
\text{Tr}(\sigma_j\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_i - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_j\sigma_i) = \text{Tr}\big[(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}(\sigma_i\sigma_j - \sigma_j\sigma_i)\big] = 2i\epsilon_{ijk}\text{Tr}\big[\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sigma_k\big] = 4i\epsilon_{ijk}n_k\,,
$$

after making use of the commutation relations, $[\sigma_i\,,\,\sigma_j] = 2i\epsilon_{ijk}\sigma_k$; the latter is a consequence of eq. (47). Inserting these traces back into eq. (52) yields,

$$
R_{ij} = \cos^2(\theta/2) + \sin^2(\theta/2)(2n_i n_j - \delta_{ij}) - 2\sin(\theta/2)\cos(\theta/2)\epsilon_{ijk}n_k\,.
$$

Finally, using the well-known trigonometric identities,

$$
\sin\theta = 2\sin(\theta/2)\cos(\theta/2)\,, \qquad \cos^2(\theta/2) = \tfrac{1}{2}(1+\cos\theta)\,, \qquad \sin^2(\theta/2) = \tfrac{1}{2}(1-\cos\theta)\,,
$$

we arrive at:

$$
R_{ij} = \delta_{ij} + (\delta_{ij} - n_i n_j)\cos\theta - \epsilon_{ijk}n_k\sin\theta\,,
$$

which are the matrix elements of the rotation matrix obtained in part (b).

(d) The set of matrices $\exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2)$ constitutes the defining representation of SU(2). Prove that this representation is pseudoreal.

To prove that $D(\theta) \equiv \exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2) = \cos(\theta/2) - i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)$ is a pseudoreal representation of SU(2), we must first prove that $D(\theta)$ and $D^*(\theta)$ are equivalent representations. Noting that

$$
D^*(\theta) = \exp(i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}^*/2) = \mathbf{I}\cos(\theta/2) + i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}^*\sin(\theta/2)\,,
$$

it is straightforward to check that

$$
\sigma_2 D(\theta)\sigma_2 = D^*(\theta)\,. \tag{53}
$$

This is a consequence of properties (i) and (ii) of part (a). Since $\sigma_2^{-1} = \sigma_2$, we can rewrite eq. (53) as $D^*(\theta) = S^{-1}D(\theta)S$, where $S = \sigma_2$. Consequently, $D(\theta)$ and $D^*(\theta)$ are equivalent representations.

To show that $D(\theta)$ is pseudoreal, we can employ the theorem (proved in class) that states that if $D^* = ADA^{-1}$ where $A^*A = -\mathbf{I}$ then D is pseudoreal. In the present case, $A = \sigma_2$ and $\sigma_2^*\sigma_2 = -\mathbf{I}$, which confirms that $D(\theta)$ is pseudoreal.

ADDED NOTE: A direct proof that $D(\theta)$ is pseudoreal.

If $D(\theta)$ is pseudoreal, then no basis exists in which $D(\theta) = \cos(\theta/2) - i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)$ is real. That is, no invertible matrix S exists such that $S^{-1}D(\theta)S$ is real. This requirement is equivalent to the condition that no invertible matrix S exists such that $iS^{-1}\vec{\boldsymbol{\sigma}}S$ is real. That is, no basis exists in which the $i\sigma_k$ are simultaneously real. Without loss of generality, one can assume that $\det S = 1$ by rescaling the elements of S (since the overall determinant factor cancels in $iS^{-1}\vec{\boldsymbol{\sigma}}S$). One way to prove that no such matrix S exists is by writing

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where $ad - bc = 1$ and then computing $iS^{-1}\sigma_k S$ for $k = 1, 2, 3$. Assuming that all the matrix elements of $iS^{-1}\sigma_k S$ are real, one quickly reaches a contradiction.

Admittedly, this is not a very elegant proof. If S is unitary, then I can assume that its determinant is equal to one without loss of generality (by rescaling the elements of S as above). Then S is an SU(2) matrix which can be expressed as $S = \exp(i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2)$. In this case, we can use the results of part (c) to obtain

$$iS^{-1}\sigma_k S = ie^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\sigma_k e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2} = iR_{jk}\sigma_j\,, \tag{54}$$

which is simultaneously real for $k = 1, 2, 3$ if $R_{1k} = R_{3k} = 0$. But this is impossible since the R_{kj} are matrix elements of an orthogonal matrix which cannot possess a row of zeros. Taking S to be unitary means that $S^{-1}D(\theta)S$ is unitary, so this last argument establishes that no real representation exists that is unitarily equivalent to $D(\theta)$. However, this last argument does not yield the stronger result that no real representation exists (whether unitary or not) that is equivalent to $D(\theta)$. A direct proof of the latter result, which is responsible for the theorem cited in the paragraph following eq. (53), is given below.

Suppose a nonsingular matrix S exists such that

$$A \equiv S^{-1}i\sigma_k S\,, \qquad \text{where } A \text{ is a real matrix}\,. \tag{55}$$

Then taking the complex conjugate of this equation (using $A = A^*$) yields

$$S^{-1}i\sigma_k S = -S^{*\,-1}i\sigma_k^* S^*\,,$$

which can be rewritten as

$$-i\sigma_k^* = (SS^{*\,-1})^{-1}i\sigma_k SS^{*\,-1}\,.$$

Using property (ii) of part (a) of this problem, $\sigma_k^* = -\sigma_2\sigma_k\sigma_2$, then yields:

$$SS^{*\,-1}\sigma_2 i\sigma_k = i\sigma_k SS^{*\,-1}\sigma_2\,. \tag{56}$$

Schur's second lemma states that if $ZD(\theta) = D(\theta)Z$ for all θ, then Z is a multiple of the identity. Using $D(\theta) = \cos(\theta/2) - i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)$, Schur's second lemma then implies that if $Zi\sigma_k = i\sigma_k Z$ for $k = 1, 2, 3$, then Z is a multiple of the identity. Using eq. (56) with $Z = SS^{*\,-1}\sigma_2$ and noting that $(\sigma_2)^2 = \mathbf{I}$, it follows that

$$SS^{*\,-1} = k\sigma_2\,, \qquad \text{where } k \text{ is a non-zero complex number}\,. \tag{57}$$

Note that $k = 0$ is not allowed since S and σ_2 are invertible matrices. Taking the complex conjugate of eq. (57) and multiplying the result by eq. (57) yields

$$SS^{*\,-1}S^*S^{-1} = |k|^2\sigma_2\sigma_2^*\,,$$

which simplifies to

$$\mathbf{I} = -|k|^2\mathbf{I}\,.$$

No complex k exists that satisfies this equation, so we have a contradiction. Therefore, our initial assumption that the matrix A defined in eq. (55) is real must be false. Thus, there exists no nonsingular matrix S such that $S^{-1}i\sigma_k S$ is real for $k = 1, 2, 3$. This completes the proof that no basis exists in which the $i\sigma_k$ are simultaneously real.